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Distribution of linear statistics of singular values of the product of random matrices

FRIEDRICH GÖTZE[1], ALEXEY NAUMOV[2] and ALEXANDER TIKHOMIROV[3]

[1]*Faculty of Mathematics, Bielefeld University, P. O. Box 10 01 31, 33501, Bielefeld, Germany. E-mail: goetze@math.uni-bielefeld.de*
[2]*Faculty of Computational Mathematics and Cybernetics, Moscow State University, Leninskiye Gory, 119991, Moscow, Russia; Institute for Information Transmission Problems RAS, Bolshoy Karetny per. 19, bld.1, 127051, Moscow, Russia. E-mail: anaumov@cs.msu.su*
[3]*Department of Mathematics, Komi Science Center of Ural Division of RAS, Chernova 3a, 167000, Syktyvkar, Russia. E-mail: tikhomirov@dm.komisc.ru*

In this paper we consider the product of two independent random matrices $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$. Assume that $X_{jk}^{(q)}, 1 \le j, k \le n, q = 1, 2,$ are i.i.d. random variables with $\mathbb{E}X_{jk}^{(q)} = 0, \operatorname{Var} X_{jk}^{(q)} = 1$. Denote by $s_1(\mathbf{W}), \dots, s_n(\mathbf{W})$ the singular values of $\mathbf{W} := \frac{1}{n}\mathbf{X}^{(1)}\mathbf{X}^{(2)}$. We prove the central limit theorem for linear statistics of the squared singular values $s_1^2(\mathbf{W}), \dots, s_n^2(\mathbf{W})$ showing that the limiting variance depends on $\kappa_4 := \mathbb{E}(X_{11}^{(1)})^4 - 3$.

Keywords: central limit theorem; characteristic functions; Fuss–Catalan distributions; products of random matrices

1. Introduction and main result

One of the main questions studied in Random Matrix Theory (RMT) is the asymptotic analysis of spectra of random matrices when the dimension goes to infinity. For example, it is well known since the pioneering work of Wigner [28] that the empirical spectral distribution function weakly converges to the semicircle law. Another well known case is the sample covariance matrices $\mathbf{W} = \mathbf{X}\mathbf{X}^{\mathsf{T}}$, where $\mathbf{X}$ is a matrix with independent entries, which was first studied in Marčenko and Pastur [21]. The distribution of singular values of products of random matrices with independent entries has been intensively studied, see, for example, Alexeev *et al.* [3,4] and Akemann *et al.* [1].

All these results may be regarded as laws of large numbers for linear eigenvalue statistics. Thus fluctuations of such linear statistics of eigenvalues around its mean are of interest. There is a vast literature on this question. We mention the results Jonsson [19], Bai and Silverstein [7], Sinai and Soshnikov [25], Anderson and Zeitouni [5], Lytova and Pastur [20], Shcherbina [24], where the central limit theorem was proved. The aim of this paper is to investigate the case of singular values of *products* of random matrices with independent entries. It will be shown that in this case the central limit theorem holds as well and the limiting variance will be explicitly determined.

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For any $m, n \geq 1$ we consider a family of independent real random variables $X_{j,k}^{(q)}$, $1 \leq j, k \leq n$, $q = 1, \ldots, m$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that the following conditions (C0) are fulfilled:

(a) $X_{jk}^{(q)}$ are independent and identically distributed for $1 \leq j, k \leq n$, $q = 1, \ldots, m$;
(b) for any $1 \leq j, k \leq n$

$$\mathbb{E}X_{jk}^{(q)} = 0 \quad \text{and} \quad \mathbb{E}\big(X_{jk}^{(q)}\big)^2 = 1;$$

(c) $\mathbb{E}(X_{jk}^{(q)})^4 =: \mu_4 < \infty$.

The random variables $X_{jk}^{(q)}$ may depend on n, but for simplicity we shall not make this explicit in our notations.

We introduce m independent random matrices $\mathbf{X}^{(q)}$, $q = 1, \ldots, m$, as follows

$$\mathbf{X}^{(q)} := \frac{1}{\sqrt{n}}\big[X_{jk}^{(q)}\big]_{j,k=1}^{n}.$$

Denote by $s_1^2(\mathbf{W}), \ldots, s_n^2(\mathbf{W})$ the eigenvalues of $\mathbf{W}\mathbf{W}^{\mathsf{T}}$, where $\mathbf{W} := \prod_{q=1}^{m} \mathbf{X}^{(q)}$. We will often omit the notation $\mathbf{W}$ from $s_k^2(\mathbf{W})$ and write s_k^2. Define the empirical spectral measure by

$$F_n^{\mathbf{W}}(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}\big(s_k^2 \leq x\big).$$

Here and in what follows $\mathbb{1}(B)$ denotes the indicator of the event B.

A fundamental problem in the theory of random matrices is to determine the limiting distribution of F_n as the size of the random matrix tends to infinity. It was shown in Alexeev *et al.* [4] that there exists a function $G_m(x)$ such that

$$\lim_{n\to\infty} \sup_{x\in R}\big|\mathbb{E}F_n^{\mathbf{W}}(x) - G_m(x)\big| = 0 \tag{1.1}$$

and $G_m(x)$ are defined by its moments M_k, $k \in \mathbb{N}$,

$$M_k = \int_0^\infty x^k \, dG_m(x) = \frac{1}{mk+1}\binom{k}{mk+k},$$

which are Fuss–Catalan numbers. For $m = 1$, we get the well-known result of Marchenko and Pastur for sample covariance matrices Marčenko and Pastur [21]. The Fuss–Catalan numbers satisfy the following simple recurrence relation

$$M_k = \sum_{k_0+\cdots+k_m=k-1} \prod_{\nu=0}^{m} M_{k_\nu}.$$

An explicit analytical formula for the density function, say P_m, whose moment sequence is given by Fuss–Catalan numbers, that is,

$$\int_0^{K_m} x^k P_m(x)\,dx = M_k$$

for some positive numbers $K_m \geq 1$ was given in Penson and Życzkowski [23]. For this formula, see the Appendix A. Furthermore, it was shown in Penson and Życzkowski [23] that

$$K_m := \frac{(m+1)^{m+1}}{m^m}. \tag{1.2}$$

The limiting distribution $G_m(x)$ may be also described in terms of its Stieltjes transform, say $s_m(z)$,

$$s_m(z) := \int_{-\infty}^{\infty} \frac{1}{x-z}\,dG_m(x), \qquad z = u + iv, v > 0.$$

It was proved, see Götze *et al.* [15], that $s_m(z)$ satisfies the following equation

$$1 + zs_m(z) + (-1)^{m+1} z^m s_m^{m+1}(z) = 0. \tag{1.3}$$

The result (1.1) was proved under more general conditions than (C0), it was assumed that the random variables may be non-identically distributed and satisfy a Lindeberg type condition for the second moments, for details see Alexeev *et al.* [4]. Under conditions (C0) the result (1.1) may be generalized and it can be shown that F_n weakly converges to G_m in probability. The latter may be rewritten in the following way. For all for all continuous and bounded real functions $f(\lambda)$, $\lambda \in \mathbb{R}_+$, in probability

$$\lim_{n\to\infty} \int_0^{\infty} f(\lambda)\,dF_n^{\mathbf{W}}(\lambda) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\big(s_k^2\big) = \int_0^{\infty} f(\lambda)\,dG_m(\lambda). \tag{1.4}$$

We may interpret (1.4) as the law of large numbers. The natural question is to investigate a fluctuation of linear statistic

$$S_{\mathbf{W}}[f] := \sum_{k=1}^{n} f\big(s_k^2\big)$$

around its mean for an appropriate and broad class of test functions $f(\cdot)$.

1.1. Main result

To formulate the main result of this paper, we need to specify the class of test function. Let $f(\lambda)$ be a smooth function with the Fourier transform given by

$$\widehat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)e^{-it\lambda}\,d\lambda. \tag{1.5}$$

We say that an arbitrary function $f(\lambda)$ belongs to the *class* $\mathcal{F}$ if it satisfies the following regularity condition

$$\int_{-\infty}^{\infty}\left(1+|t|^5\right)\left|\widehat{f}(t)\right|dt<\infty. \tag{1.6}$$

We will concentrate on the case of two random matrices, $m=2$, and prove the following theorem which is the main result of this paper.

Theorem 1.1. *Let $m=2$ and assume that the conditions* (C0) *hold. For any function $f\in\mathcal{F}$ the centralized linear statistic*

$$S_{\mathbf{W}}^{(0)}[f]:=S_{\mathbf{W}}[f]-\mathbb{E}S_{\mathbf{W}}[f]$$

weakly converges to a Gaussian random variable G with zero mean and variance given by

$$\begin{aligned}\operatorname{Var}[G]={}&\frac{\kappa_4}{2}\left[\int_{-a}^{a}f\left(\lambda^2\right)\left[p(\lambda)+\lambda p'(\lambda)\right]d\lambda\right]^2\\&+\frac{1}{2\pi^2}\int_{-a}^{a}\int_{-a}^{a}\frac{(f(\lambda^2)-f(\mu^2))^2}{(\lambda-\mu)^2}\\&\times\frac{[p(\lambda)-p'(\lambda)(\lambda-\mu)]}{3p(\mu)}\frac{[4p_1(\mu)^4+11p_1(\mu)^2+4]}{4p_1^2(\mu)+3}\,d\lambda\,d\mu,\end{aligned} \tag{1.7}$$

where $\kappa_4:=\mu_4-3$, $p_1(\lambda):=\pi p(\lambda)$, $p(\lambda):=|\lambda|P_2(\lambda^2)$ is the symmetrized Fuss–Catalan density, and $a:=\sqrt{K_2}$.

Remark 1.1. Let us complement the results of this theorem by the following remarks.

1. Obviously the result of Theorem 1.1 depends on the distribution of $X_{jk}^{(q)}$, $1\le j,k\le n$, $q=1,2$, in terms of the fourth cumulant rather than the second moment only. This means that the limiting behaviour is not universal in the usual sense, a fact which is typical for the central limit theorems of linear eigenvalue statistics.
2. The result of Theorem 1.1 may be extended on the case $X_{jk}^{(q)}$, $1\le j,k\le n$, $q=1,2$ are non-identically distributed. Here one has to impose additional assumptions, for example Lindeberg's condition on the tails of fourth moments of $X_{jk}^{(q)}$, see Section 3 for details.
3. The case $m>2$ is much more difficult to analyse. One may derive a formula for $Y(x,t)$ (see the definition below). But it is not yet clear whether this expression is positive, due to the fact that the formula for $P_m(x)$, $m>3$ is rather complicated. We plan to study this case in a subsequent paper.

1.2. Structure of the paper

We divide the proof of Theorem 1.1 into two parts. In Section 2, we consider the Gaussian case and prove an analogue of Theorem 1.1 (see Theorem 2.1). Our method will be based on the results Lytova and Pastur [20], Tikhomirov [26] and [27]. Namely, we will apply the method of characteristic functions and show that the limiting characteristic function satisfies a differential equation of the type (2.11). The idea used in Lytova and Pastur [20] was to reformulate the problem in terms of the process $Y_n(x,t)$ (see definition (2.16)) and derive an equation for its limit (see (2.32)). This equation is of Volterra type, which may be solved applying a generalized Fourier transform method (see Statement D.1 in the Appendix). To implement this strategy we will often apply the following well-known Stein identity which is valid for a Gaussian random variable ξ with zero mean

$$\mathbb{E}\xi f(\xi) = \mathbb{E}\xi^2 \mathbb{E} f'(\xi). \tag{1.8}$$

We mention here that the idea to derive an equation for the limiting characteristic function was first applied in Tikhomirov [26,27] for sums of weakly dependent random variables, where the author derived the equation (2.11) using (1.8). In addition, we use empirical Poincaré inequalities (see Section 2.2) to estimate the variance of some quantities (see Lemma 2.3).

In the Section 3, we investigate the difference between the case of arbitrary standardized random variables and the case of Gaussian variables proving Theorem 1.1. Here we will use the methods of Bentkus [8] who introduced the following 'variance stable' comparison procedure. For studying the limiting behavior of some functional $\mathbb{E}f(X_1,\dots,X_n)$ depending on arbitrary r.v. $X_1,\dots,X_n$ Bentkus suggested to compare $\mathbb{E}\varphi(X_1,\dots,X_n)$ with $\mathbb{E}\varphi(Y_1,\dots,Y_n)$. Here $Y_1,\dots,Y_n$ is some *special* sequence of standardized r.v. (for example, Gaussian) for which $\mathbb{E}\varphi(Y_1,\dots,Y_n)$ may be easily calculated. To compare the distance he suggested to use the interpolation $Z_j(\phi) := X_j \sin\phi + Y_j \cos\phi$, $\phi \in [0, \frac{\pi}{2}]$, $j = 1,\dots,n$ (see (3.2)). Applying this procedure and ideas from Tikhomirov [26,27] mentioned above we show that the limiting characteristic function for $S_{\mathbf{W}}^{(0)}[f]$ for arbitrary r.v. satisfies some equation and may be expressed via the characteristic function for the Gaussian case (see Theorem 3.1). The proof is based on the differential equation (3.6) for the large n limit of the interpolated process (depending on ϕ) (3.3). Actually this technique simplifies the approach of Lytova and Pastur [20], where the authors apply a comparison procedure to derive an equation for $Y_n(x,t)$, which is similar to the Gaussian case.

All auxiliary facts about Fuss–Catalan distribution, unitary matrix decomposition and its derivatives are collected in Appendix A–D.

1.3. Applications

One motivation for investigating the asymptotic distribution of products of random matrices follows from recent applications in wireless telecommunication, see, for example, Müller [22]. One may consider a toy model of MIMO channel, where the output vector $y \in \mathbb{C}^n$, at a given time, equals

$$y = \mathbf{H}x + z,$$

where $x \in \mathbb{C}^n$ is the transmit vector, $\mathbf{H} \in \mathbb{C}^{n\times n}$ is a channel matrix and $z \in \mathbb{C}^n$ is a random noise. Matrix $\mathbf{H}$ may be random and represented as a product of $m, m \ge 1$, independent random matrices, that is, $\mathbf{H} = \mathbf{W}$. For example, if $m = 2$ we may assume that there exist transmit and receive antennas and scatters. The instance capacity of the MIMO channel is given by

$$\mathcal{I}_n := \log\det\bigl(\mathbf{I} + \gamma\mathbf{W}\mathbf{W}^{\mathsf{T}}\bigr) = \sum_{j=1}^{n}\log\bigl(1 + \gamma s_j^2(\mathbf{W})\bigr) = n\int_0^\infty f(x)\,dF_n^{\mathbf{W}}(x),$$

where $f(x) := \log(1 + \gamma x)$ and γ denotes the Signal to Noise Ratio per received antenna. This means that we rewrite capacity $\mathcal{I}$ in terms of linear statistics of singular values of $\mathbf{W}$. It is natural question to investigate the limiting behavior of $\mathcal{I}_n$ as n goes to infinity. Other possible applications are in finance Bouchaud *et al.* [10] and quantum entanglement Collins *et al.* [13], Zyczkowski *et al.* [30].

1.4. History

There are many papers on the CLT for linear eigenvalue statistics of random matrices. We mention the results Jonsson [19], Bai and Silverstein [7], Sinai and Soshnikov [25], Anderson and Zeitouni [5], Lytova and Pastur [20], Shcherbina [24]. In our setting, the result for $m = 1$ was derived in Lytova and Pastur [20]. We will use their ideas in the proof of Theorem 1.1. One may also find a lot of information about the CLT for linear eigenvalues statistics in the book Bai and Silverstein [6]. We also believe that one may apply the result of Bai and Silverstein [7] together with Zheng [29] to derive Theorem 1.1 when restricting oneself to the class of *analytic* test functions with a different (implicit) representation for the variance using Cauchy integrals.

The distribution of singular values of products of random matrices with independent entries has been intensively studied in many papers. The relevant literature is much to extensive in order to describe it here in detail, see, for example, Alexeev *et al.* [3,4], Penson and Życzkowski [23], Akemann *et al.* [1], Götze *et al.* [15] and very recent result Forrester and Liu [14]. The central limit theorem for product of complex Ginibre matrices for polynomial test functions was derived in Breuer and Duits [11]. It is known that in the complex Ginibre case the squares of singular values of $\mathbf{W}$ form a determinantal point process and the joint density function is a bi-orthogonal ensemble, see Akemann *et al.* [2].

1.5. Notations

In what follows, we will use the following notations. Denote by $\|\mathbf{A}\|$, $\|\mathbf{A}\|_2$ the operator and Hilbert–Schmidt norms of $\mathbf{A}$, respectively. As usual $\operatorname{Tr}\mathbf{A} = \sum_{i=1}^n \mathbf{A}_{ii}$. We assume that all random variables are defined on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. By $\operatorname{Var}(\xi)$ we mean $\mathbb{E}\xi^2 - (\mathbb{E}\xi)^2$, where $\mathbb{E}$ is the mathematical expectation with respect to $\mathbb{P}$. By C and c we denote some constants which do not depend on n. We introduce the symmetrized version of f, that is,

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \ge 0, \\ f(-x), & \text{if } x < 0. \end{cases}$$

By $*$ we denote the convolution operation, that is, $f * g(t) = \int_0^t f(s)g(t-s)\,ds$.

Let us denote by $\mathbf{e}_j$ a unit vector with all zeros except 1 in the position given by the index j. We define the following matrices $\mathbf{E}_{j,k} := \mathbf{e}_j\mathbf{e}_k^{\mathsf{T}}$. To denote an element of arbitrary matrix $\mathbf{M}$ in the position j,k, we will use either M_{jk} or $[\mathbf{M}]_{jk}$ depending on convenience. By $\mathbf{I}$ we denote an identity matrix, omitting the dependence on dimension. By $\mathbf{I}$, we denote an identity matrix. We omit the dependence on dimension.

Together with a random variable, say ξ, we will often use the following notation $\xi^{(0)} := \xi - \mathbb{E}\xi$.

2. The Gaussian case

In this section, we consider the special case when $X_{jk}^{(q)}$, $1 \le j,k \le n$, $q = 1,2$ has the Gaussian distribution. We change our notations of matrices and denote by $\mathbf{Y}^{(q)}$, $q = 1,2$ the matrix $\mathbf{X}^{(q)}$ with $X_{jk}^{(q)}$ replaced by the Gaussian random variables. The main result of this section is the following theorem.

Theorem 2.1 (Gaussian case). *Let $\mathbf{Y}^{(q)} = n^{-1/2}[Y_{jk}^{(q)}]_{j,k=1}^n$, $q = 1,2$, be independent random matrices such that the entries $Y_{jk}^{(q)}$, $j,k = 1,\dots,n$, $q = 1,2$, are Gaussian and satisfy the conditions* (C0). *Then for any $f \in \mathcal{F}$ the normalized linear statistic $S_{\mathbf{W}}^{(0)}[f]$ weakly converges to the Gaussian random variable G with zero mean and variance given by*

$$\begin{aligned}\widehat{\operatorname{Var}}[G] = {}& \frac{1}{2\pi^2}\int_{-a}^{a}\int_{-a}^{a}\frac{(f(\lambda^2)-f(\mu^2))^2}{(\lambda-\mu)^2} \\ & \times \frac{[p(\lambda)-p'(\lambda)(\lambda-\mu)]}{3p(\mu)}\frac{[4p_1(\mu)^4+11p_1(\mu)^2+4]}{4p_1^2(\mu)+3}\,d\lambda\,d\mu,\end{aligned} \tag{2.1}$$

where $p_1(\lambda) := \pi p(\lambda)$, $p(\lambda) := |\lambda|P_2(\lambda^2)$ is the symmetrized Fuss–Catalan density, and $a := \sqrt{K_2}$.

2.1. Symmetrization

To prove Theorem 2.1, it will be convenient for the further analysis to introduce the following symmetrization. Let ξ be a positive random variable and $F(x)$ be the distribution function of ξ^2. Define $\widetilde{\xi} := \varepsilon\xi$, where ε denotes a Rademacher random variable with $\mathbb{P}\{\varepsilon = \pm 1\} = 1/2$ which is independent of ξ. Let $\widetilde{F}(x)$ denote the distribution function of $\widetilde{\xi}$. It satisfies the following equation

$$\widetilde{F}(x) = \frac{1}{2}\bigl[1 + \operatorname{sgn}(x)F\bigl(x^2\bigr)\bigr]. \tag{2.2}$$

Lemma 2.1. *For any one-sided distribution function $F(x)$ and $G(x)$ we have*

$$\sup_{x\ge 0}\bigl|F(x)-G(x)\bigr| = 2\sup_x\bigl|\widetilde{F}(x)-\widetilde{G}(x)\bigr|,$$

where $\widetilde{F}(x)$ $(\widetilde{G}(x))$ *denotes the symmetrization of* $F(x)$ $(G(x)$ *respectively) according to* (2.2).

Proof. By (2.2), we have for any $x \ge 0$

$$F(x) = 2\widetilde{F}(\sqrt{x}) - 1,$$
$$G(x) = 2\widetilde{G}(\sqrt{x}) - 1.$$

This implies

$$\sup_{x\ge 0}\bigl|F(x) - G(x)\bigr| = 2\sup_{x\ge 0}\bigl|\widetilde{F}(\sqrt{x}) - \widetilde{G}(\sqrt{x})\bigr| = 2\sup_{x}\bigl|\widetilde{F}(x) - \widetilde{G}(x)\bigr|.$$

Thus lemma is proved. □

We apply this lemma to the distribution of the squared singular values of the matrix $\mathbf{W}$. Let us denote

$$\mathbf{H}^{(\nu)} = \begin{pmatrix} \mathbf{Y}^{(\nu)} & \mathbf{O} \\ \mathbf{O} & [\mathbf{Y}^{(m-\nu+1)}]^{\mathsf{T}} \end{pmatrix} \quad \text{and} \quad \mathbf{J} := \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}. \tag{2.3}$$

Recall that $\mathbf{I}$ (with sub-index or without it) denotes the unit matrix of corresponding order, and $\mathbf{O}$ is a zero matrix. For any $1 \le a, b \le m$, put

$$\mathbf{V}_{[a,b]} := \begin{cases} \prod_{k=a}^{b} \mathbf{H}^{(k)}, & \text{for } a \le b, \\ \mathbf{I} & \text{otherwise}, \end{cases} \tag{2.4}$$

and $\mathbf{V} := \mathbf{V}_{[1,m]}$, $\widetilde{\mathbf{V}} := \mathbf{V}\mathbf{J}$. Note that $\widetilde{\mathbf{V}}$ is a symmetric matrix. The eigenvalues of the matrix $\widetilde{\mathbf{V}}$ are $\pm s_1(\mathbf{W}), \dots, \pm s_n(\mathbf{W})$. Note that the symmetrization of the distribution function $F_n^{\mathbf{W}}(x)$ is a function $F_n^{\widetilde{\mathbf{V}}}(x)$ which is the empirical distribution function of the eigenvalues of the matrix $\widetilde{\mathbf{V}}$. In the similar way, we define $\widetilde{G}_m$. According to Lemma 2.1, we get

$$\sup_x\bigl|\mathbb{E}F_n^{\mathbf{W}}(x) - G_m(x)\bigr| = 2\sup_x\bigl|\mathbb{E}F_n^{\widetilde{\mathbf{V}}}(x) - \widetilde{G}_m(x)\bigr|,$$

and (1.4) may be rewritten as follows. In probability

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\widetilde{f}(x)\,dF_n^{\widetilde{\mathbf{V}}}(x) = \lim_{n\to\infty}\frac{1}{2n}\sum_{k=1}^{n}\bigl[f(s_k) + f(-s_k)\bigr] = \int_{-\infty}^{\infty}\widetilde{f}(x)\,d\widetilde{G}_m(x).$$

Let us denote by $\widetilde{s}_m(z)$ the Stieltjes of $\widetilde{G}_m(x)$. Applying $\widetilde{s}_m(z) = zs_m(z^2)$ and equation (1.3) it is straightforward to check that $\widetilde{s}_m(z)$ satisfies the following equation

$$1 + z\widetilde{s}_m(z) + (-1)^{m+1}z^{m-1}\widetilde{s}_m^{\,m+1}(z) = 0. \tag{2.5}$$

To conclude this "linearization" procedure, we mention that

$$S_{\mathbf{W}}[f] = \int_0^\infty f(x)\, dF_n^{\mathbf{W}}(x) = \int_{-\infty}^\infty \widetilde{f}\big(x^2\big)\, dF_n^{\widetilde{\mathbf{V}}}(x) =: S_{\widetilde{\mathbf{V}}}[\widetilde{f} \circ g],$$

where $g(x) := x^2$. This means that we may substitute $f(\cdot)$ by $f \circ g(\cdot)$ and consider its symmetrization $\widetilde{f \circ g}(\cdot)$. In what follows we will consider symmetrized distribution functions only and omit the symbol "$\widetilde{\cdot}$" in the corresponding notations. We also omit the argument f and index $\widetilde{\mathbf{V}}$ from the notations of $S_{\widetilde{\mathbf{V}}}[f]$ and $S_{\widetilde{\mathbf{V}}}^{(0)}[f]$ writing S, $S^{(0)}$, respectively.

2.2. Empirical Poincaré inequalities

Assume that the random variables $X_1, \ldots, X_n$ have a joint distribution μ on $\mathbb{R}^n$, satisfying the Poicare-type inequality (2.7). Let F_n be the empirical measure, defined for observations $X_1 = x_1, \ldots, X_n = x_n$. Given a bounded smooth complex-valued function $\varphi(x)$ on the real line we shall estimate the variance

$$\mathbb{E}\left|\int_{-\infty}^\infty \varphi(x)\, dF_n(x) - \int_{-\infty}^\infty \varphi(x)\, dF(x)\right|^2, \tag{2.6}$$

where $F(x) := \mathbb{E}F_n(x)$. In the next subsection, we will often use such bounds for various functions φ.

Following Bobkov *et al.* [9], we say that a probability measure μ on $\mathbb{R}^d$ satisfies a Poincaré-type inequality with constant σ^2 if for any bounded smooth function g on $\mathbb{R}^d$ with gradient ∇g,

$$\operatorname{Var}(g) \le \sigma^2 \int_{\mathbb{R}^d} |\nabla g|^2\, d\mu, \tag{2.7}$$

where $\operatorname{Var}(g) = \int_{\mathbb{R}^d} g^2\, d\mu - (\int_{\mathbb{R}^d} g\, d\mu)^2$. In this case, we write $\mathrm{PI}(\sigma^2)$ for short. We apply (2.7) to the following function

$$g(x_1, \ldots, x_n) = \frac{\varphi(x_1) + \cdots + \varphi(x_n)}{n} = \int_{-\infty}^\infty \varphi(x)\, dF_n(x).$$

Calculating $|\nabla g|^2$ we obtain the following formula

$$|\nabla g|^2 = \frac{|\varphi'(x_1)|^2 + \cdots + |\varphi'(x_n)|^2}{n^2} = \frac{1}{n} \int_{-\infty}^\infty \left|\varphi'(x)\right|^2 dF_n(x).$$

Together (2.7) this yields an estimate for the variance (2.6) (see Bobkov *et al.* [9], Proposition 4.3).

Statement 2.1. *Assume that the joint distribution of r.v.* $X_1, \ldots, X_n$ *satisfies* $\mathrm{PI}(\sigma^2)$. *Then for any smooth* F*-integrable function* $\varphi : \mathbb{R} \to \mathbb{C}$,

$$\mathbb{E}\left|\int_{-\infty}^\infty \varphi(x)\, dF_n(x) - \int_{-\infty}^\infty \varphi(x)\, dF(x)\right|^2 \le \frac{\sigma^2}{n} \int_{-\infty}^\infty \left|\varphi'(x)\right|^2 dF(x).$$

We will use the following linearization trick from Burda *et al.* [12]. Let us consider the matrix $\widetilde{\mathbf{V}} = [\prod_{j=1}^{m-1} \mathbf{H}^{(j)}]\mathbf{H}^{(m)}\mathbf{J}$. We form the following $mn \times mn$ matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{O} & \mathbf{H}^{(1)} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{H}^{(2)} & \mathbf{O} & \dots & \mathbf{O} \\ & & & & \dots & \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{H}^{(m-1)} \\ \mathbf{H}^{(m)}\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \end{bmatrix}.$$

Then the mth power of $\mathbf{M}$ is a diagonal block matrix, there the first block is equal to $\widehat{\mathbf{V}}$, the second – $\mathbf{H}^{(2)}\mathbf{H}^{(3)}\cdots\mathbf{H}^{(m)}\mathbf{J}\mathbf{H}^{(1)}$ and so on. The eigenvalues of $\mathbf{M}^m$ are the eigenvalues of $\widetilde{\mathbf{V}}$ with multiplicity m. We denote the eigenvalues of $\mathbf{M}$ by $\lambda_1, \dots, \lambda_{mn}$ and their empirical distribution function by $G_n(\lambda)$. Then we have for an even function f

$$\int \varphi(x)\, dF_n(x) = \frac{1}{n}\sum_{j=1}^{n} \varphi(s_j) = \frac{1}{2nm}\sum_{j=1}^{2mn} \varphi\bigl(\lambda_i^m\bigr) = \int \varphi\bigl(\lambda^m\bigr)\, dG_n(\lambda).$$

Without loss of generality, we assume that $\lambda_1, \dots, \lambda_n$ are real positive eigenvalues $s_1^{1/m}, \dots, s_n^{1/m}$. All other eigenvalues may be derived by a rotation on an angle $\theta_k = \frac{k\pi}{m}$, $k = 1, \dots, 2m-1$. Let $\theta_0 = 0$. We denote the empirical spectral distribution of $e^{i\theta_k}\lambda_1, \dots, e^{i\theta_k}\lambda_n$ by $G_{n,k}$. It is easy to see that

$$\int \varphi\bigl(\lambda^m\bigr)\, dG_n(\lambda) = \frac{1}{2m}\sum_{k=0}^{2m-1} \int_{\mathbb{T}_k} \varphi\bigl(\lambda^m\bigr)\, dG_{n,k}(\lambda), \tag{2.8}$$

where $\mathbb{T}_k = e^{i\theta_k}\mathbb{R}$.

The joint distribution $\mathbb{P}^{\mathbf{y}}$ of the collection $\{Y_{jk}^{(q)}, j,k = 1, \dots, n, q = 1, \dots, m\}$ represents a product probability measure on the Euclidean space $\mathbb{R}^N$ of dimension $N = mn^2$, while the joint distribution μ of the spectral values $\lambda_1, \dots, \lambda_n$ is a probability measure on $\mathbb{R}^n$, obtained from $\mathbb{P}^{\mathbf{y}}$ as the image under the map $T = \Pi \circ \Lambda$, where Λ is the map from matrices to their eigenvalues and Π is the projector on the subspace of the dimension n. We will apply the following result.

Lemma 2.2. *Let $\mu_1, \dots, \mu_N$ be probability measures on $\mathbb{R}$, satisfying* $\mathrm{PI}(\sigma^2)$. *The image of the product measure $\mu_1 \otimes \cdots \otimes \mu_N$ under any Lipshitz map $T : \mathbb{R}^N \to \mathbb{R}^n$ satisfies* $\mathrm{PI}(\sigma^2\|T\|_{\mathrm{Lip}}^2)$, *where*

$$\|g\|_{\mathrm{Lip}} := \sup_{x \neq y} \frac{\rho_2(g(x), g(y))}{\rho_1(x, y)}$$

and ρ_1, ρ_2 are metrics in $\mathbb{R}^N$ and $\mathbb{R}^n$, respectively.

Proof. See Bobkov *et al.* [9], Lemma 7.1. □

Let us denote by $\widehat{\mathbf{M}}$ a perturbation of $\mathbf{M}$ and introduce similar notations for $\widehat{Y}_{jk}^{(q)}$. Applying the Hoffman–Wielandt inequality (see, for example, Hoffman and Wielandt [18]) we obtain

$$\sum_{j=1}^{n}\bigl|\lambda_j(\mathbf{M})-\lambda_j(\widehat{\mathbf{M}})\bigr|^2 \le \|\mathbf{M}-\widehat{\mathbf{M}}\|_2^2=\frac{2}{n}\sum_{q=1}^{m}\sum_{j,k=1}^{n}\bigl|Y_{jk}^{(q)}-\widehat{Y}_{jk}^{(q)}\bigr|^2,$$

thus $\|T\|_{\mathrm{Lip}}=\frac{\sqrt{2}}{\sqrt{n}}$. Since the distribution of $Y_{11}^{(q)}$ satisfies $\mathrm{PI}(\sigma^2)$ it follows from (2.8) and Statement 2.1 that

$$\begin{aligned}\mathbb{E}&\biggl|\int_{-\infty}^{\infty}\varphi(x)\,dF_n(x)-\int_{-\infty}^{\infty}\varphi(x)\,dF(x)\biggr|^2\\&\le\frac{\sigma^2m^2}{n^2}\int_{-\infty}^{\infty}|x|^{\frac{2m-2}{m}}\bigl|\varphi'(x)\bigr|^2\,dF(x).\end{aligned}\tag{2.9}$$

2.3. Proof of CLT in the Gaussian case

In this subsection, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. For the proof, we shall use the method of characteristic functions. Recall the convention made in Section 2.1 to use S and $S^{(0)}$ instead of $S_{\widetilde{\mathbf{V}}}[f]$ and $S_{\widetilde{\mathbf{V}}}^{(0)}[f]$, respectively. Let us denote the characteristic function of $S^{(0)}$ by $Z_n(x)$, that is,

$$Z_n(x):=\mathbb{E}e^{ixS^0}.$$

To prove Theorem 2.1, it is sufficient to derive that

$$\lim_{n\to\infty}Z_n(x)=Z(x),$$

where $Z(x)$ is a characteristic function of the Gaussian random variable G with zero mean and variance given by the formula (2.1), that is,

$$Z(x):=\mathbb{E}e^{ixG}=e^{-\widehat{\mathrm{Var}}[G]\frac{x^2}{2}}.\tag{2.10}$$

One has to show that

$$Z(x)=1-\widehat{\mathrm{Var}}[G]\int_0^x yZ(y)\,dy.\tag{2.11}$$

It is obvious that

$$Z_n(x)=1+\int_0^x Z_n'(y)\,dy.$$

Similarly to Lytova and Pastur [20] it is sufficient to prove that any converging subsequences $\{Z_{n_l}\}$ and $\{Z'_{n_l}\}$ satisfy

$$\lim_{n_l\to\infty} Z_{n_l}(x) = Z(x), \qquad \lim_{n_l\to\infty} Z'_{n_l}(x) = -xZ(x)\widehat{\mathrm{Var}}[G] \tag{2.12}$$

and show that $\widehat{\mathrm{Var}}[G]$ is given by the formula (2.1).

We now implement the approach outlined before. Using the derivative of $Z_n(x)$ and applying the Fourier inverse formula

$$f(\lambda) = \int_{-\infty}^{\infty} \widehat{f}(t)e^{it\lambda}\,dt \tag{2.13}$$

we obtain the following representation for $Z'_n(x)$

$$Z'_n(x) = i\mathbb{E}S^{(0)}e^{ixS^{(0)}} = \frac{i}{2}\int_{-\infty}^{\infty} \widehat{f}(t)\mathbb{E}\big[\operatorname{Tr}\mathbf{U}(t) - \mathbb{E}\operatorname{Tr}\mathbf{U}(t)\big]e^{ixS^{(0)}}\,dt, \tag{2.14}$$

where $\mathbf{U}(t)$ denotes unitary transform of $\mathbf{W}$, that is,

$$\mathbf{U}(t) := e^{it\widetilde{\mathbf{V}}}. \tag{2.15}$$

Furthermore, we use the notations

$$\begin{aligned} &u_n(t) := \frac{1}{2}\operatorname{Tr}\mathbf{U}(t), \qquad u_n^{(0)}(t) := u_n(t) - \mathbb{E}u_n(t), \qquad e_n(x) := e^{ixS^{(0)}}, \\ &Y_n(x,t) := \mathbb{E}u_n^{(0)}(t)e_n(x). \end{aligned} \tag{2.16}$$

From the unitary matrix representation, see (B.5) in the Appendix, it follows that

$$u_n(t) = \sum_{j=1}^{n} U_{jj}(t) = \sum_{j=1}^{n} U_{j+n,j+n}(t).$$

In these notations we may rewrite (2.14) as follows

$$Z'_n(x) = i\int_{-\infty}^{\infty} \widehat{f}(t)Y_n(x,t)\,dt. \tag{2.17}$$

The next lemma gives estimates for the variance of $u_n(t)$ and its derivative $u'_n(t)$ with respect to the argument t, and $Y_n(x,t)$.

Lemma 2.3. *Under condition of Theorem* 2.1, *we have*

$$\operatorname{Var}\big(u_n(t)\big) \le C_1t^2, \qquad \operatorname{Var}\big(u'_n(t)\big) \le C_2\big(1+t^2\big), \qquad \big|Y_n(x,t)\big| \le \sqrt{C_1}t.$$

Proof. The statement of this lemma for $u_n(t)$ and $u_n'(t)$ follows from (2.9) applied to $\varphi(x) = \cos(tx)$ and $\varphi(x) = -x\sin(tx)$ respectively. From the Cauchy–Schwarz inequality, we conclude that

$$\left|Y_n(x,t)\right| = \left|\mathbb{E}\left(\left(u_n(t) - \mathbb{E}u_n(t)\right)e_n(x)\right)\right| \le \operatorname{Var}^{1/2}\left(u_n(t)\right) \le \sqrt{C_1}t. \qquad \square$$

From Lemma 2.3, we may conclude that

$$\left|\frac{\partial Y_n(x,t)}{\partial t}\right| \le \operatorname{Var}^{1/2}\left(u_n'(t)\right) \le C_2^{1/2}\sqrt{1+t^2}$$

and

$$\left|\frac{\partial Y_n(x,t)}{\partial x}\right| \le \operatorname{Var}^{1/2}\left(u_n(t)\right)\operatorname{Var}^{1/2}\left(S^0\right) \le C_1^{1/2} t \sup_{\lambda\in\mathbb{R}} f'(\lambda).$$

One may see that $Y_n(x,t)$ is bounded and equicontinues on any finite set of $\mathbb{R}^2$. Similarly to Lytova and Pastur [20] it is sufficient to show that any uniformly converging subsequence of $\{Y_n\}$ has the same limit Y, together with (2.17) leading to (2.12). In the rest of the proof, we derive an equation for $Y(x,t)$ and solve it.

We start from the Duhamel formula

$$\mathbf{U}(t) = \mathbf{I} + i\int_0^t \widetilde{\mathbf{V}}\mathbf{U}(s)\,ds$$

and obtain the following representation for $Y_n(x,t)$

$$Y_n(x,t) = \frac{i}{2}\int_0^t \mathbb{E}\left[\operatorname{Tr}\widetilde{\mathbf{V}}\mathbf{U}(s) - \mathbb{E}\operatorname{Tr}\widetilde{\mathbf{V}}\mathbf{U}(s)\right]e_n(x)\,ds = \frac{1}{2}\mathcal{A}_1 + \frac{1}{2}\mathcal{A}_2,$$

where

$$\mathcal{A}_1 := \frac{i}{\sqrt{n}}\int_0^t \sum_{j,k=1}^n \mathbb{E}\left[Y_{jk}^{(1)}\left[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\right]_{kj} - \mathbb{E}Y_{jk}^{(1)}\left[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\right]_{kj}\right]e_n(x)\,ds,$$

$$\mathcal{A}_2 := \frac{i}{\sqrt{n}}\int_0^t \sum_{j,k=1}^n \mathbb{E}\left[Y_{jk}^{(2)}\left[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\right]_{j+n,k+n} - \mathbb{E}Y_{jk}^{(2)}\left[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\right]_{j+n,k+n}\right]e_n(x)\,ds.$$

Let us consider the term $\mathcal{A}_1$. We will often apply the following well-known identity which is valid for a Gaussian random variable ξ with zero mean

$$\mathbb{E}\xi f(\xi) = \mathbb{E}\xi^2\mathbb{E}f'(\xi). \tag{2.18}$$

Applying (2.18) to $\mathcal{A}_1$ we rewrite it as a sum of two terms

$$\mathcal{A}_1 = \mathcal{I}_1 + \mathcal{I}_2, \tag{2.19}$$

where

$$\mathcal{I}_1 := \frac{i}{\sqrt{n}} \int_0^t \sum_{j,k=1}^n \mathbb{E}\left[\left[\mathbf{H}^{(2)}\mathbf{J}\frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(1)}}\right]_{kj} - \mathbb{E}\left[\mathbf{H}^{(2)}\mathbf{J}\frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(1)}}\right]_{kj}\right] e_n(x)\, ds,$$

$$\mathcal{I}_2 := -\frac{x}{\sqrt{n}} \int_0^t \sum_{j,k=1}^n \mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{kj} \frac{\partial S}{\partial Y_{jk}^{(1)}} e_n(x)\, ds.$$

From Lemma B.3 in the Appendix it follows that

$$\begin{aligned}
\sum_{j,k=1}^n \left[\mathbf{H}^{(2)}\mathbf{J}\frac{\partial \mathbf{U}(s)}{\partial Y_{jk}^{(1)}}\right]_{kj} &= \frac{i}{\sqrt{n}} \sum_{j,k=1}^n \sum_{l=1}^{2n} \big[\mathbf{H}^{(2)}\mathbf{J}\big]_{kl} \big[\mathbf{U}\mathbf{H}^{(1)}\big]_{l,k+n} * [\mathbf{U}]_{jj}(s) \\
&\quad + \frac{i}{\sqrt{n}} \sum_{j,k=1}^n \sum_{l=1}^{2n} \big[\mathbf{H}^{(2)}\mathbf{J}\big]_{kl} \big[\mathbf{U}\mathbf{H}^{(1)}\big]_{j,k+n} * [\mathbf{U}]_{lj}(s) \\
&= \frac{i}{\sqrt{n}} \int_0^s u_n(s-s_1) \sum_{k=1}^n \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s_1)\mathbf{H}^{(1)}\big]_{k,k+n}\, ds_1 \\
&\quad + \frac{i}{\sqrt{n}} \int_0^s \sum_{j,k=1}^n \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s_1)\big]_{kj} \big[\mathbf{U}(s-s_1)\mathbf{H}^{(1)}\big]_{j,k+n}\, ds_1,
\end{aligned}$$

where we used convolution notation $*$ given in Section 1.5. We introduce further notations

$$t_n(s) := \sum_{k=1}^n \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k,k+n}, \qquad t_n^{(0)}(s) := t_n(s) - \mathbb{E}t_n(s).$$

In these notations we may write, applying Lemma C.1 in the Appendix,

$$\mathcal{I}_1 = -\frac{1}{n} \int_0^t ds \int_0^s \mathbb{E}\big[u_n(s-s_1)t_n(s_1) - \mathbb{E}u_n(s-s_1)t_n(s_1)\big] e_n(x)\, ds_1 + r_n(t),$$

where

$$\big|r_n(t)\big| \le C\frac{t^3}{\sqrt{n}}.$$

In what follows for simplicity we will not specify the term $r_n(t)$, but one should have in mind that $r_n(t)$ goes to zero as n goes to infinity. Let us rewrite the difference $t_n(s_1)u_n(s-s_1) - \mathbb{E}t_n(s_1)u_n(s-s_1)$. We obtain

$$\begin{aligned}
t_n(s_1)u_n(s-s_1) &= t_n^{(0)}(s_1)u_n^{(0)}(s-s_1) + t_n^{(0)}(s_1)\mathbb{E}u_n(s-s_1) \\
&\quad + u_n^{(0)}(s-s_1)\mathbb{E}t_n(s_1) + \mathbb{E}t_n(s_1)\mathbb{E}u_n(s-s_1).
\end{aligned} \tag{2.20}$$

Substituting $\mathbb{E}t_n(s_1)u_n(s-s_1)$ from both sides of (2.20) we arrive at the identity

$$\begin{aligned}
&t_n(s_1)u_n(s-s_1)-\mathbb{E}t_n(s_1)u_n(s-s_1)\\
&\quad=t_n^{(0)}(s_1)u_n^{(0)}(s-s_1)+t_n^{(0)}(s_1)\mathbb{E}u_n(s-s_1)+u_n^{(0)}(s-s_1)\mathbb{E}t_n(s_1)\\
&\qquad-\mathbb{E}t_n^{(0)}(s_1)u_n^{(0)}(s-s_1).
\end{aligned}\tag{2.21}$$

Applying (2.21), we may rewrite the term $\mathcal{I}_1$ as follows

$$\begin{aligned}
\mathcal{I}_1&=-\frac{1}{n}\int_0^t ds\int_0^s \mathbb{E}u_n(s-s_1)\mathbb{E}t_n^{(0)}(s_1)e_n(x)\,ds_1\\
&\quad-\frac{1}{n}\int_0^t ds\int_0^s \mathbb{E}t_n(s-s_1)Y_n(x,s_1)\,ds_1+r_n(t)=:\mathcal{I}_{11}+\mathcal{I}_{12}+r_n(t).
\end{aligned}$$

Let us investigate $t_n(s)$. Using (2.18), we obtain

$$\begin{aligned}
\mathbb{E}t_n(s)&=\frac{1}{\sqrt{n}}\sum_{j,k=1}^n \mathbb{E}Y_{jk}^{(2)}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,j+n}\\
&=\frac{1}{\sqrt{n}}\sum_{j,k=1}^n \mathbb{E}\bigg[\frac{\partial\mathbf{U}(s)}{\partial Y_{jk}^{(2)}}\mathbf{H}^{(1)}\bigg]_{k+n,j+n}\\
&\quad+\frac{1}{\sqrt{n}}\sum_{j,k=1}^n \mathbb{E}\bigg[\mathbf{U}(s)\frac{\partial\mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}}\bigg]_{k+n,j+n}.
\end{aligned}\tag{2.22}$$

From Lemma B.4 in the Appendix, we conclude

$$\begin{aligned}
&\sum_{j,k=1}^n\bigg[\frac{\partial\mathbf{U}(s)}{\partial Y_{jk}^{(2)}}\mathbf{H}^{(1)}\bigg]_{k+n,j+n}\\
&\quad=\frac{i}{\sqrt{n}}\int_0^s\sum_{j,k=1}^n\sum_{l=1}^{2n}\big[\mathbf{U}(s_1)\big]_{k+n,k+n}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s-s_1)\big]_{j+n,l}\big[\mathbf{H}^{(1)}\big]_{l,j+n}\,ds_1\\
&\qquad+\frac{i}{\sqrt{n}}\int_0^s\sum_{j,k=1}^n\sum_{l=1}^{2n}\big[\mathbf{U}(s_1)\big]_{l,k+n}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s-s_1)\big]_{j+n,k+n}\big[\mathbf{H}^{(1)}\big]_{l,j+n}\,ds_1\\
&\quad=\frac{i}{\sqrt{n}}\int_0^s u_n(s_1)\sum_{j=1}^n\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s-s_1)\mathbf{H}^{(1)}\big]_{j+n,j+n}\,ds_1\\
&\qquad+\frac{is}{\sqrt{n}}\sum_{k=1}^n\big[\mathbf{U}(s)\widetilde{\mathbf{V}}\big]_{k+n,k+n}.
\end{aligned}\tag{2.23}$$

It is easy to see that

$$\begin{aligned}\sum_{j=1}^{n}&\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s-s_1)\mathbf{H}^{(1)}\big]_{j+n,j+n}\\ &=\sum_{j=1}^{n}\big[\widetilde{\mathbf{V}}\mathbf{U}(s-s_1)\big]_{j+n,j+n}\\ &=-i\sum_{j=1}^{n}\big[\mathbf{U}'(s-s_1)\big]_{j+n,j+n}=-iu_n'(s-s_1)\end{aligned}\tag{2.24}$$

and

$$\sum_{k=1}^{n}\big[\mathbf{U}(s)\widetilde{\mathbf{V}}\big]_{k+n,k+n}=-iu_n'(s).\tag{2.25}$$

For the second term in (2.22), we have

$$\sum_{j,k=1}^{n}\mathbb{E}\bigg[\mathbf{U}(s)\frac{\partial\mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}}\bigg]_{k+n,j+n}=\sqrt{n}\mathbb{E}u_n(s).$$

It follows from (2.22)–(2.25) that

$$\mathbb{E}t_n(s)=\frac{1}{n}\int_0^s\mathbb{E}u_n(s_1)u_n'(s-s_1)\,ds_1+\mathbb{E}u_n(s)+\frac{s}{n}\mathbb{E}u_n'(s).\tag{2.26}$$

Using (2.26), we may rewrite the term $\mathcal{I}_{12}$ as follows

$$\begin{aligned}\mathcal{I}_{12}=&-\frac{1}{n^2}\int_0^t ds\int_0^s Y_n(x,s_1)\,ds_1\int_0^{s-s_1}\mathbb{E}u_n(s_2)u_n'(s-s_1-s_2)\,ds_2\\ &-\frac{1}{n}\int_0^t ds\int_0^s Y_n(x,s_1)\mathbb{E}u_n(s-s_1)\,ds_1\\ &-\frac{1}{n^2}\int_0^t ds\int_0^s Y_n(x,s_1)\mathbb{E}u_n'(s-s_1)\,ds_1.\end{aligned}$$

Since $|Y_n(x,s)|\le C$ (see Lemma 2.3) and $\mathbb{E}|u_n'(s-s_1)|\le n\sqrt{n}$ we get

$$\begin{aligned}\mathcal{I}_{12}=&-\frac{1}{n^2}\int_0^t ds\int_0^s Y_n(x,s_1)\,ds_1\int_0^{s-s_1}\mathbb{E}u_n(s_2)u_n'(s-s_1-s_2)\,ds_2\\ &-\frac{1}{n}\int_0^t ds\int_0^s Y_n(x,s_1)\mathbb{E}u_n(s-s_1)\,ds_1+r_n(t).\end{aligned}$$

Applying Lemma 2.3, we obtain

$$\mathcal{I}_{12} = -\frac{1}{n^2}\int_0^t ds\int_0^s Y_n(x,s_1)\,ds_1\int_0^{s-s_1}\mathbb{E}u_n(s_2)\mathbb{E}u_n'(s-s_1-s_2)\,ds_2$$
$$-\frac{1}{n}\int_0^t ds\int_0^s Y_n(x,s_1)\mathbb{E}u_n(s-s_1)\,ds_1 + r_n(t).$$

Changing the limits of integration, we get

$$\mathcal{I}_{12} = -\frac{1}{n^2}\int_0^t Y_n(x,s)\,ds\int_0^{t-s}\mathbb{E}u_n(s_1)\mathbb{E}u_n(t-s-s_1)\,ds_1 + r_n(t).$$

We investigate now the quantity $\mathbb{E}t_n^{(0)}(s)e_n(x)$. Applying (2.18) we come to the following representation

$$\begin{aligned}
\mathbb{E}t_n^{(0)}(s)e_n(x) &= \frac{1}{\sqrt{n}}\sum_{j,k=1}^n \mathbb{E}\big[Y_{jk}^{(2)}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,j+n} - \mathbb{E}Y_{jk}^{(2)}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,j+n}\big]e_n(x)\\
&= \frac{1}{\sqrt{n}}\sum_{j,k=1}^n \mathbb{E}\left[\left[\frac{\partial\mathbf{U}(s)}{\partial Y_{jk}^{(2)}}\mathbf{H}^{(1)}\right]_{k+n,j+n} - \mathbb{E}\left[\frac{\partial\mathbf{U}(s)}{\partial Y_{jk}^{(2)}}\mathbf{H}^{(1)}\right]_{k+n,j+n}\right]e_n(x)\\
&\quad + \frac{1}{\sqrt{n}}\sum_{j,k=1}^n \mathbb{E}\left[\left[\mathbf{U}(s)\frac{\partial\mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}}\right]_{k+n,j+n} - \mathbb{E}\left[\mathbf{U}(s)\frac{\partial\mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}}\right]_{k+n,j+n}\right]e_n(x)\\
&\quad + \frac{ix}{\sqrt{n}}\sum_{j,k=1}^n \mathbb{E}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,j+n}\frac{\partial S}{\partial Y_{jk}^{(2)}}e_n(x) =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}$$

For the first term $\mathcal{J}_1$ we may use (2.23) and get

$$\mathcal{J}_1 = \frac{1}{n}\int_0^s \mathbb{E}\big[u_n(s_1)u_n'(s-s_1) - \mathbb{E}u_n(s_1)u_n'(s-s_1)\big]e_n(x)\,ds_1$$
$$+\frac{s}{n}\mathbb{E}\big[u_n'(s) - \mathbb{E}u_n'(s)\big]e_n(x).$$

Repeating the step (2.20) and (2.21) the last relation may be rewritten in the following way

$$\mathcal{J}_1 = \frac{1}{n}\int_0^s \big[\mathbb{E}u_n(s_1)\mathbb{E}\big(u_n^{(0)}\big)'(s-s_1)e_n(x) + \mathbb{E}u_n'(s-s_1)\mathbb{E}u_n^{(0)}(s_1)e_n(x)\big]\,ds_1$$
$$+\frac{s}{n}\mathbb{E}\big[u_n'(s) - \mathbb{E}u_n'(s)\big]e_n(x).$$

For the second term $\mathcal{J}_2$ we have

$$\mathcal{J}_2 = Y_n(x,s).$$

Let us consider now the term $\mathcal{J}_3$. We may write

$$
\begin{aligned}
\mathcal{J}_3 &= \frac{ix}{\sqrt{n}} \sum_{j,k=1}^{n} \mathbb{E}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,j+n} \frac{\partial S}{\partial Y_{jk}^{(2)}} e_n(x) \\
&= \frac{ix}{n} \sum_{j,k=1}^{n} \mathbb{E}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,j+n} \big[\mathbf{H}^{(2)}\mathbf{J} f'(\widetilde{\mathbf{V}})\big]_{j+n,k+n} e_n(x) \\
&= \frac{x}{2n} \mathbb{E}\operatorname{Tr}\mathbf{U}'(s) f'(\widetilde{\mathbf{V}}) e_n(x),
\end{aligned}
$$

where we applied the unitary matrix block decomposition (B.5) in the Appendix and used the following fact

$$
\int_{-\infty}^{\infty} u\widehat{f}(u) \sum_{k=1}^{n} \big[\mathbf{U}_3(s)\mathbf{W}\mathbf{H}\big(\Lambda(u)+\Lambda(-u)\big)\mathbf{H}^*\big]_{kk} = 0,
$$

which is valid since $f(\lambda)$ is an even function. Finally, we will have

$$
\begin{aligned}
\mathcal{I}_{11} = {}& -\frac{1}{n^2} \int_0^t ds \int_0^s \mathbb{E}u_n(s-s_1) \int_0^{s_1} \big[\mathbb{E}u_n(s_2)\mathbb{E}\big(u_n^{(0)}\big)'(s_1-s_2)e_n(x) \\
& + \mathbb{E}u_n'(s_1-s_2)\mathbb{E}u_n^{(0)}(x,s_2)e_n(x)\big]\,ds_2\,ds_1 \\
& - \frac{1}{n^2} \int_0^t ds \int_0^s s_1 \mathbb{E}u_n(s-s_1)\mathbb{E}\big(u_n^{(0)}(s_1)\big)'\,ds_1 \\
& - \frac{1}{n} \int_0^t ds \int_0^s \mathbb{E}u_n(s_1) Y_n(x, s-s_1)\,ds_1 \\
& - \frac{x}{2n^2} \int_0^t ds \int_0^s \mathbb{E}u_n(s_1)\mathbb{E}\operatorname{Tr}\mathbf{U}'(s-s_1) f'(\widetilde{\mathbf{V}}) e_n(x)\,ds_1.
\end{aligned}
$$

Changing the limits of integration, applying Lemma 2.3 and $\mathbb{E}|u_n(t)| \le n$, we get

$$
\begin{aligned}
\mathcal{I}_{11} = {}& -\frac{2}{n^2} \int_0^t Y_n(x,s)\,ds \int_0^{t-s} \mathbb{E}u_n(s_1)\mathbb{E}u_n(t-s-s_1)\,ds_1 \\
& - \frac{x}{2n^2} \int_0^t \mathbb{E}u_n(s)\mathbb{E}\operatorname{Tr}\big(\mathbf{U}(t-s)-\mathbf{I}\big) f'(\widehat{\mathbf{V}}) e_n(x)\,ds + r_n(t).
\end{aligned}
\tag{2.27}
$$

Using (2.27) we may write the following representation for the term $\mathcal{I}_1$

$$
\begin{aligned}
\mathcal{I}_1 = {}& -\frac{3}{n^2} \int_0^t Y_n(x,s)\,ds \int_0^{t-s} \mathbb{E}u_n(s_1)\mathbb{E}u_n(t-s-s_1)\,ds_1 \\
& - \frac{xZ_n(x)}{2n^2} \int_0^t \mathbb{E}u_n(s)\mathbb{E}\operatorname{Tr}\big(\mathbf{U}(t-s)-\mathbf{I}\big) f'(\widetilde{\mathbf{V}})\,ds + r_n(t).
\end{aligned}
$$

It remains to calculate the term $\mathcal{I}_2$. From Lemma B.5, we conclude that

$$\mathcal{I}_2 = -\frac{x}{\sqrt{n}}\int_0^t \sum_{j,k=1}^n \mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{kj}\frac{\partial S}{\partial Y_{jk}^{(1)}}e_n(x)\,ds$$

$$= -\frac{x}{n}\int_0^t \sum_{j,k=1}^n \mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{kj}\big[f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{j,k+n}e_n(x)\,ds$$

$$= -\frac{xZ_n(x)}{n}\int_0^t \sum_{k=1}^n \mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{k,k+n}\,ds,$$

where we used the following observation. First, we may write

$$\begin{aligned}
&\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\mathbf{U}(u)\mathbf{H}^{(1)}\big]_{k,k+n}\\
&= \sum_{j=1}^{2n}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{kj}\big[\mathbf{U}(u)\mathbf{H}^{(1)}\big]_{j,k+n}\\
&= \operatorname{Tr}\big[\big(\mathbf{Y}^{(2)}\big)^{\mathsf{T}}\mathbf{Y}^{(2)}\mathbf{U}_3(s)\mathbf{U}_2(u)\big] + \operatorname{Tr}\big[\big(\mathbf{Y}^{(2)}\big)^{\mathsf{T}}\mathbf{Y}^{(2)}\mathbf{U}_4(s)\mathbf{U}_4(u)\big]\\
&= \sum_{j=1}^{n}\mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{kj}\big[\mathbf{U}(u)\mathbf{H}^{(1)}\big]_{j,k+n} + \operatorname{Tr}\big[\big(\mathbf{Y}^{(2)}\big)^{\mathsf{T}}\mathbf{Y}^{(2)}\mathbf{U}_4(s)\mathbf{U}_4(u)\big].
\end{aligned}$$

From the representation (B.5) in the Appendix it follows that

$$\mathbf{U}_4(s)\mathbf{U}_4(u) = 4\mathbf{H}\mathbf{D}(s,u)\mathbf{H}^*,$$

where $\mathbf{D}(s,u)$ is a diagonal matrix with $D_{jj}(s,u) = \cos(s_js)\cos(s_ju)$, $j = 1,\dots,n$. Since $\widehat{f}(t)$ is an even function we obtain

$$\int_{-\infty}^{\infty} u\widehat{f}(u)\int_0^t \mathbb{E}\operatorname{Tr}\big[\big(\mathbf{Y}^{(2)}\big)^{\mathsf{T}}\mathbf{Y}^{(2)}\mathbf{H}\mathbf{D}(s,u)\mathbf{H}^*\big]\,ds\,du = 0.$$

We investigate now the behavior of

$$\sum_{k=1}^n\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t)f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{k,k+n}. \tag{2.28}$$

Applying the same arguments as before, we get

$$\begin{aligned}
&\mathbb{E}\sum_{k=1}^n\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t)f'(\widehat{\mathbf{V}})\mathbf{H}^{(1)}\big]_{k,k+n}\\
&= \frac{1}{\sqrt{n}}\sum_{j,k=1}^n \mathbb{E}Y_{jk}^{(2)}\big[\mathbf{U}(t)f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{k+n,j+n}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{n}} \sum_{j,k=1}^{n} \mathbb{E}\Big[\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(2)}} f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\Big]_{k+n,j+n} \\
&+ \frac{1}{\sqrt{n}} \sum_{j,k=1}^{n} \mathbb{E}\Big[\mathbf{U}(t) f'(\widetilde{\mathbf{V}}) \frac{\partial \mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}}\Big]_{k+n,j+n} \\
&=: \mathcal{T}_1 + \mathcal{T}_2.
\end{aligned}
$$

The term $\mathcal{T}_1$ may be expanded in the sum of two terms

$$
\begin{aligned}
\mathcal{T}_1 &= \frac{1}{\sqrt{n}} \sum_{j,k=1}^{n} \sum_{l=1}^{2n} \mathbb{E}\Big[\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(2)}}\Big]_{k+n,l} \big[f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{l,j+n} \\
&= \frac{i}{n} \int_0^t \sum_{k=1}^{n} \mathbb{E} U_{k+n,k+n}(s) \sum_{j=1}^{n} \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t-s) f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{j+n,j+n}\, ds \\
&\quad + \frac{i}{n} \int_0^t \sum_{k,j=1}^{n} \mathbb{E}\big[\mathbf{U}(s) f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{k+n,j+n} \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t-s)\big]_{j+n,k+n}\, ds \\
&= \frac{1}{2n} \int_0^t \mathbb{E} u_n(s) \operatorname{Tr} \mathbf{U}'(t-s) f'(\widetilde{\mathbf{V}})\, ds \\
&\quad + \frac{i}{n} \int_0^t \sum_{k,j=1}^{n} \mathbb{E}\big[\mathbf{U}(s) f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{k+n,j+n} \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t-s)\big]_{j+n,k+n}\, ds.
\end{aligned}
$$

For the term $\mathcal{T}_2$ we get

$$
\mathcal{T}_2 = \sum_{k=1}^{n} \mathbb{E}\big[\mathbf{U}(t) f'(\widetilde{\mathbf{V}})\big]_{k+n,k+n} = \frac{1}{2} \mathbb{E} \operatorname{Tr} \mathbf{U}(t) f'(\widetilde{\mathbf{V}}).
$$

We get the following decomposition for (2.28)

$$
\begin{aligned}
&\mathbb{E} \sum_{k=1}^{n} \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t) f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{k,k+n} \\
&\quad = \frac{1}{n} \int_0^t \mathbb{E} u_n(s) \operatorname{Tr} \mathbf{U}'(t-s) f'(\widetilde{\mathbf{V}})\, ds \\
&\qquad + \frac{1}{2} \operatorname{Tr} \mathbf{U}(t) f'(\widetilde{\mathbf{V}}) \\
&\qquad + \frac{i}{n} \int_0^t \sum_{k,j=1}^{n} \mathbb{E}\big[\mathbf{U}(s) f'(\widetilde{\mathbf{V}})\mathbf{H}^{(1)}\big]_{k+n,j+n} \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t-s)\big]_{j+n,k+n}\, ds.
\end{aligned}
$$

Inserting this equation to $\mathcal{I}_2$ we will have

$$\mathcal{I}_2 = -\frac{xZ_n(x)}{2n^2}\int_0^t\int_0^s \mathbb{E}u_n(s_1)\,\mathrm{Tr}\,\mathbf{U}'(s-s_1)f'(\widetilde{\mathbf{V}})\,ds_1\,ds$$
$$-\frac{xZ_n(x)}{2n}\int_0^t \mathrm{Tr}\,\mathbf{U}(s)f'(\widetilde{\mathbf{V}})\,ds + r_n(t).$$

Changing the limits of integration and applying Lemma 2.3, we get

$$\mathcal{I}_2 = -\frac{xZ_n(x)}{2n^2}\int_0^t \mathbb{E}u_n(s)\mathbb{E}\,\mathrm{Tr}\big[\mathbf{U}(t-s)f'(\widetilde{\mathbf{V}})\big]\,ds$$
$$-\frac{xZ_n(x)}{2n}\int_0^t \mathrm{Tr}\,\mathbf{U}(s)f'(\widetilde{\mathbf{V}})\,ds + r_n(t),$$

where we also used that $f(\lambda)$ and $\mathbb{E}u(s)$ are even functions. It follows from (2.19) that we have derived representation for $\mathcal{A}_1$. The same arguments are valid for $\mathcal{A}_2$.

To simplify our notations let us introduce the following quantity

$$A_n(t) := -\frac{1}{2n}\mathbb{E}\,\mathrm{Tr}\big[\mathbf{U}(t)f'(\widetilde{\mathbf{V}})\big].$$

One may see that $A_n(t)$ depends on t only, but $Z_n(x)$ depends on x only. We derive an equation for $Y_n(x,t)$:

$$\begin{aligned} &Y_n(x,t) + 3\int_0^t Y_n(x,s)v_n^{2*}(t-s)\,ds \\ &\quad = xZ_n(x)\int_0^t \big[v_n(s)A_n(t-s) + A_n(s)\big]\,ds + r_n(x,t), \end{aligned} \tag{2.29}$$

where

$$v_n(t) := \frac{1}{n}\mathbb{E}u_n(t).$$

As n goes to infinity the sequence $v_n(t)$ uniformly converges to the following function

$$v(t) = \int_{-a}^{a} e^{itx}p(x)\,dx, \tag{2.30}$$

where

$$p(x) := |x|P_2\big(x^2\big) \quad \text{and} \quad a := \sqrt{K_2}, \tag{2.31}$$

with $P_2(x)$, K_2 defined in Appendix A (see also the Introduction). This function is a characteristic function of the Fuss–Catalan distribution. The same arguments lead to

$$A(t) := \lim_{n\to\infty} A_n(t) = -\int_{-a}^{a} e^{it\lambda}f'(\lambda)p(\lambda)\,d\lambda.$$

Taking a limit in (2.29) with respect to $n_l \to \infty$, we get

$$Y(x,t) + 3\int_0^t Y(x,s)v^{2*}(t-s)\,ds = xZ(x)\int_0^t \big[2v(s)A(t-s) + A(s)\big]\,ds. \tag{2.32}$$

Denote by $F(z)$, $V(z)$ and $R(z)$ the generalized Fourier transform of $Y(x,t)$, $v(t)$ and $A(t)$ respectively (see Appendix D). Applying Statement D.1 in the Appendix, we get from (2.32)

$$F(z) - 3F(z)V^2(z) = -2ixZ(x)R(z)V(z) + \frac{ixZ(x)R(z)}{z}$$

and it follows that

$$F(z) = \frac{ixZ(x)(-2R(z)V(z) + R(z)/z)}{1 - 3V^2(z)}. \tag{2.33}$$

It is easy to check that

$$V(z) = s(z),$$

where $s(z)$ is the Stieltjes transform of $p(x)$. In these notations, we may rewrite (2.33) as follows

$$F(z) = \frac{ixZ(x)(-2R(z)s(z) + R(z)/z)}{1 - 3s^2(z)}. \tag{2.34}$$

By Lemma D.1 in the Appendix, the inverse Fourier transform of

$$\frac{1/z - 2s(z)}{1 - 3s^2(z)}$$

is given by

$$T(t) = \frac{1}{\pi}\int_{-a}^{a} \frac{e^{it\mu}}{3p_1(\mu)} \frac{4p_1(\mu)^4 + 11p_1(\mu)^2 + 4}{4p_1(\mu)^2 + 3}\,d\mu, \tag{2.35}$$

where $p_1(\mu) := \pi p(\lambda)$. From (2.34) and (2.35), we conclude

$$\begin{aligned} Y(x,t) = {} & -\frac{xZ(x)}{\pi^2}\int_0^t \int_{-a}^{a} e^{is\lambda} f'(\lambda)p(\lambda)\,d\lambda \\ & \times \int_{-a}^{a} \frac{e^{i(t-s)\mu}}{3p(\mu)} \frac{4p_1(\mu)^4 + 11p_1(\mu)^2 + 4}{4p_1(\mu)^2 + 3}\,dv. \end{aligned}$$

Simple calculation yields

$$\begin{aligned} Y(x,t) = {} & \frac{ixZ(x)}{\pi^2}\int_{-a}^{a} p(\lambda)\,d\lambda \\ & \times \int_{-a}^{a} \frac{e^{it\lambda} - e^{it\mu}}{\lambda - \mu} f'(\lambda)\frac{1}{3p(\mu)} \frac{4p_1(\mu)^4 + 11p_1(\mu)^2 + 4}{4p_1(\mu)^2 + 3}\,d\mu. \end{aligned} \tag{2.36}$$

Finally, we get from (2.17) and (2.36)

$$\lim_{n_l\to\infty} Z_n'(x,t) = -\frac{xZ(x)}{\pi^2}\int_{-a}^{a} p(\lambda)\,d\lambda \\ \times \int_{-a}^{a} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} f'(\lambda)\frac{1}{3p(\mu)} \frac{4p_1(\mu)^4+11p_1(\mu)^2+4}{4p_1(\mu)^2+3}\,d\mu. \tag{2.37}$$

One may see that

$$\big(f(\lambda)-f(\mu)\big)f'(\lambda) = \frac{1}{2}\frac{\partial}{\partial\lambda}\big(f(\lambda)-f(\mu)\big)^2,$$

and (2.37) may be rewritten applying integration by parts in the following way

$$\lim_{n_l\to\infty} Z_n' = -\frac{xZ(x)}{2\pi^2}\int_{-a}^{a}\int_{-a}^{a} \frac{(f(\lambda)-f(\mu))^2}{(\lambda-\mu)^2} \\ \times \frac{[p(\lambda)-p'(\lambda)(\lambda-\mu)]}{3p(\mu)} \frac{[4p_1(\mu)^4+11p_1(\mu)^2+4]}{4p_1(\mu)^2+3}\,d\lambda\,d\mu.$$

Comparing this with (2.12), we may conclude the proof of Theorem 2.1. □

3. The general case

In this section, we finish the proof of Theorem 1.1. Applying the method from Bentkus [8] and the method from Tikhomirov [26,27] we show that one may substitute the general matrix by the matrix with i.i.d. Gaussian random variables and express the characteristic function in the general case via the characteristic function in the Gaussian case. These methods have been applied several times in random matrix theory, see, for example, Götze *et al.* [17] and Götze *et al.* [16].

3.1. Truncation

In this subsection, we show by standard arguments that we may truncate the entries of $\mathbf{X}^{(q)}$, $q=1,2$. For all $1\le j,k\le n$, $q=1,2$, we introduce truncated random variables $X_{jk}^{(q,c)} := X_{jk}^{(q)}\mathbb{1}(|X_{jk}^{(q)}|\le\tau\sqrt{n})$. Denote by $\mathbf{X}^{(q,c)} := [X_{jk}^{(q,c)}]_{j,k=1}^{n}$. One checks that

$$\mathbb{P}\big(\mathbf{X}^{(q,c)}\ne\mathbf{X}^{(q)}\big) \le \sum_{j,k=1}^{n}\mathbb{E}\mathbb{1}\big(\big|X_{jk}^{(q)}\big|\ge\tau\sqrt{n}\big) \le \frac{1}{\tau^4}\mathbb{E}X_{11}^4\mathbb{1}\big(\big|X_{11}^{(q)}\big|\ge\tau\sqrt{n}\big). \tag{3.1}$$

Let $\widehat{S}^{(0)}$ denote $S^{(0)}$ with all entries $X_{jk}^{(q)}$ replaced by $X_{jk}^{(q,c)}$. It follows from (3.1) and the existence of the fourth moment of $X_{11}^{(q)}$ that

$$\lim_{n\to\infty}\big|\mathbb{E}e_n(x)-\mathbb{E}e^{ix\widehat{S}^{(0)}}\big| = 0.$$

By standard arguments one may also show that we may assume that $\mathbb{E}X_{jk}^{(q,c)} = 0$ and $\operatorname{Var}(X_{jk}^{(q,c)}) = 1$. We omit the details. In what follows, we will assume that $X_{jk}^{(q)}$ satisfy the assumptions (C0) and $|X_{jk}^{(q)}| \le \tau\sqrt{n}$ for all $1 \le j, k \le n$, $q = 1, 2$.

Remark 3.1. It is easy to see that one may assume $X_{jk}^{(q)}$ are non i.i.d., but satisfy the following Lendeberg type condition on the fourth moments

$$\text{for all } \tau > 0 \lim_{n\to\infty} \frac{1}{\tau^4 n^2} \sum_{j,k=1}^{n} \mathbb{E}\bigl(X_{jk}^{(q)}\bigr)^4 \mathbb{1}\bigl(\bigl|X_{jk}^{(q)}\bigr| \ge \tau\sqrt{n}\bigr) = 0.$$

3.2. From the general case to the Gaussian case

Let $\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}$ be $n \times n$ independent random matrices with independent Gaussian entries $n^{-1/2} Y_{jk}^{(q)}$ such that

$$\mathbb{E}Y_{jk}^{(q)} = 0, \qquad \mathbb{E}\bigl(Y_{jk}^{(q)}\bigr)^2 = 1 \qquad \text{for any } q = 1, 2,\ j, k = 1, \dots, n.$$

Following Bentkus [8] for any $\phi \in [0, \frac{\pi}{2}]$ and any $\nu = 1, 2$, we introduce the following matrices

$$\mathbf{Z}^{(q)}(\phi) := \mathbf{X}^{(q)} \sin\phi + \mathbf{Y}^{(q)} \cos\phi,$$

where

$$\bigl[\mathbf{Z}^{(q)}(\phi)\bigr]_{jk} := \frac{1}{\sqrt{n}} Z_{jk}^{(q)}(\phi) := \frac{1}{\sqrt{n}}\bigl(X_{jk}^{(q)} \sin\phi + Y_{jk}^{(q)} \cos\phi\bigr). \tag{3.2}$$

We remark here that in Lytova and Pastur [20] the so called Slepian interpolant is used, that is, $Z_{jk}^{(q)}(t) := \frac{1}{\sqrt{n}}(tX_{jk}^{(q)} + \sqrt{1-t^2}Y_{jk}^{(q)})$, $t \in [0, 1]$. Since we are going to differentiate with respect to the parameter, in this case w.r.t. to t, up to fourth order, we get rather involved expressions. In this respect the parametrization of Bentkus (3.2), see (3.8)–(3.10) below, is much more convenient.

We define the matrices $\mathbf{H}^{(q)}(\phi)$, $\mathbf{V}(\phi)$, $\widehat{\mathbf{V}}(\phi)$, $\mathbf{U}(\phi, t)$ as follows (compare with (2.3) and (2.4))

$$\mathbf{H}^{(q)}(\phi) = \begin{bmatrix} \mathbf{Z}^{(q)}(\phi) & \mathbf{O} \\ \mathbf{O} & \bigl[\mathbf{Z}^{(m-q+1)}(\phi)\bigr]^{\mathsf{T}} \end{bmatrix}, \qquad \mathbf{V}(\phi) = \prod_{q=1}^{2} \mathbf{H}^{(q)}(\phi),$$

$$\widetilde{\mathbf{V}}(\phi) = \mathbf{V}(\phi)\mathbf{J}, \qquad \mathbf{U}(\phi, t) = e^{it\widetilde{\mathbf{V}}(\phi)}.$$

Let $S(\phi) := \operatorname{Tr} f(\widetilde{\mathbf{V}}(\phi))$, $S^{(0)}(\phi) = S(\phi) - \mathbb{E}S(\phi)$ and

$$Z_n(x, \phi) = \mathbb{E}e^{ixS^{(0)}(\phi)}. \tag{3.3}$$

We also define the limiting characteristic function $Z(x,\phi) := \lim_{n\to\infty} Z_n(x,\phi)$. To simplify all notations, we will often omit the argument ϕ. The following theorem is the main result of this section.

Theorem 3.1. *Assume that the conditions* (C0) *hold. For an arbitrary function $f \in \mathcal{F}$ the limiting characteristic function $Z(x,\phi)$ satisfies the following equation*

$$Z(x,\pi/2) = Z(x,0)e^{-x^2\kappa_4\Psi^2/4},$$

where

$$\Psi = \int_{-a}^{a} f(\lambda)\big[p(\lambda) + \lambda p'(\lambda)\big]\,d\lambda \tag{3.4}$$

and $\kappa_4 := \mu_4 - 3$.

Proof of Theorem 1.1. Applying Theorem 3.1, we get

$$Z(x,\pi/2) = Z(x,0)e^{-x^2\kappa_4\Psi^2/4}. \tag{3.5}$$

We know that in the Gaussian case (see (2.10))

$$Z(x,0) = e^{-\widehat{\mathrm{Var}}[G]\frac{x^2}{2}}$$

and $\widehat{\mathrm{Var}}[G]$ is given by (2.1). From the last equation and (3.5), it follows that

$$Z(x,\pi/2) = e^{-\mathrm{Var}[G]\frac{x^2}{2}},$$

where

$$\mathrm{Var}[G] := \widehat{\mathrm{Var}}[G] + \kappa_4\Psi^2/2.$$

□

Proof of Theorem 3.1. We prove that the function $Z(x,\phi)$ satisfies the following equation

$$\frac{\partial Z(x,\phi)}{\partial\phi} = -\kappa_4 x^2 \sin^3\phi\cos\phi\,\Psi^2 Z(x,\phi). \tag{3.6}$$

It follows from this equation that

$$Z(x,\pi/2) = Z(x,0)\exp\left\{-\kappa_4 x^2\Psi^2\int_0^{\pi/2}\sin^3\alpha\cos\alpha\,d\alpha\right\}.$$

Note that

$$Z_n(x,\pi/2) - Z_n(x,0) = \int_0^{\pi/2}\frac{\partial Z_n(x,\phi)}{\partial\phi}\,d\phi.$$

Similarly to the Section 2 it is sufficient to prove that any converging subsequences $\{Z_{n_l}\}$ and $\{\frac{\partial Z_{n_l}}{\partial \phi}\}$ satisfy

$$\lim_{n_l\to\infty} Z_{n_l}(x,\phi) = Z(x,\phi), \qquad \lim_{n_l\to\infty} \frac{\partial Z_{n_l}(x,\phi)}{\partial \phi} = -\kappa_4 x^2 \sin^3\phi \cos\phi \Psi^2 Z(x,\phi).$$

By Lemma B.5, we get

$$\frac{\partial Z_n(x,\phi)}{\partial \phi} = \frac{ix}{\sqrt{n}} \sum_{q=1}^{2} \sum_{j,k=1}^{n} \mathbb{E}\widehat{Z}_{jk}^{(q)} \big[\mathbf{V}_{[m-q+2,m]} \mathbf{J} f'(\widetilde{\mathbf{V}}) \mathbf{V}_{[1,m-q]}\big]_{j+n,k+n} e_n(x), \tag{3.7}$$

where

$$\widehat{Z}_{jk}^{(q)} := \frac{d}{d\phi} Z_{jk}^{(q)} = X_{jk}^{(q)} \cos\phi - Y_{jk}^{(q)} \sin\phi.$$

It is straightforward to check that

$$\mathbb{E}\widehat{Z}_{jk}^{(q)} \big(Z_{jk}^{(q)}\big)^p = 0 \qquad \text{for } p = 0, 1; \tag{3.8}$$

$$\mathbb{E}\widehat{Z}_{jk}^{(q)} \big(Z_{jk}^{(q)}\big)^2 = \mathbb{E}\big(X_{jk}^{(q)}\big)^3 \cos^3\phi; \tag{3.9}$$

$$\mathbb{E}\widehat{Z}_{jk}^{(q)} \big(Z_{jk}^{(q)}\big)^3 = \kappa_4 \sin^3\phi \cos\phi. \tag{3.10}$$

Let us introduce further notations. Denote by $\mathbf{V}_{[\alpha,\beta]}^{j,k,q}(y)$ the corresponding matrix $\mathbf{V}_{[\alpha,\beta]}$ with $Z_{jk}^{(q)}$ replaced by y. Let us also denote

$$\Phi_{jkq}(y) := \big[\mathbf{V}_{[m-q+2,m]}^{(j,k,q)}(y) \mathbf{J} f'\big(\widetilde{\mathbf{V}}^{(j,k,q)}(y)\big) \mathbf{V}_{[1,m-q]}^{(j,k,q)}(y)\big]_{j+n,k+n} e^{itS^{(0)}(\mathbf{V}^{(j,k,q)}(y))}.$$

Applying Taylor's formula, we get

$$\Phi_{jkq}\big(Z_{jk}^{(q)}\big) = \sum_{p=0}^{3} \frac{1}{p!} \big(Z_{jk}^{(q)}\big)^p \Phi_{jkq}^{(p)}(0) + \frac{1}{3!} \big(Z_{jk}^{(q)}\big)^4 \mathbb{E}_\theta (1-\theta)^3 \Phi_{jkq}^{(4)}\big(\theta Z_{jk}^{(q)}\big).$$

This equation and (3.7) together imply

$$\begin{aligned}
\frac{\partial Z_n(x,\phi)}{\partial \phi} &= \frac{ix}{n^{1/2}} \sum_{p=1}^{3} \frac{1}{p!} \sum_{q=1}^{2} \sum_{j,k=1}^{n} \mathbb{E}\widehat{Z}_{jk}^{(q)} \big(Z_{jk}^{(q)}\big)^p \mathbb{E}\Phi_{jkq}^{(p)}(0) \\
&\quad + \frac{ix}{3! n^{1/2}} \sum_{q=1}^{2} \sum_{j,k=1}^{n} \mathbb{E}(1-\theta)^3 \widehat{Z}_{jk}^{(q)} \big(Z_{jk}^{(q)}\big)^4 \Phi_{jkq}^{(4)}\big(\theta Z_{jk}^{(q)}\big) \\
&=: \mathcal{T}_1 + \cdots + \mathcal{T}_4.
\end{aligned}$$

It follows from (3.8) that $\mathcal{T}_1 = 0$. In the next subsections we will investigate the term $\mathcal{T}_k, k = 2, 3, 4$.

3.3. The second derivative

First, we note that

$$\big[\mathbf{V}^{(j,k,q)}_{[m-q+2,m]}\mathbf{J}\mathbf{A}\mathbf{V}^{(j,k,q)}_{[1,m-q]}\big]_{j+n,k+n} = [\mathbf{V}_{[m-q+2,m]}\mathbf{J}\mathbf{A}\mathbf{V}_{[1,m-q]}]_{j+n,k+n}.$$

for an arbitrary matrix $\mathbf{A}$. It is straightforward to check that

$$\Phi^{(2)}_{jkq}(0) = L^1_{jkq} + L^2_{jkq} + L^3_{jkq},$$

where

$$L^1_{jkq} := \bigg[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial^2 f'(\widetilde{\mathbf{V}})}{\partial (Z^{(q)}_{jk})^2}\Big|_{Z^{(q)}_{jk}=0}\mathbf{V}_{[1,m-q]}\bigg]_{j+n,k+n} e_n(x),$$

$$L^2_{jkq} := \frac{3ix}{\sqrt{n}}\bigg[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z^{(q)}_{jk}}\Big|_{Z^{(q)}_{jk}=0}\mathbf{V}_{[1,m-q]}\bigg]_{j+n,k+n}$$
$$\times \big[\mathbf{V}_{[m-q+2,m]}\mathbf{J} f'(\widetilde{\mathbf{V}})|_{Z^{(q)}_{jk}=0}\mathbf{V}_{[1,m-q]}\big]_{j+n,k+n} e_n(x),$$

$$L^3_{jkq} := -\frac{x^2}{n}\big[\mathbf{V}_{[m-q+2,m]}\mathbf{J} f'(\widetilde{\mathbf{V}})|_{Z^{(q)}_{jk}=0}\mathbf{V}_{[1,m-q]}\big]^3_{j+n,k+n} e_n(x).$$

Let us consider, for example, the term $\frac{1}{\sqrt{n}}L^2_{jkq}$. We have, for $q = 1$,

$$\frac{1}{n^{1/2}}\sum_{j,k=1}^{n}\mathbb{E}L^2_{jkq} = \mathcal{I}_1 + \mathcal{I}_2,$$

where

$$\mathcal{I}_1 := \frac{3x}{n^{3/2}}\int_{\mathbb{R}^2} uv\widehat{f}(u)\widehat{f}(v)\int_0^u \mathbb{E}\sum_{j,k=1}^{n}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{j,k+n}$$
$$\times \big[\mathbf{U}(u-s)\mathbf{H}^{(1)}\big]_{j,k+n}\big[\mathbf{U}(v)\mathbf{H}^{(1)}\big]_{j,k+n}\, ds\, du\, dv,$$

$$\mathcal{I}_2 := \frac{3x}{n^{3/2}}\int_{\mathbb{R}^2} uv\widehat{f}(u)\widehat{f}(v)\int_0^u \mathbb{E}\sum_{j,k=1}^{n}\big[\big(\mathbf{H}^{(1)}\big)^{\mathsf{T}}\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,k+n}$$
$$\times \big[\mathbf{U}(u-s)\big]_{jj}\big[\mathbf{U}(v)\mathbf{H}^{(1)}\big]_{j,k+n}\, ds\, du\, dv.$$

We estimate the term $\mathcal{I}_1$. Applying the Cauchy–Schwarz inequality and orthogonality properties of $\mathbf{U}$, we get

$$\begin{aligned}
&\frac{1}{n^{3/2}}\mathbb{E}\left|\sum_{j,k=1}^{n}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{j,k+n}\big[\mathbf{U}(u-s)\mathbf{H}^{(1)}\big]_{j,k+n}\big[\mathbf{U}(v)\mathbf{H}^{(1)}\big]_{j,k+n}\right| \\
&\quad\le \frac{1}{n^3}\mathbb{E}\left[\sum_{l_1,l_2,l_3=1}^{n}\left(\sum_{k=1}^{n}Z_{kl_1}^{(2)}Z_{kl_2}^{(2)}Z_{kl_3}^{(2)}\right)\left(\sum_{j=1}^{n}[\mathbf{U}_2]_{jl_1}(s)[\mathbf{U}_2]_{jl_2}(u-s)[\mathbf{U}_2]_{jl_3}(v)\right)\right] \\
&\quad\le \frac{1}{n^{5/2}}\mathbb{E}^{1/2}\left[\sum_{l_1,l_2,l_3=1}^{n}\left(\sum_{k=1}^{n}Z_{kl_1}^{(2)}Z_{kl_2}^{(2)}Z_{kl_3}^{(2)}\right)^2\right]\le C(\tau+1)n^{-1/2}.
\end{aligned}$$

For the term $\mathcal{I}_2$ we may apply the same arguments. Finally

$$|\mathcal{I}_1+\mathcal{I}_2|\le\frac{Cx(\tau+1)}{n^{1/2}}\int_{\infty}^{\infty}|u|^3\big|\widehat{f}(u)\big|\,du\int_{\infty}^{\infty}|v|\big|\widehat{f}(v)\big|\,dv.$$

Analogously one may show that $\frac{1}{n^{1/2}}\sum_{j,k=1}^{n}L_{jkq}^{1}$ and $\frac{1}{n^{1/2}}\sum_{j,k=1}^{n}L_{jkq}^{3}$ goes to zero as n goes to infinity. It follows that $\mathcal{T}_2=o(1)$.

3.4. The third derivative

We investigate now the term $\mathcal{T}_3$. Direct computations yield

$$\begin{aligned}
\Phi_{jkq}^{(3)}(0)&=\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial^3 f'(\widetilde{\mathbf{V}})}{\partial(Z_{jk}^{(q)})^3}\Big|_{Z_{jk}^{(q)}=0}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n}e_n(x) \\
&\quad+\frac{4ix}{\sqrt{n}}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial^2 f'(\widetilde{\mathbf{V}})}{\partial(Z_{jk}^{(q)})^2}\Big|_{Z_{jk}^{(q)}=0}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n} \\
&\quad\times\big[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})|_{Z_{jk}^{(q)}=0}\mathbf{V}_{[1,m-q]}\big]_{j+n,k+n}e_n(x) \\
&\quad+\frac{3ix}{\sqrt{n}}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z_{jk}^{(q)}}\Big|_{Z_{jk}^{(q)}=0}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n}^2 e_n(x) \\
&\quad-\frac{6x^2}{n}\big[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})\big|_{Z_{jk}^{(q)}=0}\mathbf{V}_{[1,m-q]}\big]_{j+n,k+n}^2 \\
&\quad\times\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z_{jk}^{(q)}}\Big|_{Z_{jk}^{(q)}=0}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n}e_n(x) \\
&\quad-\frac{ix^3}{n^{3/2}}\big[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})|_{Z_{jk}^{(q)}=0}\mathbf{V}_{[1,m-q]}\big]_{j+n,k+n}^4 e_n(x).
\end{aligned}$$

It is straightforward to check that all terms except the third are of order $o(1)$. These may be done similarly to the previous section. Let us denote

$$\Psi_n^q = \frac{1}{n}\sum_{j,k=1}^{n}\mathbb{E}\Big[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z_{jk}^{(q)}}\Big|_{Z_{jk}^{(q)}=0}\mathbf{V}_{[1,m-q]}\Big]_{j+n,k+n}^2.$$

Our aim is to show that

$$\lim_{n_l\to\infty}\Psi_{n_l}^q = \Big[\int_{-a}^{a} f(\lambda)\big[p(\lambda)+\lambda p'(\lambda)\big]\,d\lambda\Big]^2, \tag{3.11}$$

where $p(\lambda)$ is defined in (2.31). Consider the case $q=1$. By Lemma B.3 in the Appendix, we get

$$\begin{aligned}
\Big[\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z_{jk}^{(1)}}\mathbf{H}^{(1)}\Big]_{j,k+n} &= \sum_{l=1}^{2n}\Big[\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z_{jk}^{(1)}}\Big]_{jl}\big[\mathbf{H}^{(1)}\big]_{l,k+n} \\
&= -\frac{1}{\sqrt{n}}\int_{-\infty}^{\infty} s\widehat{f}(s)\Big[\sum_{l=1}^{2n}\big[\mathbf{U}\mathbf{H}^{(1)}\big]_{j,k+n} * [\mathbf{U}]_{lj}(s)\big[\mathbf{H}^{(1)}\big]_{l,k+n} \\
&\quad + \sum_{l=1}^{2n}\big[\mathbf{U}\mathbf{H}^{(1)}\big]_{l,k+n} * [\mathbf{U}]_{j,j}(s)\big[\mathbf{H}^{(1)}\big]_{l,k+n}\Big]\,ds \\
&= -\frac{1}{\sqrt{n}}\int_{-\infty}^{\infty} s\widehat{f}(s)\big[\big[\mathbf{U}\mathbf{H}^{(1)}\big]_{j,k+n} * \big[\mathbf{U}\mathbf{H}^{(1)}\big]_{j,k+n}(s) \\
&\quad + \big[\big[\mathbf{H}^{(1)}\big]^{\mathsf{T}}\mathbf{U}\mathbf{H}^{(1)}\big]_{k+n,k+n} * [\mathbf{U}]_{jj}(s)\big]\,ds.
\end{aligned}$$

Similarly to the previous sections we may show that the first term in the last equation has the zero impact. It is straightforward to check

$$\big[\big[\mathbf{H}^{(1)}\big]^{\mathsf{T}}\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,k+n} = \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k,k+n} =: T_{n,k}(s). \tag{3.12}$$

Let us investigate the following integral

$$\int_{-\infty}^{\infty} s\widehat{f}(s)\mathbb{E}T_{n,k} * \mathbb{E}[\mathbf{U}]_{jj}(s)\,ds.$$

Applying (2.18) we obtain

$$\begin{aligned}
\mathbb{E}T_{n,k}(s) &= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\mathbb{E}Z_{kl}^{(2)}\big[\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{l+n,k+n} \\
&= \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\mathbb{E}\Big[\frac{\partial\mathbf{U}(s)}{\partial Z_{kl}^{(2)}}\mathbf{H}^{(1)}\Big]_{l+n,k+n} + \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\mathbb{E}\Big[\mathbf{U}(s)\frac{\partial\mathbf{H}^{(1)}(s)}{\partial Z_{kl}^{(2)}}\Big]_{l+n,k+n}.
\end{aligned}$$

It follows from Lemma B.4 in the Appendix that

$$\begin{aligned}\sum_{l=1}^{n}\mathbb{E}\left[\frac{\partial \mathbf{U}(s)}{\partial Z_{kl}^{(2)}}\mathbf{H}^{(1)}\right]_{l+n,k+n} &= \sum_{l=1}^{n}\sum_{p=1}^{2n}\mathbb{E}\left[\frac{\partial \mathbf{U}(s)}{\partial Z_{kl}^{(2)}}\right]_{l+n,p}\big[\mathbf{H}^{(1)}\big]_{p,k+n}\\
&= \frac{i}{\sqrt{n}}\sum_{l=1}^{n}\sum_{p=1}^{2n}\mathbb{E}[\mathbf{U}]_{l+n,l+n} * \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}\big]_{k+n,p}\big[\mathbf{H}^{(1)}\big]_{p,k+n}\\
&\quad + \frac{i}{\sqrt{n}}\sum_{l=1}^{n}\sum_{p=1}^{2n}\mathbb{E}[\mathbf{U}]_{p,l+n} * \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}\big]_{k+n,l+n}\big[\mathbf{H}^{(1)}\big]_{p,k+n}\\
&= \frac{i}{\sqrt{n}}\sum_{l=1}^{n}\mathbb{E}[\mathbf{U}]_{l+n,l+n} * \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}\mathbf{H}^{(1)}\big]_{k+n,k+n}\\
&\quad + \frac{i}{\sqrt{n}}\sum_{l=1}^{n}\mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}\big]_{k+n,l+n} * \big[\mathbf{U}\mathbf{H}^{(1)}\big]_{l+n,k+n}.\end{aligned}$$

It is straightforward to check that the second term has the zero impact. Hence,

$$\mathbb{E}T_{n,k}(s) = \frac{i}{n}\mathbb{E}u_n * \mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}\mathbf{H}^{(1)}\big]_{k+n,k+n}(s) + \frac{1}{n}\mathbb{E}u_n(s) + r_n(s), \tag{3.13}$$

where we applied Lemma C.2. Here and in what follows $\lim_{n\to\infty} r_n(s) = 0$ and $r_n(s)$ depends polynomially on s. Finally, we will have for $q = 1$

$$\begin{aligned}\Psi_n^q = \frac{1}{n^2}\sum_{j,k=1}^{n}\bigg\{\int_{-\infty}^{\infty} s\widehat{f}(s)\bigg[&\frac{1}{n}\mathbb{E}u_n * \mathbb{E}[\mathbf{U}]_{j,j}(s)\\
&+ \frac{i}{n}\mathbb{E}u_n * \mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}\mathbf{H}^{(1)}\big]_{k+n,k+n} * \mathbb{E}[\mathbf{U}]_{j,j}(s)\bigg]ds\bigg\}^2 + o(1).\end{aligned} \tag{3.14}$$

Let us introduce further notations and denote

$$V_{n,k}(s) := \mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\mathbf{H}^{(1)}\big]_{k+n,k+n}. \tag{3.15}$$

We may write, applying Lemma B.3 in the Appendix and (2.18) that

$$\begin{aligned}V_{n,k}(s) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\mathbb{E}Z_{jk}^{(1)}\big[\mathbf{U}\mathbf{H}^{(1)}\big]_{j,k+n}\\
&= \sum_{j=1}^{n}\mathbb{E}\left[\frac{\partial \mathbf{U}(s)}{\partial Z_{jk}^{(1)}}\right]_{j,k+n}\end{aligned}$$

$$= \frac{i}{n}\sum_{j=1}^{n}\sum_{l=1}^{2n}\mathbb{E}\big[\mathbf{U}\mathbf{H}^{(1)}\big]_{j,k+n} * [\mathbf{U}]_{l,j}(s)\big[\mathbf{H}^{(1)}\big]_{l,k+n}$$

$$+ \frac{i}{n}\sum_{j=1}^{n}\sum_{l=1}^{2n}\mathbb{E}\big[\mathbf{U}\mathbf{H}^{(1)}\big]_{l,k+n} * [\mathbf{U}]_{j,j}(s)\big[\mathbf{H}^{(1)}\big]_{l,k+n}.$$

The same arguments as before yield that

$$V_{n,k}(s) = \frac{i}{n}\mathbb{E}u_n(s) * \mathbb{E}T_{n,k}(s) + r_n(s). \tag{3.16}$$

Applying (3.13), we get

$$V_{n,k}(s) = \frac{i}{n^2}\mathbb{E}u_n * \mathbb{E}u_n(s) - \frac{1}{n^2}\mathbb{E}u_n * \mathbb{E}u_n * V_{n,k}(s) + r_n(s).$$

This means that $\lim_{n\to\infty} V_{n,k}(s)$ satisfies the following equation

$$h(s) = iu * u(s) - u * u * h(s).$$

The same equation holds for $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} V_{n,k}(s)$. It is easy to see that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} V_{n,k}(s) = -iv'(s),$$

where $v(s)$ was defined in (2.30). That means that

$$\lim_{n\to\infty} V_{n,k}(vs) = -iv'(s).$$

Taking the limit with respect to $n_l \to \infty$ we get in (3.14)

$$\lim_{n_l\to\infty}\Psi_{n_l}^{q} = \left\{\int_{-\infty}^{\infty} s\widehat{f}(s)\big[v * v(s) + v * v * v'(s)\big]\,ds\right\}^2. \tag{3.17}$$

Let us consider the following integral

$$\int_{-\infty}^{\infty} s\widehat{f}(s)\big[v * v(s) + v * v * v'(s)\big]\,ds.$$

Let us denote the Stieltjes transform of $p(x)$ by $s(z)$. The Fourier transform of $v * v(s) + v * v * v'(s)$ is given by

$$is^2(z) - i\big(1 + zs(z)\big)s^2(z) = -izs^3(z).$$

By Proposition D.1 in the Appendix, we obtain

$$v * v(t) + v * v * v'(s) = \frac{1}{2\pi}\int_L e^{isz} zs^3(z)\,dz. \tag{3.18}$$

It follows from (2.5) with $m=2$ that $s(z)$ satisfies an equation $1+zs(z)=zs^3(z)$. Hence, the right-hand side of (3.18) may be rewritten as

$$\frac{1}{2\pi}\int_L e^{isz}\bigl(1+zs(z)\bigr)\,dz.$$

Similarly to the proof of Lemma D.1 in the Appendix, we get

$$\frac{1}{2\pi}\int_L e^{isz}\bigl(1+zs(z)\bigr)\,dz = i\int_{-a}^{a} e^{is\lambda}\lambda p(\lambda)\,d\lambda.$$

Integrating by parts, we will have

$$i\int_{-a}^{a} e^{is\lambda}\lambda p(\lambda)\,d\lambda = -\frac{1}{s}\int_{-a}^{a} e^{is\lambda}\bigl[p(\lambda)+\lambda p'(\lambda)\bigr]\,d\lambda.$$

Finally, we conclude that

$$\int_{-\infty}^{\infty} t\widehat{f}(t)\bigl[v*v(s)+v*v*v'(s)\bigr]\,ds = \int_{-a}^{a} f(\lambda)\bigl[p(\lambda)+\lambda p'(\lambda)\bigr]\,d\lambda$$

and finish the proof of (3.11). If we show that for all $1\le j,k\le n$

$$\lim_{n\to\infty}\mathbb{E}\biggl[\int_{-\infty}^{\infty} s\widehat{f}(s)\bigl[T_k*[\mathbf{U}]_{jj}(s)-\mathbb{E}T_k*\mathbb{E}[\mathbf{U}]_{jj}(s)\bigr]\,ds\biggr]^2 e_n(x)=0 \tag{3.19}$$

then from (3.10) and (3.11) we will have

$$\lim_{n_l\to\infty}\frac{ix}{3!n_l^{1/2}}\sum_{q=1}^{2}\sum_{j,k=1}^{n_l}\mathbb{E}\widehat{Z}_{jk}^{(q)}\bigl(Z_{jk}^{(q)}\bigr)^3\mathbb{E}\Phi_{jkq}^{(p)}(0) = -\kappa_4 x^2\sin^3\phi\cos\phi\Psi^2 Z(x,\phi),$$

where Ψ is given by (3.4). To prove (3.19), it is enough to show that for all $k=1,\dots,n$,

$$\operatorname{Var}\bigl[T_{n,k}(s)\bigr]=r_n(s),\qquad \operatorname{Var}(U_{kk})=r_n(s).$$

These bounds follow from Lemma C.2 in the Appendix.

3.5. The remainder term

To conclude the proof of Theorem 3.1 it remains to estimate the remainder term $\mathcal{T}_4$. One checks that $\mathbb{E}\widehat{Z}_{jk}^{(q)}(Z_{jk}^{(q)})^4\le C\tau\sqrt{n}\mu_4$. Let Z be a random variable which has the same distribution as $Z_{11}^{(1)}$. We estimate $\mathbb{E}\sup_Z\Phi_{jkq}^{(3)}(Z)$. Simple calculations yield that

$$\Phi_{jkq}^{(3)}(Z)=L_{jkq}^1+\cdots+L_{jkq}^7,$$

where

$$L^1_{jkq} := \frac{1}{n^2}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial^4 f'(\widetilde{\mathbf{V}})}{\partial (Z^{(q)}_{jk})^4}\Big|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n} e_n(x),$$

$$L^2_{jkq} := \frac{5ix}{\sqrt{n}}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial^3 f'(\widetilde{\mathbf{V}})}{\partial (Z^{(q)}_{jk})^3}\Big|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n}$$
$$\times \left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n} e_n(x),$$

$$L^3_{jkq} := \frac{10ix}{\sqrt{n}}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial^2 f'(\widetilde{\mathbf{V}})}{\partial (Z^{(q)}_{jk})^2}\Big|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n}$$
$$\times \left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z^{(q)}_{jk}}\Big|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n} e_n(x),$$

$$L^4_{jkq} := -\frac{10x^2}{n}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial^2 f'(\widetilde{\mathbf{V}})}{\partial (Z^{(q)}_{jk})^2}\Big|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n}$$
$$\times \left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]^2_{j+n,k+n} e_n(x),$$

$$L^5_{jkq} := -\frac{15x^2}{n}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z^{(q)}_{jk}}\Big|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]^2_{j+n,k+n}$$
$$\times \left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n} e_n(x),$$

$$L^6_{jkq} := -\frac{10ix^3}{n^{3/2}}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}\frac{\partial f'(\widetilde{\mathbf{V}})}{\partial Z^{(q)}_{jk}}\Big|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]_{j+n,k+n}$$
$$\times \left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]^3_{j+n,k+n} e_n(x),$$

$$L^7_{jkq} := \frac{x^4}{n^2}\left[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})|_{Z^{(q)}_{jk}=Z}\mathbf{V}_{[1,m-q]}\right]^5_{j+n,k+n} e_n(x).$$

Applying the same arguments as before in Sections 3.3 and 3.4 we get that

$$|T_4| \le C\tau.$$

It is possible to change τ in the definition of $X^{(c)}$ by a sequence τ_n, such that $\lim_{n\to\infty}\tau_n = 0$ and $\lim_{n\to\infty}\sqrt{n}\tau_n = \infty$. This fact finishes the proof of Theorem 3.1. □

Appendix A: Fuss–Catalan distribution

For any $m \in \mathbb{N}$ let us consider the sequence of numbers

$$M_k = \frac{1}{mk+1}\binom{k}{mk+k}, \qquad k \in \mathbb{N} \cup \{0\}.$$

These numbers are called Fuss–Catalan numbers. In Penson and Życzkowski [23] the density function $P_m(x)$ which satisfies

$$\int_0^{K_m} x^k P_m(x)\,dx = M_k,$$

was found. Here $K_m := (m+1)^{m+1}/m^m$. The explicit formula for $P_m(x)$ is given by the following formula in terms of hypergeometric functions

$$\begin{aligned} P_m(x) &= \sum_{k=1}^{m} \Lambda_{k,m} x^{\frac{k}{m+1}-1} \\ &\quad \times {}_mF_{m-1}\left(\left[\left\{1 - \frac{1+j}{m} + \frac{k}{m+1}\right\}_{j=1}^{m}\right],\right. \\ &\quad \left.\left[\left\{1 + \frac{k-j}{m+1}\right\}_{j=1}^{k-1}, \left\{1 + \frac{k-j}{m+1}\right\}_{j=k+1}^{m}\right]; \frac{m^m}{(m+1)^{m+1}} x\right), \end{aligned}$$

where the coefficients $\Lambda_{k,m}$ are given for $k = 1, 2, \ldots, m$ by

$$\Lambda_{k,m} := m^{-3/2}\sqrt{\frac{m+1}{2\pi}}\left(\frac{m^{m/(m+1)}}{m+1}\right)^k \frac{[\prod_{j=1}^{k-1}\Gamma(\frac{j-k}{m+1})][\prod_{j=k+1}^{m}\Gamma(\frac{j-k}{m+1})]}{\prod_{j=1}^{m}\Gamma(\frac{j+1}{m} - \frac{k}{m+1})}.$$

For example,

$$P_1(x) = \frac{\sqrt{1-x/4}}{\pi\sqrt{x}}$$

and

$$P_2(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi}\frac{[\sqrt[3]{2}(27+3\sqrt{81-12x})^{\frac{2}{3}} - 6\sqrt[3]{x}]}{x^{\frac{2}{3}}(27+3\sqrt{81-12x})^{\frac{1}{3}}},$$

valid for $x \in [0, 27/4]$.

Appendix B: Unitary matrix derivatives

In this section, we collect useful facts about matrix derivatives and exponential functions of matrices. Let us consider a function $f(\lambda)$ and recall the definition (1.5) of its Fourier transform

$$\widehat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\lambda)e^{-it\lambda}\,d\lambda.$$

The function $f(\lambda)$ may be reconstructed from $\widehat{f}(t)$ via the inverse Fourier transform (see (2.13))

$$f(\lambda) = \int_{-\infty}^{\infty} \widehat{f}(t)e^{it\lambda}\,dt. \tag{B.1}$$

Let $f^{(k)}(\lambda)$ denote the kth derivative of $f(\lambda)$. Then

$$f^{(k)}(\lambda) = i^k\int_{-\infty}^{\infty} t^k\widehat{f}(t)e^{it\lambda}\,dt. \tag{B.2}$$

Recall the definition of the matrix $\mathbf{U}(t)$, see (2.15), $\mathbf{U}(t) := e^{it\widetilde{\mathbf{V}}}$. Applying (B.1) we get the following representation for $f(\widetilde{\mathbf{V}})$,

$$f(\widetilde{\mathbf{V}}) = \int_{-\infty}^{\infty} \widehat{f}(t)\mathbf{U}(t)\,dt. \tag{B.3}$$

Using Duhamel's formula (see, for example, Lytova and Pastur [20]), we arrive at

$$e^{(\mathbf{M}_1+\mathbf{M}_2)t} = e^{\mathbf{M}_1 t} + \int_0^t e^{\mathbf{M}_1(t-s)}\mathbf{M}_2 e^{(\mathbf{M}_1+\mathbf{M}_2)s}\,ds, \tag{B.4}$$

valid for arbitrary matrices $\mathbf{M}_1, \mathbf{M}_2$ and $t \in \mathbb{R}$.

In what follows, we shall use matrix notation (2.3) and (2.4). Consider the singular value decomposition of the matrix $\mathbf{X}$ (or $\mathbf{Y}$ in the Gaussian case) of dimension $n \times n$. Let $\mathbf{L}$ and $\mathbf{H}$ be unitary matrices of dimension $n \times n$. Let $\mathbf{\Lambda}$ be a diagonal matrix whose entries are the singular values of the matrix $\mathbf{X}$. We have the following representation

$$\mathbf{X} = \mathbf{L}\mathbf{\Lambda}\mathbf{H}^*.$$

We introduce the following matrix

$$\mathbf{Z}^* = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{L}^* & \mathbf{H}^*\\ \mathbf{L}^* & -\mathbf{H}^*\end{bmatrix}.$$

It is straightforward to check that

$$\mathbf{Z}^*\mathbf{V}\mathbf{Z} = \begin{bmatrix}\mathbf{\Lambda} & \mathbf{O}\\ \mathbf{O} & -\mathbf{\Lambda}\end{bmatrix}$$

and

$$\mathbf{Z}^*\mathbf{U}(s)\mathbf{Z} = \mathbf{Z}^*\begin{bmatrix}\boldsymbol{\Lambda}(s) & \mathbf{O}\\ \mathbf{O} & \boldsymbol{\Lambda}(-s)\end{bmatrix}\mathbf{Z},$$

where $\boldsymbol{\Lambda}(s)$ is a diagonal matrix such that $[\boldsymbol{\Lambda}(s)]_{jj} = e^{[is\Lambda_{jj}]}$, $j = 1, \dots, n$. A simple calculation yields that

$$\mathbf{U}(s) = \begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\\ \mathbf{U}_3 & \mathbf{U}_4\end{bmatrix} = \begin{bmatrix}\mathbf{L}\big(\boldsymbol{\Lambda}(s)+\boldsymbol{\Lambda}(-s)\big)\mathbf{L}^* & \mathbf{L}\big(\boldsymbol{\Lambda}(s)-\boldsymbol{\Lambda}(-s)\big)\mathbf{H}^*\\ \mathbf{H}\big(\boldsymbol{\Lambda}(s)-\boldsymbol{\Lambda}(-s)\big)\mathbf{L}^* & \mathbf{H}\big(\boldsymbol{\Lambda}(s)+\boldsymbol{\Lambda}(-s)\big)\mathbf{H}^*\end{bmatrix}. \tag{B.5}$$

We also denote by $\mathbf{M}^{(j,k)}$ or $[\mathbf{M}]^{(j,k)}$ a matrix $\mathbf{M}$ with M_{jk} removed. To calculate derivatives of $\mathbf{U}(s)$, we need the following lemma.

Lemma B.1. *Let $1 \le j, k \le n$ and $m \ge 2$. Then*

$$\left[\frac{\partial\widehat{\mathbf{V}}}{\partial Y_{jk}^{(1)}}\right]_{ab} = \frac{1}{\sqrt{n}}[\mathbf{V}_{[1,m-1]}]_{a,k+n}\mathbb{1}(b=j) + \frac{1}{\sqrt{n}}[\mathbf{V}_{[2,m]}\mathbf{J}]_{kb}\mathbb{1}(a=j) \tag{B.6}$$

for any $1 \le a, b \le 2n$.

Proof. We decompose $\widehat{\mathbf{V}}$ in the following way

$$\widehat{\mathbf{V}} = \left(\big[\mathbf{H}^{(1)}\big]^{(j,k)} + \frac{Y_{jk}^{(1)}}{\sqrt{n}}\mathbf{E}_{j,k}\right)\mathbf{V}_{[2,m-1]}\left(\big[\mathbf{H}^{(m)}\big]^{(j,k)} + \frac{Y_{jk}^{(1)}}{\sqrt{n}}\mathbf{E}_{k+n,j+n}\right)\mathbf{J}.$$

It is easy to see

$$[\mathbf{V}_{[1,m-1]}\mathbf{E}_{k+n,j+n}\mathbf{J}]_{ab} = [\mathbf{V}_{[1,m-1]}]_{a,k+n}\mathbb{1}(b=j),$$
$$[\mathbf{E}_{j,k}\mathbf{V}_{[2,m]}\mathbf{J}]_{ab} = [\mathbf{V}_{[2,m]}\mathbf{J}]_{kb}\mathbb{1}(a=j)$$

and

$$[\mathbf{E}_{j,k}\mathbf{V}_{[2,m-1]}\mathbf{E}_{k+n,j+n}\mathbf{J}]_{ab} = [\mathbf{V}_{[2,m-1]}]_{k,k+n}\mathbb{1}(a=b=j) = 0. \qquad \square$$

We may generalize the last lemma to the case where the derivatives are taken with respect to $Y_{jk}^{(q)}$, $q = 2, \dots, m$. We have the following lemma.

Lemma B.2. *Let $1 \le j, k \le n$ and $m \ge 2$. Then*

$$\begin{aligned}\left[\frac{\partial\widetilde{\mathbf{V}}}{\partial Y_{jk}^{(q)}}\right]_{ab} &= \frac{1}{\sqrt{n}}[\mathbf{V}_{[1,m-q]}]_{a,k+n}[\mathbf{V}_{[m-q+2,m]}\mathbf{J}]_{j+n,b}\\ &\quad + \frac{1}{\sqrt{n}}[\mathbf{V}_{[1,q-1]}]_{aj}[\mathbf{V}_{[q+1,m]}\mathbf{J}]_{kb}.\end{aligned} \tag{B.7}$$

Proof. The proof is similar. □

Lemma B.3. *Let* $1 \le j, k \le n$ *and* $m \ge 2$. *Then*

$$\left[\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(1)}}\right]_{xy} = \frac{i}{\sqrt{n}}[\mathbf{U}\mathbf{V}_{[1,m-1]}]_{x,k+n} * [\mathbf{U}]_{yj}(t) + \frac{i}{\sqrt{n}}[\mathbf{U}\mathbf{V}_{[1,m-1]}]_{y,k+n} * [\mathbf{U}]_{xj}(t).$$

Proof. Using the chain rule, we will have

$$\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(1)}} = \sum_{a=1}^{n}\sum_{b=n+1}^{2n} \frac{\partial \mathbf{U}(t)}{\partial \widetilde{V}_{ab}} \frac{\partial \widetilde{V}_{ab}}{\partial Y_{jk}^{(1)}} + \sum_{a=n+1}^{2n}\sum_{b=1}^{n} \frac{\partial \mathbf{U}(t)}{\partial \widetilde{V}_{ab}} \frac{\partial \widetilde{V}_{ab}}{\partial Y_{jk}^{(1)}}.$$

Applying Lemma B.1, we will have

$$\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(1)}} = \frac{1}{\sqrt{n}} \sum_{b=n+1}^{2n} \frac{\partial \mathbf{U}(t)}{\partial \widetilde{V}_{jb}} [\mathbf{V}_{[2,m]}\mathbf{J}]_{k,b} + \frac{1}{\sqrt{n}} \sum_{a=n+1}^{2n} \frac{\partial \mathbf{U}(t)}{\partial \widetilde{V}_{aj}} [\mathbf{V}_{[1,m-1]}]_{a,k+n}.$$

From (B.4), it follows that

$$\begin{aligned}\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(1)}} &= \frac{i}{\sqrt{n}} \sum_{b=n+1}^{2n} U_{xj} * U_{by}(t)[\mathbf{V}_{[2,m]}\mathbf{J}]_{kb} \\ &\quad + \frac{i}{\sqrt{n}} \sum_{a=n+1}^{2n} U_{xa} * U_{jy}(t)[\mathbf{V}_{[1,m-1]}]_{a,k+n}.\end{aligned}$$

Since $[\mathbf{U}\mathbf{V}_{[1,m-1]}]_{y,k+n} = [\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{U}]_{k,y}$ we get the statement of lemma. □

Lemma B.4. *Let* $1 \le j, k \le n$ *and* $m \ge 2$. *Then*

$$\begin{aligned}\left[\frac{\partial \mathbf{U}(t)}{\partial Y_{jk}^{(q)}}\right]_{x,y} &= \frac{i}{\sqrt{n}}[\mathbf{U}\mathbf{V}_{1,[m-q]}]_{x,k+n} * [\mathbf{V}_{[m-q+2,m]}\mathbf{J}\mathbf{U}]_{j+n,y}(t) \\ &\quad + \frac{i}{\sqrt{n}}[\mathbf{U}\mathbf{V}_{1,[m-q]}]_{y,k+n} * [\mathbf{V}_{[m-q+2,m]}\mathbf{J}\mathbf{U}]_{j+n,x}(t).\end{aligned}$$

Proof. The proof is similar. □

The following lemma gives an expression for the derivative of $S(\widehat{\mathbf{V}}) := \frac{1}{2}\operatorname{Tr} f(\widehat{\mathbf{V}})$ with respect to $Y_{jk}^{(1)}$.

Lemma B.5. *Let* $1 \le j, k \le n$ *and* $m \ge 2$. *Then*

$$\frac{\partial S}{\partial Y_{jk}^{(1)}} = \frac{1}{\sqrt{n}}\bigl[f'(\widetilde{\mathbf{V}})\mathbf{V}_{[1,m-1]}\bigr]_{j,k+n}. \tag{B.8}$$

Proof. It is easy to see that

$$\frac{\partial S}{\partial Y_{jk}^{(1)}} = \frac{1}{2}\int_{-\infty}^{\infty} \widehat{f}(u)\,\mathrm{Tr}\,\frac{\partial \mathbf{U}(u)}{\partial Y_{jk}^{(1)}}\,du.$$

Applying Lemma B.3, we get

$$\begin{aligned}\frac{\partial S}{\partial Y_{jk}^{(1)}} &= \frac{i}{2\sqrt{n}}\int_{-\infty}^{\infty} s\widehat{f}(s)\big[\mathbf{U}(s)\mathbf{V}_{[1,m-1]}\big]_{j,k+n}\,ds \\ &\quad + \frac{i}{2\sqrt{n}}\int_{-\infty}^{\infty} s\widehat{f}(s)\big[\mathbf{V}_{[2,m]}\mathbf{J}\mathbf{U}(s)\big]_{kj}\,ds.\end{aligned}$$

Applying the properties of $\mathbf{V}$ and $\mathbf{U}$, we get (B.8). □

Lemma B.6. *Let* $1 \le j, k \le n$ *and* $m \ge 2$. *Then*

$$\frac{\partial S}{\partial Y_{jk}^{(q)}} = \frac{1}{\sqrt{n}}\big[\mathbf{V}_{[m-q+2,m]}\mathbf{J}f'(\widetilde{\mathbf{V}})\mathbf{V}_{[1,m-q]}\big]_{j+n,k+n}. \tag{B.9}$$

Proof. The proof is similar to the proof of the previous lemma. □

Appendix C: Auxiliary lemmas

In this section, we prove some auxiliary lemmas. The following lemma gives an estimate for the variance of

$$T_n(s,t) := \frac{1}{n}\sum_{j,k=1}^{n}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{kj}\big[\mathbf{U}(t-s)\mathbf{H}^{(1)}\big]_{j,k+n}.$$

Lemma C.1. *Under condition of Theorem* 2.1, *we have*

$$\mathrm{Var}\big(T_n(t,s)\big) \le \frac{C\max(t^2,(t-s)^2)}{n}.$$

Proof. Let us introduce the following matrices removing rth row and column

$$\mathbf{H}^{(q,l)} = \mathbf{H}^{(q)} - \mathbf{E}_{l,l}\mathbf{H}^{(q)} - \mathbf{H}^{(q)}\mathbf{E}_{l,l}, \qquad \widetilde{\mathbf{H}}^{(q,l)} = \mathbf{H}^{(q)} - \mathbf{E}_{l+n,l+n}\mathbf{H}^{(q)} - \mathbf{H}^{(q)}\mathbf{E}_{l+n,l+n},$$

where $q = 1, 2$ and $l = 1, \dots, n$. We define the following filtration

$$\mathfrak{F}_{1,l} := \sigma\big\{Y_{j,k}^{(1)}, l < j,k \le n, Y_{pq}^{(2)}, p,q = 1,\dots,n\big\}, \qquad \mathfrak{F}_{2,l} := \sigma\big\{Y_{j,k}^{(2)}, l < j,k \le n\big\}.$$

We may rewrite the difference

$$\mathbb{E}\sum_{j,k=1}^{n}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{kj}\big[\mathbf{U}(t-s)\mathbf{H}^{(1)}\big]_{j,k+n}-\sum_{k=1}^{n}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{k,j}\big[\mathbf{U}(t-s)\mathbf{H}^{(1)}\big]_{j,k+n}$$
$$=\sum_{q=1}^{2}\sum_{l=1}^{n}(\mathbb{E}_{q,l}-\mathbb{E}_{q,l-1}),$$

where $\mathbb{E}_{q,l}$ denotes the mathematical expectation with respect to $\mathcal{F}_{q,l}$. It is easy to see that $\mathcal{F}_{1,n}=\mathcal{F}_{2,0}$ and

$$\mathbb{E}_{1,l}\sum_{j,k=1}^{n}\big[\widetilde{\mathbf{H}}^{(2,l)}\mathbf{J}\mathbf{U}^{(1,l)}(s)\big]_{kj}\big[\mathbf{U}^{(1,l)}(t-s)\mathbf{H}^{(1,l)}\big]_{j,k+n}$$
$$=\mathbb{E}_{1,l-1}\sum_{j,k=1}^{n}\big[\widetilde{\mathbf{H}}^{(2,l)}\mathbf{J}\mathbf{U}^{(1,l)}(s)\big]_{kj}\big[\mathbf{U}^{(1,l)}(t-s)\mathbf{H}^{(1,l)}\big]_{j,k+n},$$
$$\mathbb{E}_{2,l}\sum_{j,k=1}^{n}\big[\mathbf{H}^{(2,l)}\mathbf{J}\mathbf{U}^{(2,l)}(s)\big]_{k,j}\big[\mathbf{U}^{(2,l)}(t-s)\widetilde{\mathbf{H}}^{(1,l)}\big]_{j,k+n}$$
$$=\mathbb{E}_{2,l-1}\sum_{j,k=1}^{n}\big[\mathbf{H}^{(2,l)}\mathbf{J}\mathbf{U}^{(2,l)}(s)\big]_{kj}\big[\mathbf{U}^{(2,l)}(t-s)\widetilde{\mathbf{H}}^{(1,l)}\big]_{j,k+n}.$$

We consider the case $q=1$ only. The case $q=2$ is similar. We may write

$$\sum_{j,k=1}^{n}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(s)\big]_{kj}\big[\mathbf{U}(t-s)\mathbf{H}^{(1)}\big]_{j,k+n}$$
$$-\sum_{j,k=1}^{n}\big[\widetilde{\mathbf{H}}^{(2,l)}\mathbf{J}\big]_{kj}\big[\mathbf{U}^{(1,l)}(t-s)\mathbf{H}^{(1,l)}\big]_{j,k+n}=\Theta_{1,l}+\Theta_{2,l}+\Theta_{3,l}+\Theta_{4,l},$$

where we denoted

$$\Theta_{1,l}:=\sum_{j,k=1}^{n}\big[\big(\mathbf{H}^{(2)}-\widetilde{\mathbf{H}}^{(2,l)}\big)\mathbf{J}\mathbf{U}(s)\big]_{kj}\big[\mathbf{U}(t-s)\mathbf{H}^{(1)}\big]_{j,k+n},$$
$$\Theta_{2,l}:=\sum_{j,k=1}^{n}\big[\widetilde{\mathbf{H}}^{(2,l)}\mathbf{J}\big(\mathbf{U}(s)-\mathbf{U}^{(1,l)}(s)\big)\big]_{jk}\big[\mathbf{U}(t-s)\mathbf{H}^{(1)}\big]_{j,k+n},$$
$$\Theta_{3,l}:=\sum_{j,k=1}^{n}\big[\widetilde{\mathbf{H}}^{(2,l)}\mathbf{J}\mathbf{U}^{(1,l)}(s)\big]_{jk}\big[\big(\mathbf{U}(t-s)-\mathbf{U}^{(1,l)}(t-s)\big)\mathbf{H}^{(1)}\big]_{j,k+n},$$

$$\Theta_{4,l} := \sum_{j,k=1}^{n} \big[\widetilde{\mathbf{H}}^{(2,l)}\mathbf{J}\mathbf{U}^{(1,l)}(s)\big]\big[\mathbf{U}^{(1,l)}(t-s)\big(\mathbf{H}^{(1)} - \mathbf{H}^{(1,l)}\big)\big]_{k,k+n}.$$

It is easy to check that $\Theta_{1,l} = \Theta_{4,l} = 0$. We consider the term $\Theta_{2,l}$. The term $\Theta_{3,l}$ is similar. Applying (B.4) we get

$$\Theta_{2,l} = \mathcal{I}_{1,l} + \mathcal{I}_{2,l},$$

where

$$\mathcal{I}_{1,l} := \int_0^s \sum_{j,k=1}^{n} \big[\widetilde{\mathbf{H}}^{(2,l)}\mathbf{J}\mathbf{U}^{(1,l)}(s_1)\mathbf{E}_{l,l}\mathbf{V}\mathbf{J}\mathbf{U}(s-s_1)\big]_{kj}\big[\mathbf{U}(t-s)\mathbf{H}^{(1)}\big]_{j,k+n}\,ds_1,$$

$$\mathcal{I}_{2,l} := \int_0^s \sum_{j,k=1}^{n} \big[\widetilde{\mathbf{H}}^{(2,l)}\mathbf{J}\mathbf{U}^{(1,l)}(s_1)\mathbf{V}\mathbf{J}\mathbf{E}_{l,l}\mathbf{U}(s-s_1)\big]_{kj}\big[\mathbf{U}(t-s)\mathbf{H}^{(1)}\big]_{j,k+n}\,ds_1.$$

By simple calculations, we get

$$\mathcal{I}_{1,l} = \int_0^s \big[\mathbf{W}\mathbf{U}_3(s-s_1)\mathbf{U}_2(t-s)\big[\mathbf{Y}^{(2)}\big]^{\mathsf{T}}\mathbf{Y}^{(2)}\mathbf{U}_3^{(1,l)}(s_1)\big]_{ll}\,ds_1.$$

It is easy to derive the following estimate

$$\sum_{l=1}^{n} \mathbb{E}\mathcal{I}_{1,l}^2 \le Cs^2\mathbb{E}\big\|\mathbf{W}\big[\mathbf{Y}^{(2)}\big]^{\mathsf{T}}\mathbf{Y}^{(2)}\big\|_2^2 \le Cs^2 n.$$

The same is true for $\sum_{l=1}^{n}\mathbb{E}\mathcal{I}_{2,l}^2$. This fact finishes the proof of the lemma. □

Recall that, see definitions (3.15) and (3.13),

$$V_{n,j}(t) := \mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t)\mathbf{H}^{(1)}\big]_{j+n,j+n},$$
$$T_{n,j}(t) := \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t)\mathbf{H}^{(1)}\big]_{j,j+n}.$$

The following lemma gives an estimate for the variance of $\frac{1}{n}u_n(t)$, $V_{n,j}(t)$ and $T_{n,j}(t)$.

Lemma C.2. *Under conditions of Theorem 1.1 we have*

$$\operatorname{Var}\left[\frac{1}{n}u_n(t)\right] \le \frac{Ct^2}{n}, \tag{C.1}$$

and for all $j = 1, \dots, n$,

$$\operatorname{Var}\big[V_{n,j}(t)\big] \le C\big(1+t^2\big)\varepsilon_n, \tag{C.2}$$

$$\operatorname{Var}\big[T_{n,j}(t)\big] \le C\big(1+t^2\big)\varepsilon_n, \tag{C.3}$$

where $\lim_{n\to\infty}\varepsilon_n = 0$.

Proof. The proof of the first statement (C.1) is similar to the proof of Lemma C.1. Here one uses the results for the matrix resolvent and the Stieltjes transforms as well. We present the proof of (C.3) only. The proof of (C.2) is similar. Let us denote

$$K_{j,n}(t_1,t_2) := \mathbb{E}\big[T_{n,j}(t_1)\big(T_{n,j}(t_2)-\mathbb{E}T_{n,j}(t_2)\big)\big] = \mathbb{E}T_{n,j}(t_1)T_{n,j}^{(0)}(t_2),$$

where $T_{n,j}^{(0)}(t) := T_{n,j}(t) - \mathbb{E}T_{n,j}(t)$. We have

$$K_{j,n}(t_1,t_2) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_{jk}^{(2)}\big[\mathbf{U}(t_1)\mathbf{H}^{(2)}\big]_{k+n,j+n}T_{n,j}^{(0)}(t_2).$$

By Taylor's formula,

$$\begin{aligned}
K_{j,n}(t_1,t_2) &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbb{E}\bigg[\frac{\partial\mathbf{U}(t_1)}{\partial Y_{jk}^{(2)}}\mathbf{H}^{(1)}\bigg]_{k+n,j+n}T_{n,j}^{(0)}(t_2)\\
&\quad+\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbb{E}\bigg[\mathbf{U}(t_1)\frac{\partial\mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}}\bigg]_{k+n,j+n}T_{n,j}^{(0)}(t_2)\\
&\quad+\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbb{E}\big[\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{k+n,j+n}\bigg[\frac{\partial\mathbf{H}^{(2)}}{\partial Y_{jk}^{(2)}}\mathbf{J}\mathbf{U}(t_2)\mathbf{H}^{(1)}\bigg]_{j,j+n}\\
&\quad+\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbb{E}\big[\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{k+n,j+n}\bigg[\mathbf{H}^{(2)}\mathbf{J}\frac{\partial\mathbf{U}(t_2)}{\partial Y_{jk}^{(2)}}\mathbf{H}^{(1)}\bigg]_{j,j+n}\\
&\quad+\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\mathbb{E}\big[\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{k+n,j+n}\bigg[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t_2)\frac{\partial\mathbf{H}^{(1)}}{\partial Y_{jk}^{(2)}}\bigg]_{j,j+n}\\
&\quad+r_n(t_1,t_2),
\end{aligned}$$

where $r_n(t_1,t_2)$ denotes a remainder term, which polynomially depends on t_1,t_2. It is straightforward to check $\lim_{n\to\infty} r_n(t_1,t_2)=0$. By Lemma B.3, we get

$$\begin{aligned}
K_{j,n}(t_1,t_2) &= \frac{i}{n}\mathbb{E}u_n * \big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{j+n,j+n}T_{n,j}^{(0)}(t_2)\\
&\quad+\frac{i}{n}\sum_{k=1}^{n}\mathbb{E}\big[\mathbf{U}\mathbf{H}^{(1)}\big]_{k+n,j+n} * \big[\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{k+n,j+n}T_{n,j}^{(0)}(t_2)\\
&\quad+\frac{1}{n}\mathbb{E}u_n(t_1)T_{n,j}^{(0)}(t_2)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{n}\sum_{k=1}^{n}\mathbb{E}\big[\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{k+n,j+n}\big[\mathbf{U}(t_2)\mathbf{H}^{(1)}\big]_{k+n,j+n}\\
&+\frac{2i}{n}\sum_{k=1}^{n}\mathbb{E}\big[\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{k+n,j+n}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}\big]_{j,k+n}*\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t_2)\mathbf{H}^{(1)}\big]_{j+n,j+n}\\
&+r_n(t_1,t_2).
\end{aligned}
$$

Similarly to the previous estimates it is not very difficult to check that all terms except the first one are of order $o(1)$. Let us consider the first term

$$
\begin{aligned}
&\frac{i}{n}\mathbb{E}u_n*\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{j+n,j+n}T_{n,j}^{(0)}(t_2)\\
&\quad=\frac{i}{n}\mathbb{E}u_n*\mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{j+n,j+n}T_{n,j}^{0}(t_2)+r_n(t_1,t_2).
\end{aligned}
$$

From (3.16), we have

$$
V_{n,k}(s)=\frac{i}{n}\mathbb{E}u_n(s)*\mathbb{E}T_{n,k}(s)+r_n(s).
$$

We may conclude that

$$
\begin{aligned}
&\frac{i}{n}\sum_{k=1}^{n}\mathbb{E}[\mathbf{U}]_{k+n,k+n}*\mathbb{E}\big[\mathbf{H}^{(2)}\mathbf{J}\mathbf{U}(t_1)\mathbf{H}^{(1)}\big]_{j+n,j+n}T_{n,j}^{0}(t_2)\\
&\quad=-\frac{1}{n^2}(\mathbb{E}u_n)^{*2}*\mathbb{E}T_{n,j}(t_1)T_{n,j}^{0}(t_2)+r_n(t_1,t_2).
\end{aligned}
$$

Taking the limit with respect to $n_l\to\infty$ we get that $K_j:=\lim_{n_l\to\infty}K_{j,n_l}$ satisfies the following equation

$$
K_j(t_1,t_2)=-\int_0^{t_1}v^{*2}(t_1-s)K_j(s,t_2)\,ds.
$$

Since $K_j(t_1,t_2)=0$ is a unique solution of the last equation this means that

$$
K_{j,n}(t_1,t_2)=r_n(t_1,t_2).
$$

Taking $t_2=t_1$ finishes the proof of the lemma. □

Appendix D: Laplace transform

In this section, we recall several results from the theory of Laplace transforms. We will follow Lytova and Pastur [20], Proposition 2.1.

Statement D.1. *Let $f : \mathbb{R}_+ \to \mathbb{C}$ denote a local Lipshitz function such that for some $\delta > 0$*

$$\sup_{t \geq 0} e^{-\delta t} \left| f(t) \right| < \infty$$

and let $\breve{f} : \{z \in C : \operatorname{Im} z < -\delta\} \to \mathbb{C}$ denotes its generalized Fourier transform

$$\breve{f}(z) = \frac{1}{i} \int_0^\infty e^{-izt} f(t)\, dt.$$

The inversion formula is given by

$$f(t) = \frac{i}{2\pi} \int_L e^{izt} \breve{f}(z)\, dz, \qquad t \geq 0,$$

where $L = (-\infty - i\varepsilon, \infty - i\varepsilon)$, $\varepsilon > \delta$, and the principal value of the integral at infinity is used. Denote the correspondence between functions and their generalized Fourier transforms by $f \leftrightarrow \breve{f}$. Then we have

$$f'(t) \leftrightarrow i\big(f(+0) + z\breve{f}(z)\big);$$

$$\int_0^t f(s)\, ds \leftrightarrow (iz)^{-1} \breve{f}(z);$$

$$f * g(t) \leftrightarrow i\breve{f}(z)\breve{g}(z).$$

Recall that, see definition (2.31), $p(x) := |x| P_2(x^2)$ and $a := \sqrt{K}_2$. Let $s(z)$ be the Stieltjes transform of $p(x)$. It satisfies the following equation (see (2.5) with $m = 2$)

$$1 + zs(z) = zs^3(z). \tag{D.1}$$

Lemma D.1. *The inverse Fourier transform of*

$$K(z) = \frac{1/z - 2s(z)}{1 - 3s^2(z)}$$

is given by

$$T(t) = \frac{1}{\pi} \int_{-a}^{a} \frac{e^{itx}}{3p_1(x)} \frac{4p_1^4(x) + 11p_1^2(x) + 4}{4p_1^2(x) + 3}\, dx,$$

where $p_1(x) := \pi p(x)$.

Proof. By definition, see Statement D.1,

$$T(t) = \frac{i}{2\pi} \int_L e^{itz} K(z)\, dz,$$

where $L = (-\infty - i\varepsilon, \infty - i\varepsilon)$, $\varepsilon > 0$. We introduce the following contour $\mathbf{K}$:

$$\mathbf{K} := \mathbf{K}_1 \cup \cdots \cup \mathbf{K}_8,$$

where

$$\mathbf{K}_1 := \{z = u + iv, |u| \le T, v = -\varepsilon\}, \qquad \mathbf{K}_2 := \{z = u + iv : |z| = T, v \ge 0\},$$
$$\mathbf{K}_{3,4} := \{z = u + iv : |u| \le a + \varepsilon/2, v = \pm\varepsilon/2\},$$
$$\mathbf{K}_{5,6} := \{z = u + iv : u = \pm(a + \varepsilon/2), -\varepsilon/2 \le v \le \varepsilon/2\},$$
$$\mathbf{K}_{7,8} := \{z = u + iv : u = \pm T, -\varepsilon \le v \le 0\}.$$

We may write

$$T(t) = \lim_{T\to\infty} \frac{i}{2\pi} \int_{\mathbf{K}_1} e^{itz} K(z)\, dz$$

and

$$\int_{\mathbf{K}} e^{itz} K(z)\, dz = 0.$$

Furthermore, we note

$$\lim_{T\to\infty} \int_{\mathbf{K}_2 \cup \mathbf{K}_7 \cup \mathbf{K}_8} e^{itz} K(z)\, dz = 0.$$

We compute the integrals

$$\mathcal{K}_1 := \left(\int_{\mathbf{K}_3} - \int_{\mathbf{K}_4}\right) e^{itz} K(z)\, dz.$$

Let $s(z) := if(z) + g(z)$ for $z = u + iv$. Note that by definition

$$\operatorname{Im} s(z) = \begin{cases} f(z), & \text{if } \operatorname{Im} z > 0, \\ -f(z), & \text{if } \operatorname{Im} z < 0. \end{cases}$$

Let us calculate $K(z)$ for $z \in K_3$. Applying (D.1) we obtain that

$$\begin{aligned} K(z) &= \frac{s(z)(s^2(z) - 3)}{1 - 3s^2(z)} = \frac{1}{3} \frac{(g + if)(f^2 + 1 - 3f^2 + 6ifg - 9)}{2f(f - 3ig)} \\ &= \frac{1}{6f} \frac{(g + if)(6ifg - 2f^2 - 8)}{f - 3ig} = \frac{1}{6f} \frac{(g + if)(6ifg - 2f^2 - 8)(f + 3ig)}{|f - 3ig|^2}. \end{aligned}$$

The enumerator is equal to

$$(g + if)\big(6ifg - 2f^2 - 8\big)(f + 3ig) = -2\big(3if + 4f^2 g + 4g\big)(f + 3ig).$$

The imaginary part of the enumerator is given by

$$-6f^2 - 24f^2g^2 - 24g^2 = -6f^2 - 8f^2 - 8f^4 - 8 - 8f^2 = -2\bigl(4f^4 + 11f^2 + 4\bigr).$$

Finally,

$$\operatorname{Im} K(z) = -\frac{1}{3f}\frac{4f^4 + 11f^2 + 4}{4f^2 + 3}.$$

The real part is equal to

$$\operatorname{Re} K(z) = -\frac{1}{3}\frac{g(5 - 4f^2)}{4f^2 + 3}.$$

It is easy to see that for $z \in K_4$ we will have

$$\operatorname{Im} K(z) = \frac{1}{3f}\frac{4f^4 + 11f^2 + 4}{4f^2 + 3}, \qquad \operatorname{Re} K(z) = -\frac{1}{3}\frac{g(5 - 4f^2)}{4f^2 + 3}.$$

Since ε is an arbitrary number and

$$\lim_{\varepsilon \to 0} f(u + i\varepsilon) = \pi p(u)$$

then integrating $\operatorname{Re} K(z)$ in the opposite directions we get zero. Finally

$$T(t) = \frac{1}{\pi}\int_{-a}^{a} \frac{e^{itx}}{3\pi p(x)} \frac{4(\pi p(x))^4 + 11(\pi p(x))^2 + 4}{4(\pi p(x))^2 + 3}\, dx. \qquad \square$$

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