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Conditions for a Lévy process to stay positive near 0, in probability

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A necessary and sufficient condition for a Lévy process X to stay positive, in probability, near 0, which arises in studies of Chung-type laws for X near 0, is given in terms of the characteristics of X.

Keywords: Lévy process; staying positive

1. Introduction

Let $(X_t)_{t\geq 0}$ be a real valued Lévy process with canonical triplet (γ, σ^2, Π), thus having characteristic function $Ee^{\mathrm{i}\theta X_t} = e^{t\Psi(\theta)}$, $t \geq 0$, $\theta \in \mathbb{R}$, with characteristic exponent

$$\Psi(\theta) := \mathrm{i}\theta\gamma - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} \bigl(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|\leq 1\}}\bigr)\Pi(\mathrm{d}x). \tag{1.1}$$

Here, $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and Π is a Borel measure on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}\setminus\{0\}}(x^2 \wedge 1)\Pi(\mathrm{d}x) < \infty$.

The condition

$$\liminf_{t\downarrow 0} P(X_t \leq 0) \wedge P(X_t \geq 0) > 0 \tag{1.2}$$

was shown by Wee [10] to imply a Chung-type law at 0 for X. Attention is drawn to this in a recent paper of Aurzada, Döring and Savov [2], who give extended and refined versions of the Chung law using a quite different approach to that of Wee. The difference between (1.2) and the conditions imposed by Aurzada et al. [2] is not at all clear, though based on some examples they suggest that theirs are weaker than (1.2). Our aim in this paper is to give necessary and sufficient conditions for X to stay positive near 0, or to stay negative near 0, and hence to characterise (1.2).

We need some more notation. The positive, negative and two-sided tails of Π are

$$\begin{aligned} &\overline{\Pi}^+(x) := \Pi\bigl\{(x,\infty)\bigr\}, \qquad \overline{\Pi}^-(x) := \Pi\bigl\{(-\infty,-x)\bigr\} \quad \text{and} \\ &\overline{\Pi}(x) := \overline{\Pi}^+(x) + \overline{\Pi}^-(x), \qquad x > 0. \end{aligned} \tag{1.3}$$

The restriction of Π to $(0,\infty)$ is denoted by $\Pi^{(+)}$, and we define $\Pi^{(-)}$ on $(0,\infty)$ by $\Pi^{(-)}(\mathrm{d}x) := -\Pi(-\mathrm{d}x)$, for $x > 0$. We are only interested in small time behaviour of X_t, and we eliminate the compound Poisson case by assuming $\Pi(\mathbb{R}) = \infty$ throughout.

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Define truncated and Winsorised moments as

$$\begin{aligned}\nu(x) &= \gamma - \int_{x<|y|\le 1} y\Pi(\mathrm{d}y),\\ A(x) &= \gamma + \overline{\Pi}^{+}(1) - \overline{\Pi}^{-}(1) - \int_x^1 \bigl(\overline{\Pi}^{+}(y) - \overline{\Pi}^{-}(y)\bigr)\,\mathrm{d}y\end{aligned} \tag{1.4}$$

and

$$V(x) = \sigma^2 + \int_{0<|y|\le x} y^2\Pi(\mathrm{d}y), \qquad U(x) = \sigma^2 + 2\int_0^x y\overline{\Pi}(y)\,\mathrm{d}y, \qquad x>0. \tag{1.5}$$

These functions are defined and finite for all $x>0$ by virtue of property $\int_{0<|y|\le 1} y^2\Pi(\mathrm{d}y) < \infty$ of the Lévy measure Π but only their behaviour as $x \downarrow 0$ will be relevant for us. Integration by parts shows that

$$A(x) = \nu(x) + x\bigl(\overline{\Pi}^{+}(x) - \overline{\Pi}^{-}(x)\bigr), \qquad x>0. \tag{1.6}$$

Doney [5], Lemma 9, gives the following version of the Itô decomposition of X which caters for positive and negative jumps separately. Take constants $h_+>0$ and $h_->0$. Then for $t\ge 0$,

$$\begin{aligned}X_t &= t\gamma - t\nu_+(h_+) + t\nu_-(h_-)\\ &\quad + \sigma Z_t + X_t^{(S,h_+,+)} + X_t^{(S,h_-,-)} + X_t^{(B,h_+,+)} + X_t^{(B,h_-,-)},\end{aligned} \tag{1.7}$$

where γ and σ are as in (1.1), and the functions $\nu_\pm$ are

$$\nu_+(h_+) := \int_{(h_+,1]} x\Pi(\mathrm{d}x) \quad \text{and} \quad \nu_-(h_-) := \int_{(h_-,1]} x\Pi^{(-)}(\mathrm{d}x). \tag{1.8}$$

Again, only their behaviour for small values of $h_\pm$ will be relevant. We can keep $h_\pm \in (0,1)$. Note that $\nu(x) = \gamma - \nu_+(x) + \nu_-(x)$. In (1.7), $(X_t^{(S,h_+,+)})_{t\ge 0}$ is a compensated sum of small *positive* jumps, that is,

$$X_t^{(S,h_+,+)} = \text{a.s.}\lim_{\varepsilon\downarrow 0}\Biggl(\sum_{0<s\le t} \Delta X_s 1_{\{\varepsilon<\Delta X_s\le h_+\}} - t\int_{\varepsilon<x\le h_+} x\Pi(\mathrm{d}x)\Biggr),$$

$(X_t^{(S,h_-,-)})_{t\ge 0}$ is a compensated sum of small *negative* jumps, that is,

$$X_t^{(S,h_-,-)} = \text{a.s.}\lim_{\varepsilon\downarrow 0}\Biggl(\sum_{0<s\le t} \Delta X_s 1_{\{-h_-\le\Delta X_s<-\varepsilon\}} - t\int_{-h_-\le x<-\varepsilon} x\Pi(\mathrm{d}x)\Biggr),$$

where the almost sure limits exist; and $(X_t^{(B,h_\pm,\pm)})_{t\ge 0}$ are the processes of positive and negative big jumps, thus,

$$X_t^{(B,h_+,+)} = \sum_{0<s\le t} \Delta X_s 1_{\{\Delta X_s>h_+\}} \quad \text{and} \quad X_t^{(B,h_-,-)} = \sum_{0<s\le t} \Delta X_s 1_{\{\Delta X_s<-h_-\}}, \qquad t>0.$$

Finally, $(Z_t)_{t\ge 0}$ is a standard Brownian motion independent of the jump processes, all of which are independent from each other.

To motivate our approach, we quote part of a result due to Doney [5]. It gives an equivalence for X to remain positive at small times, with probability approaching 1, in terms of the functions $A(x)$, $U(x)$ and the negative tail of Π. The condition reflects the positivity of X at small times in that the function $A(x)$ remains positive for small values of x, and dominates $U(x)$ and the negative tail of Π in a certain way.

Theorem 1.1. *Suppose* $\Pi(\mathbb{R}) = \infty$. (i) *Suppose also that* $\overline{\Pi}^{-}(0+) > 0$. *Then*

$$\lim_{t\downarrow 0} P(X_t > 0) = 1 \tag{1.9}$$

if and only if

$$\lim_{x\downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}^{-}(x)}} = \infty. \tag{1.10}$$

(ii) *Suppose alternatively that X is spectrally positive, that is,* $\overline{\Pi}^{-}(x) = 0$ *for all* $x > 0$. *Then* (1.9) *is equivalent to*

$$\sigma^2 = 0 \quad \text{and} \quad A(x) \ge 0 \qquad \text{for all small } x, \tag{1.11}$$

and this happens if and only if X is a subordinator. Furthermore, we then have $A(x) \ge 0$, *not only for small x, but for all* $x > 0$.

Remarks. (i) Other equivalences for (1.9) are in Theorem 1 of Doney [5] (and his remark following the theorem). He assumes a priori that $\sigma^2 = 0$ but this is not necessary as it follows from the inequality:

$$\limsup_{x\downarrow 0} \frac{A(x)}{\sqrt{\overline{\Pi}^{-}(x)}} < \infty, \tag{1.12}$$

whenever $\overline{\Pi}^{-}(0+) > 0$, which is proved in Buchmann, Fan and Maller [4].

(ii) When $\Pi(\mathbb{R}) < \infty$, X is compound Poisson and its behaviour near 0 is simply determined by the sign of the shift constant γ. We eliminate this case throughout.

The next section contains our main result which is essentially a subsequential version of Theorem 1.1.

2. Staying positive near 0, subsequential version

Denote the jump process of X by $(\Delta X_t)_{t\ge 0}$, where $\Delta X_t = X_t - X_{t-}$, $t > 0$, with $\Delta X_0 \equiv 0$, and define $\Delta X_t^{+} = \max(\Delta X_t, 0)$, $\Delta X_t^{-} = \max(-\Delta X_t, 0)$, $(\Delta X^{+})_t^{(1)} = \sup_{0<s\le t} \Delta X_s^{+}$, $(\Delta X^{-})_t^{(1)} = \sup_{0<s\le t} \Delta X_s^{-}$.

Theorem 2.1. *Assume* $\Pi(\mathbb{R}) = \infty$.

(i) *Suppose* $\overline{\Pi}^{-}(0+) > 0$. *Then the following are equivalent*:
there is a non-stochastic sequence $t_k \downarrow 0$ *such that*

$$P(X_{t_k} > 0) \to 1; \tag{2.1}$$

there is a non-stochastic sequence $t_k \downarrow 0$ *such that*

$$\frac{X_{t_k}}{(\Delta X^{-})^{(1)}_{t_k}} \xrightarrow{\mathrm{P}} \infty \qquad \text{as } k \to \infty; \tag{2.2}$$

$$\limsup_{x\downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}^{-}(x)}} = \infty. \tag{2.3}$$

(ii) *Suppose alternatively that* X *is spectrally positive*, *that is*, $\overline{\Pi}^{-}(x) = 0$ *for all* $x > 0$. *Then* (2.1) *is equivalent to* $\lim_{t\downarrow 0} P(X_t > 0) \to 1$, *thus to* (1.11), *equivalently*, X_t *is a subordinator, and* $A(x) \geq 0$ *for all* $x > 0$.

(iii) *Suppose* $\overline{\Pi}^{-}(0+) > 0$. *Then* $X_{t_k}/t_k \xrightarrow{\mathrm{P}} \infty$ *for a non-stochastic sequence* $t_k \downarrow 0$ *if and only if*

$$\limsup_{x\downarrow 0} \frac{A(x)}{1 + \sqrt{U(x)\overline{\Pi}^{-}(x)}} = \infty. \tag{2.4}$$

Remarks. (i) When $\overline{\Pi}^{-}(0+) > 0$, $\sup_{0<s\leq t} \Delta X_s^{-} > 0$ a.s. for all $t > 0$, so the ratio in (2.2) is well defined.

(ii) Sato [9], page 65, shows that $P(X_t \leq x)$ is a continuous function of x for all $t > 0$ when $\Pi(\mathbb{R}) = \infty$. So $P(X_t > 0) = P(X_t \geq 0)$ for all $t > 0$ and $P(X_{t_k} > 0)$ can be replaced by $P(X_{t_k} \geq 0)$ in (2.1) without changing the result (and similarly in Theorem 1.1).

(iii) Assuming $\overline{\Pi}^{+}(0+) = \infty$ and $\overline{\Pi}^{-}(0+) > 0$, the contrapositive of (2.1) shows that there is no sequence $t_k \downarrow 0$ such that $P(X_{t_k} > 0) \to 1$, or, equivalently, $\liminf_{t\downarrow 0} P(X_t \leq 0) > 0$, if and only if

$$\limsup_{x\downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}^{-}(x)}} < \infty. \tag{2.5}$$

By a symmetrical argument, when $\overline{\Pi}^{-}(0+) = \infty$ and $\overline{\Pi}^{+}(0+) > 0$, then $\liminf_{t\downarrow 0} P(X_t \geq 0) > 0$ if and only if

$$\liminf_{x\downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}^{+}(x)}} > -\infty. \tag{2.6}$$

Combining these gives the following.

Corollary 2.2. *Assume $\overline{\Pi}^{+}(0+) = \overline{\Pi}^{-}(0+) = \infty$. Then (1.2) holds if and only if*

$$-\infty < \liminf_{x\downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}^{+}(x)}} \quad \textit{and} \quad \limsup_{x\downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}^{-}(x)}} < \infty. \tag{2.7}$$

When one of $\overline{\Pi}^{+}(0+)$ or $\overline{\Pi}^{-}(0+)$ is infinite but the other is zero, conditions for (1.2) can also be read from Theorem 2.1.

(ii) A random walk version of Theorem 2.1 is in Kesten and Maller [8]. Andrew [1], Theorem 4, has results related to Theorem 2.1, including the equivalence of (2.1) and (2.2).

3. Some inequalities for the distribution of X

For the proof of Theorem 2.1, some lemmas are needed. The first gives a non-uniform Berry–Esseen bound for a small jump component of X. The proof is rather similar to that of Lemma 4.3 of Bertoin, Doney and Maller [3], so we omit details.

Lemma 3.1. *Fix $h_- \ge 0, h_+ \ge 0, h_- \vee h_+ > 0$. Let $(X_t^{(-h_-,h_+)})_{t\ge 0}$ be the small jump martingale obtained from X as the compensated sum of jumps with magnitudes in $(-h_-, h_+)$:*

$$X_t^{(-h_-,h_+)} = \text{a.s.} \lim_{\varepsilon\downarrow 0}\left(\sum_{0<s\le t} \Delta X_s 1_{\{\Delta X_s \in (-h_-,-\varepsilon)\cup(\varepsilon,h_+)\}} - t\int_{x\in(-h_-,-\varepsilon)\cup(\varepsilon,h_+)} x\Pi(\mathrm{d}x)\right).$$

(*Interpret integrals over intervals of the form* $(0,-\varepsilon)$, *and* $(\varepsilon, 0)$, $\varepsilon > 0$, *as* 0.) *Define absolute moments $m_k^{(-h_-,h_+)} := \int_{-h_-<x<h_+} |x|^k \Pi(\mathrm{d}x)$, $k = 2, 3, \dots$, and assume $\sigma^2 + m_2^{(-h_-,h_+)} > 0$. Then we have the non-uniform bound: for any $x \in \mathbb{R}$, $t > 0$,*

$$\left| P\left(\frac{\sigma Z_t + X_t^{(-h_-,h_+)}}{\sqrt{t(\sigma^2 + m_2^{(-h_-,h_+)})}} \le x\right) - \Phi(x)\right| \le \frac{C m_3^{(-h_-,h_+)}}{\sqrt{t}(\sigma^2 + m_2^{(-h_-,h_+)})^{3/2}(1+|x|)^3}, \tag{3.1}$$

where C is an absolute constant and $\Phi(x)$ is the standard normal c.d.f.

Next, we use Lemma 3.1 to develop other useful bounds. Define

$$V_+(x) = \int_{0<y\le x} y^2\Pi(\mathrm{d}y) \quad \text{and} \quad V_-(x) = \int_{-x\le y<0} y^2\Pi(\mathrm{d}y), \qquad x > 0. \tag{3.2}$$

In the next lemma, the "+" and "−" signs are to be taken together. When $\overline{\Pi}^{+}(0+) = 0$ we have $V_+ \equiv 0$, and interpret $(X_t^{(S,d_+,+)})_{t\ge 0}$ as 0; similarly with "−" replacing "+" when $\overline{\Pi}^{-}(0+) = 0$.

Lemma 3.2. (i) *Suppose $d_\pm > 0$, $\kappa_\pm > 0$ and $K_\pm$ are constants satisfying*

$$K_\pm \ge 4C \max\left(\frac{\kappa_\pm}{\Phi(-\kappa_\pm)}, \frac{1}{\Phi(-\kappa_\pm)\sqrt{1-\Phi(-\kappa_\pm)/2}}\right), \tag{3.3}$$

where C is the absolute constant in (3.1). Then for each $t>0$

$$P\big(X_t^{(S,d_\pm,\pm)} \le K_\pm d_\pm - \kappa_\pm\sqrt{tV_\pm(d_\pm)}\big) \ge \Phi(-\kappa_\pm)/2. \tag{3.4}$$

(ii) *Suppose, for each $t>0$, $d_\pm = d_\pm(t)>0$ satisfy*

$$t\overline{\Pi}^+(d_+) \le c_+ \quad \textit{and} \quad t\overline{\Pi}^-(d_- -) \ge c_- \tag{3.5}$$

for some $c_+>0$, $c_->0$. Assume $\kappa_\pm>0$ and $K_\pm$ are constants satisfying (3.3).

(a) *Suppose $\overline{\Pi}^+(0+)>0$. Then for each $t>0$ and $L\ge 0$*

$$\begin{aligned} &P\big(X_t \le t\gamma - t\nu_+(d_+) + t\nu_-(d_-) + K_+d_+ - Ld_- - \kappa_+\sqrt{tV_+(d_+)} - \kappa_-\sqrt{tV_-(d_-)}\big) \\ &\quad \ge e^{-c_+}\Phi(-\kappa_+)\Phi(-\kappa_-)P\big(N(c_-)\ge K_- + L\big)/8, \end{aligned} \tag{3.6}$$

where $N(c_-)$ is a Poisson rv with expectation c_-.

(b) *When $\overline{\Pi}^+(0+)=0$, (3.6) remains true with $\nu_+(d_+)=V_+(d_+)=d_+=c_+=0$.*

(iii) *Suppose $0\le\overline{\Pi}^-(0+)<\infty=\overline{\Pi}^+(0+)$ and, for $t>0$, $d_+=d_+(t)>0$ is such that $t\overline{\Pi}^+(d_+(t))\le c_+$. Suppose $\kappa_+>0$ and K_+ are constants satisfying (3.3). Then*

$$P\big(X_t \le t\gamma - t\nu_+(d_+) + t\nu_-(0) + K_+d_+ - \kappa_+\sqrt{tV_+(d_+)}\big) \ge e^{-c_+}\Phi(-\kappa_+)/4, \tag{3.7}$$

where $\nu_-(0)\equiv 0$ when $\overline{\Pi}^-(0+)=0$.

Proof. (i) We give the proof just for the "+" signs. Fix $t>0$ and take any constants $d_+>0$, $\kappa_+>0$ and K_+, with κ_+ and K_+ satisfying (3.3).

(a) Assume $V_+(d_+)>0$. Apply the bound (3.1) in Lemma 3.1 to $X_t^{(S,d_+,+)}$, which has Lévy measure Π restricted to $(0,d_+)$. Noting that $\int_{0<y\le x} y^3\Pi(\mathrm{d}y)\le xV_+(x)$, $x>0$, (3.1) then gives, for each $t>0$,

$$\sup_{x\in\mathbb{R}}\big|P\big(X_t^{(S,d_+,+)} \le x\sqrt{tV_+(d_+)}\big) - \Phi(x)\big| \le \frac{Cd_+}{\sqrt{tV_+(d_+)}}. \tag{3.8}$$

Substitute $x=-\kappa_+$ in this to get

$$P\big(X_t^{(S,d_+,+)} \le -\kappa_+\sqrt{tV_+(d_+)}\big) \ge \Phi(-\kappa_+) - \frac{Cd_+}{\sqrt{tV_+(d_+)}}.$$

When $2Cd_+ \le \Phi(-\kappa_+)\sqrt{tV_+(d_+)}$, this inequality implies

$$P\big(X_t^{(S,d_+,+)} \le -\kappa_+\sqrt{tV_+(d_+)}\big) \ge \tfrac{1}{2}\Phi(-\kappa_+). \tag{3.9}$$

When $2Cd_+ > \Phi(-\kappa_+)\sqrt{tV_+(d_+)}$, we have

$$2\kappa_+\sqrt{tV_+(d_+)} < 4Cd_+\kappa_+/\Phi(-\kappa_+) \le K_+d_+,$$

since K_+ satisfies (3.3). Apply Chebychev's inequality, noting that $X_t^{(S,d_+,+)}$ has mean 0 and variance $tV_+(d_+)$, to get

$$\begin{aligned} P\bigl(X_t^{(S,d_+,+)} \le K_+d_+ - \kappa_+\sqrt{tV_+(d_+)}\bigr) &\ge 1 - \frac{tV_+(d_+)}{(K_+d_+ - \kappa_+\sqrt{tV_+(d_+)})^2} \\ &\ge 1 - \frac{4tV_+(d_+)}{K_+^2 d_+^2}. \end{aligned}$$

Also when $2Cd_+ > \Phi(-\kappa_+)\sqrt{tV_+(d_+)}$, by choice of K_+ in (3.3) we have

$$\frac{4tV_+(d_+)}{K_+^2 d_+^2} \le \frac{16C^2}{\Phi^2(-\kappa_+)K_+^2} \le 1 - \frac{\Phi(-\kappa_+)}{2},$$

giving

$$P\bigl(X_t^{(S,d_+,+)} \le K_+d_+ - \kappa_+\sqrt{tV_+(d_+)}\bigr) \ge \tfrac{1}{2}\Phi(-\kappa_+). \tag{3.10}$$

The same inequality holds when $2Cd_+ \le \Phi(-\kappa_+)\sqrt{tV_+(d_+)}$, by (3.9), so it holds in general.

(b) When $V_+(d_+) = 0$, $\Pi(\cdot)$ has no mass in $(0, d_+)$, and (3.4) with a "+" sign remains valid in the sense that $X_t^{(S,d_+,+)} = 0$ a.s. and the left-hand side of (3.4) equals 1. This proves (3.4) with a "+" sign, and the same argument goes through with "−" in place of "+".

(ii) We use the Itô representation in (1.7). Fix $t > 0$ and take any constants $d_\pm > 0$ satisfying (3.5). Let $\kappa_\pm > 0$ be any constants and choose $K_\pm$ to satisfy (3.3). For the small jump processes, we have the bounds in (3.4). Note that these remain true if $\overline{\Pi}^+(0+) = 0$ or $\overline{\Pi}^-(0+) = 0$. For the big positive jumps, we have

$$\begin{aligned} P\bigl(X_t^{(B,d_+,+)} = 0\bigr) &\ge P(\text{no } \Delta X_s \text{ exceeds } d_+ \text{ up till time } t) \\ &= e^{-t\overline{\Pi}^+(d_+)} \\ &\ge e^{-c_+} \qquad \bigl(\text{by } (3.5)\bigr). \end{aligned} \tag{3.11}$$

Equation (3.11) remains true with $c_+ = 0$ when $\overline{\Pi}^+(0+) = 0$. By (1.7), the probability on the left-hand side of (3.6) is, for any $L \ge 0$,

$$\begin{aligned} &P\bigl(\sigma Z_t + X_t^{(S,d_+,+)} + X_t^{(B,d_+,+)} + X_t^{(S,d_-,-)} + X_t^{(B,d_-,-)} \\ &\qquad \le K_+d_+ - Ld_- - \kappa_+\sqrt{tV_+(d_+)} - \kappa_-\sqrt{tV_-(d_-)}\bigr) \\ &\quad \ge P\bigl(Z_t \le 0, X_t^{(S,d_+,+)} \le K_+d_+ - \kappa_+\sqrt{tV_+(d_+)},\ X_t^{(B,d_+,+)} = 0, \\ &\qquad X_t^{(S,d_-,-)} \le K_-d_- - \kappa_-\sqrt{tV_-(d_-)},\ X_t^{(B,d_-,-)} \le -(K_- + L)d_-\bigr) \\ &\quad \ge e^{-c_+}\Phi(-\kappa_+)\Phi(-\kappa_-)P\bigl(X_t^{(B,d_-,-)} \le -(K_- + L)d_-\bigr)/8. \end{aligned} \tag{3.12}$$

In the last inequality, we used (3.4) (twice; once with "+" and once with "−"), (3.11) and the independence of the Z_t and the $X_t^{(\cdot)}$ processes. No jump in $X_t^{(B,d_-,-)}$ is larger than $-d_-$, so we have the upper bound $X_t^{(B,d_-,-)} \le -d_- N_t^-(d_-)$, where $N_t^-(d_-)$ is the number of jumps of X_t less than or equal in size to $-d_-$ which occur by time t. $N_t^-(d_-)$ is distributed as Poisson with expectation $t\overline{\Pi}^-(d_- -)$, and $t\overline{\Pi}^-(d_- -) \ge c_-$ by (3.5). (Note that this implies $\overline{\Pi}^-(0+) > 0$.) The Poisson distribution is stochastically monotone in the sense that if $N(\mu_1)$ and $N(\mu_2)$ are Poisson rvs with means $\mu_1 > \mu_2$, then $P(N(\mu_1) \ge x) \ge P(N(\mu_2) \ge x)$ for all $x \ge 0$. So, letting $N(c_-)$ be a Poisson rv with expectation c_-, we have

$$P\big(N_t^-(d_-) \ge K_- + L\big) \ge P\big(N(c_-) \ge K_- + L\big). \tag{3.13}$$

Then using

$$P\big(X_t^{(B,d_-,-)} \le -(K_- + L)d_-\big) \ge P\big(N_t^-(d_-) \ge K_- + L\big) \tag{3.14}$$

and (3.12) we arrive at (3.6). When $\overline{\Pi}^+(0+) = 0$, we can take all the "+" terms in (3.12) as 0 to get (3.6) with all the "+" terms 0.

(iii) Assume $0 \le \overline{\Pi}^-(0+) < \infty = \overline{\Pi}^+(0+)$. In this case, we do not define d_- but still have $d_+ = d_+(t) > 0$ and assume $t\overline{\Pi}^+(d_+) \le c_+$ as in (3.5). From (1.7), write

$$X_t = t\gamma - t\nu_+(d_+) + t\nu_-(0) + X_t^{(S,d_+,+)} + X_t^{(B,d_+,+)} + X_t^{(0,-)}, \tag{3.15}$$

where the negative jump components have been amalgamated into

$$X_t^{(0,-)} := \sum_{0<s\le t} \Delta X_s 1_{\{\Delta X_s \le 0\}}, \qquad t > 0,$$

which is a compound Poisson process comprised of non-positive jumps. This term and the term $t\nu_-(0)$ are absent from (3.15) when $\overline{\Pi}^-(0+) = 0$. Using (3.4), (3.11) and (3.15), write

$$\begin{aligned}
&P\big(X_t \le t\gamma - t\nu_+(d_+) + t\nu_-(0+) + K_+ d_+ - \kappa_+\sqrt{tV_+(d_+)}\big)\\
&\quad \ge P\big(Z_t \le 0, X_t^{(S,d_+,+)} \le K_+ d_+ - \kappa_+\sqrt{tV_+(d_+)}, X_t^{(B,d_+,+)} = 0, X_t^{(0,-)} \le 0\big)\\
&\quad \ge e^{-c_+}\Phi(-\kappa_+)P\big(X_t^{(0,-)} \le 0\big)/4 = e^{-c_+}\Phi(-\kappa_+)/4
\end{aligned}$$

and this gives (3.7). □

4. Proof of Theorem 2.1

Part (i). Assume $\overline{\Pi}^-(0+) > 0$ throughout this part.

(2.3) $\Longrightarrow$ (2.1): Assume (2.3). $\overline{\Pi}^-(0+) > 0$ implies $\overline{\Pi}^-(x) > 0$ in a neighbourhood of 0 so we can assume $\overline{\Pi}^-(x) > 0$ for all $0 < x < 1$. Choose $1 > x_k \downarrow 0$ such that

$$\frac{A(x_k)}{\sqrt{U(x_k)\overline{\Pi}^-(x_k)}} \to \infty$$

as $k \to \infty$. This implies $\sigma^2 = 0$ by (1.12) (because $U(x) \ge \sigma^2$). It also means that $A(x_k) > 0$ for all large k, and without loss of generality we may assume it to be so for all k. Let

$$s_k := \sqrt{\frac{U(x_k)}{\overline{\Pi}^-(x_k) A^2(x_k)}},$$

then

$$s_k \overline{\Pi}^-(x_k) = \frac{\sqrt{U(x_k)\overline{\Pi}^-(x_k)}}{A(x_k)} \to 0$$

and since $\overline{\Pi}^-(0+) > 0$, also $s_k \to 0$ as $k \to \infty$. In addition, we have

$$\frac{U(x_k)}{s_k A^2(x_k)} = \frac{\sqrt{U(x_k)\overline{\Pi}^-(x_k)}}{A(x_k)} \to 0$$

and

$$\frac{s_k A(x_k)}{x_k} = \sqrt{\frac{U(x_k)}{x_k^2 \overline{\Pi}^-(x_k)}} \ge 1.$$

Set

$$t_k := \sqrt{\frac{s_k}{\overline{\Pi}^-(x_k)}},$$

so $t_k / s_k \to \infty$, but still $t_k \overline{\Pi}^-(x_k) \to 0$, as $k \to \infty$. Then

$$\frac{U(x_k)}{t_k A^2(x_k)} = \frac{s_k}{t_k} \frac{U(x_k)}{s_k A^2(x_k)} \to 0, \tag{4.1}$$

and

$$\frac{t_k A(x_k)}{x_k} = \frac{t_k}{s_k} \frac{s_k A(x_k)}{x_k} \to \infty, \tag{4.2}$$

as $k \to \infty$.

Recall (1.6) and use the Itô decomposition in (1.7) with $\sigma^2 = 0$ and $h_+ = h_- = h > 0$ to write

$$X_t = tA(h) + X_t^{(S,h)} + X_t^{(B,h,+)} - th\overline{\Pi}^+(h) + X_t^{(B,h,-)} + th\overline{\Pi}^-(h), \qquad t > 0. \tag{4.3}$$

Here, $X_t^{(S,h)} = X_t^{(S,h,+)} + X_t^{(S,h,-)}$ is the compensated small jump process, and $X_t^{(B,h,\pm)}$ are the positive and negative big jump processes.

Case (a): Suppose $\overline{\Pi}^+(0+) > 0$. Since each jump in $X_t^{(B,h,+)}$ is at least h, we have the lower bound $X_t^{(B,h,+)} \ge hN_t^+(h)$, where $N_t^+(h)$ is Poisson with expectation $t\overline{\Pi}^+(h)$ (and variance

$t\overline{\Pi}^{+}(h)$). Using this and substituting in (4.3) with $t = t_k$ and $h = x_k$ we get

$$X_{t_k} \geq t_k A(x_k) + X_{t_k}^{(S,x_k)} + x_k\big(N_{t_k}^{+}(x_k) - t_k\overline{\Pi}^{+}(x_k)\big) + X_{t_k}^{(B,x_k,-)}. \tag{4.4}$$

Since $t_k\overline{\Pi}^{-}(x_k) \to 0$, we have $P(X_{t_k}^{(B,x_k,-)} = 0) \to 1$ as $k \to \infty$. Also, for $\varepsilon \in (0,1)$,

$$\begin{aligned} P\big(X_{t_k}^{(S,x_k)} + x_k\big(N_{t_k}^{+}(x_k) - t_k\overline{\Pi}^{+}(x_k)\big) \leq -\varepsilon t_k A(x_k)\big) &\leq \frac{t_k V(x_k) + t_k x_k^2\overline{\Pi}^{+}(x_k)}{\varepsilon^2 t_k^2 A^2(x_k)} \\ &\leq \frac{U(x_k)}{\varepsilon^2 t_k A^2(x_k)} \to 0, \end{aligned}$$

as $k \to \infty$ by (4.1), so

$$P\left(\frac{X_{t_k}}{x_k} \geq (1-\varepsilon)\frac{t_k A(x_k)}{x_k}\right) \to 1, \tag{4.5}$$

and hence, by (4.2), $X_{t_k}/x_k \xrightarrow{\mathrm{P}} \infty$ as $k \to \infty$. Thus, (2.1) holds.

Case (b): Alternatively, if $\overline{\Pi}^{+}(0+) = 0$, we can omit the term containing $N_{t_k}^{+}(x_k) - t_k\overline{\Pi}^{+}(x_k)$ in (4.4) and in what follows it, and again obtain (4.5), and hence (2.1).[1]

(2.3) $\Longrightarrow$ (2.2): Continuing the previous argument, $t_k\overline{\Pi}^{-}(x_k) \to 0$ implies

$$P\big((\Delta X^{-})_{t_k}^{(1)} > x_k\big) = P\Big(\sup_{0<s\leq t_k} \Delta X_s^{-} > x_k\Big) = 1 - e^{-t_k\overline{\Pi}^{-}(x_k)} \to 0 \qquad \text{as } k \to \infty,$$

so, using (4.5), (2.2) also holds when (2.3) holds and $\overline{\Pi}^{-}(0+) > 0$.

(2.2) $\Longrightarrow$ (2.1): This is obvious when $\overline{\Pi}^{-}(0+) > 0$.

(2.1) $\Longrightarrow$ (2.3): Assume $\Pi(\mathbb{R}) = \infty$ as well as $\overline{\Pi}^{-}(0+) > 0$, and that (2.1) holds. Suppose (2.3) fails, so we can choose $1 < a < \infty$, $x_0 > 0$, such that

$$A(x) \leq a\sqrt{U(x)\overline{\Pi}^{-}(x)}, \tag{4.6}$$

for all $0 < x \leq x_0$. We will obtain a contradiction. Note that (2.1) implies $\sigma^2 = 0$, because $X_t/\sqrt{t} \xrightarrow{\mathrm{D}} N(0,\sigma^2)$, a non-degenerate normal rv, when $\sigma^2 > 0$. So we assume $\sigma^2 = 0$ in what follows. We consider 3 cases.

Case (a): Assume in fact that $\overline{\Pi}^{-}(0+) = \infty = \overline{\Pi}^{+}(0+)$. In this situation, we can introduce quantile versions for the $d_{\pm}$ in (3.5). Define the non-decreasing function

$$d_{+}(t) := \inf\big\{x > 0 : \overline{\Pi}^{+}(x) \leq t^{-1}\big\}, \qquad t > 0, \tag{4.7}$$

[1] Observe that the assumption $\Pi(\mathbb{R}) = \infty$ was not used in this part of the proof. The trivial case, $X_t = t\gamma$, $\gamma > 0$, when $A(x) \equiv \gamma$, is included if we interpret (2.3) as holding then.

and set $d_+(0) = 0$. Since $\overline{\Pi}^+(0+) = \infty$, we have $0 < d_+(t) < \infty$ for all $t > 0$, $d_+(t) \downarrow 0$ as $t \downarrow 0$, and

$$t\overline{\Pi}^+\big(d_+(t)\big) \le 1 \le t\overline{\Pi}^+\big(d_+(t)-\big) \qquad \text{for all } t > 0. \tag{4.8}$$

Analogously, define $d_-(0) = 0$, and

$$d_-(t) := \inf\{x > 0 : \overline{\Pi}^-(x) \le t^{-1}\}, \qquad t > 0, \tag{4.9}$$

having, since $\overline{\Pi}^-(0+) = \infty$, $0 < d_-(t) < \infty$, $d_-(t) \downarrow 0$ as $t \downarrow 0$, and

$$t\overline{\Pi}^-\big(d_-(t)\big) \le 1 \le t\overline{\Pi}^-\big(d_-(t)-\big). \tag{4.10}$$

With a as in (4.6), set $\kappa_+ = \kappa_- = \kappa = 2a$, then choose $K_\pm$ to satisfy (3.3). Then (2.1) together with (3.6) shows that we must have

$$0 \le t_k\big(\gamma - \nu_+(d_+) + \nu_-(d_-)\big) + K_+d_+ - Ld_- - \kappa\sqrt{t_k\big(V_+(d_+) + V_-(d_-)\big)}, \tag{4.11}$$

for all large k. Here, d_+ and d_- are any positive numbers and we used the inequality $\sqrt{a} + \sqrt{b} \ge \sqrt{a+b}$, $a, b > 0$, in (3.6). Take $\lambda > 0$ and set

$$d_+ = d_+(\lambda t_k) \quad \text{and} \quad d_- = d_-(t_k),$$

where $d_+(\cdot)$ and $d_-(\cdot)$ are defined in (4.7) and (4.9). By (4.8) and (4.10), we then have

$$t\overline{\Pi}^+\big(d_+(\lambda t)\big) \le \lambda^{-1} \quad \text{and} \quad t\overline{\Pi}^-\big(d_-(t)-\big) \ge 1,$$

so (3.5) holds with $c_+ = \lambda^{-1}$ and $c_- = 1$. With t_k as the sequence in (2.1), let $d = d(t_k) := \max(d_+(\lambda t_k), d_-(t_k))$.

Equation (4.11) implies

$$\begin{aligned} &t_k\bigg(\gamma - \int_{(d_+,1]} y\Pi(\mathrm{d}y) + \int_{(d_-,1]} y\Pi^{(-)}(\mathrm{d}y)\bigg) \\ &\quad \ge -K_+d_+ + Ld_- + \kappa\sqrt{t_k\big(V_+(d_+) + V_-(d_-)\big)}. \end{aligned} \tag{4.12}$$

Adding the quantity

$$t_k\bigg(\int_{(d_+,d]} y\Pi(\mathrm{d}y) - \int_{(d_-,d]} y\Pi^{(-)}(\mathrm{d}y)\bigg)$$

to both sides of (4.12) gives $t_k\nu(d)$ on the left, and a quantity no smaller than

$$t_kd_+\big(\overline{\Pi}^+(d_+) - \overline{\Pi}^+(d)\big) - t_kd\big(\overline{\Pi}^-(d_-) - \overline{\Pi}^-(d)\big) - K_+d_+ + Ld_- + \kappa\sqrt{t_k\big(V_+(d_+) + V_-(d_-)\big)}$$

on the right. Further adding $t_k d(\overline{\Pi}^+(d) - \overline{\Pi}^-(d))$ to both sides gives $t_k A(d)$ on the left (see (1.6)), and then after some cancellation we arrive at

$$t_k A(d) \ge t_k d_+ \overline{\Pi}^+(d_+) - t_k d \overline{\Pi}^-(d_-) - K_+ d_+ + L d_- + \kappa \sqrt{t_k \big(V_+(d_+) + V_-(d_-)\big)}. \quad (4.13)$$

At this stage, it is helpful to assume that $\overline{\Pi}^+(x)$ is a continuous function on $(0, \infty)$. It then follows from (4.8) that $t_k \overline{\Pi}^+(d_+(\lambda t_k)) = 1/\lambda$, while $t_k \overline{\Pi}^-(d_-(t_k)) \le 1$ by (4.10). Also, $d \le d_+ + d_-$. Thus, we deduce

$$t_k A(d) \ge (1/\lambda - K_+ - 1) d_+ + (L - 1) d_- + \kappa \sqrt{t_k \big(V_+(d_+) + V_-(d_-)\big)}. \quad (4.14)$$

Next, write

$$\begin{aligned} V_+(d_+) + V_-(d_-) &= V_+(d) - \int_{(d_+, d]} y^2 \Pi(\mathrm{d}y) + V_-(d) - \int_{(d_-, d]} y^2 \Pi^{(-)}(\mathrm{d}y) \\ &\ge V(d) - d^2\big(\overline{\Pi}^+(d_+) - \overline{\Pi}^+(d)\big) - d^2\big(\overline{\Pi}^-(d_-) - \overline{\Pi}^-(d)\big) \quad (4.15) \\ &= U(d) - d^2\big(\overline{\Pi}^+(d_+) + \overline{\Pi}^-(d_-)\big). \end{aligned}$$

So

$$t_k\big(V_+(d_+) + V_-(d_-)\big) \ge t_k U(d) - d^2(1/\lambda + 1)$$

giving

$$\sqrt{t_k\big(V_+(d_+) + V_-(d_-)\big)} \ge \sqrt{t_k U(d)} - d(1/\sqrt{\lambda} + 1).$$

Substituting into (4.14), we obtain

$$\begin{aligned} t_k A(d) \ge \kappa \sqrt{t_k U(d)} &+ (1/\lambda - K_+ - 1 - \kappa/\sqrt{\lambda} - \kappa) d_+ \\ &+ (L - 1 - \kappa/\sqrt{\lambda} - \kappa) d_-. \end{aligned}$$

Choose λ small enough for the first expression in brackets on the right-hand side to be positive. Then choose L large enough for the second expression in brackets on the right-hand side to be positive. This gives

$$t_k A(d) \ge \kappa \sqrt{t_k U(d)}, \quad (4.16)$$

for all large k, where $d = d(t_k) = \max(d_+(\lambda t_k), d_-(t_k)) \downarrow 0$ as $k \to \infty$. Inequality (4.16) implies

$$\frac{A(d)}{\sqrt{U(d)\overline{\Pi}^-(d)}} \ge \frac{\kappa}{\sqrt{t_k \overline{\Pi}^-(d)}} \ge \frac{\kappa}{\sqrt{t_k \overline{\Pi}^-(d_-)}} \ge \kappa, \quad (4.17)$$

giving a contradiction with (4.6), since $\kappa = 2a$.

This proves (2.3) from (2.1) in case $\overline{\Pi}^+(0+) = \overline{\Pi}^-(0+) = \infty$ and $\overline{\Pi}^+(x)$ is continuous for $x > 0$. To complete the proof of part (i), case (a), of the theorem we remove the assumption of continuity made in deriving (4.14). This can be done using the following lemma.

Lemma 4.1. *Let* Π *be any Lévy measure with* $\overline{\Pi}^+(0+) = \infty$. *Then there exists a sequence of Lévy measures* Π_n, *absolutely continuous with respect to Lebesgue measure, and having strictly positive* C^∞*-densities on* $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$, *satisfying* $\overline{\Pi}_n^+(0+) = \infty$ *and* $\Pi_n \xrightarrow{v} \Pi$ *as* $n \to \infty$.

Proof. ($\xrightarrow{v}$ refers to vague convergence in $\overline{\mathbb{R}}_*$; see, for example, Chapter 15 in Kallenberg [7].) We extend Π to a Borel measure on $\mathbb{R}$ by setting $\Pi(\{0\}) := 0$. Assume $\Pi \neq 0$, so $C := \int x^2\Pi(\mathrm{d}x)/(1+x^2) \in (0,\infty)$. Observe that $P(\mathrm{d}x) := x^2\Pi(\mathrm{d}x)/C(1+x^2)$ defines a Borel probability measure on $\mathbb{R}$. For all $n \in \mathbb{N}$, the convolved probability measure $P_n := P \star N(0,1/n)$ admits a strictly positive C^∞-Lebesgue density, when $N(0,1/n)$ is a normal rv with expectation 0 and variance $1/n$. Set $\Pi_n(\mathrm{d}x) := C(1+x^2)P_n(\mathrm{d}x)/x^2$, $n = 1, 2, \ldots$. It is easily verified that $(\Pi_n)_{n\in\mathbb{N}}$ is a sequence of Lévy measures with the desired properties. $\square$

Now to complete the proof of part (i), assume (2.1), that is, $P(X_{t_k} \geq 0) \to 1$ for some $t_k \downarrow 0$, for a general X with Lévy measure Π. Using Lemma 4.1, construct a sequence of approximating Lévy measures Π_n, converging vaguely to Π, such that their positive tails $\overline{\Pi}_n^+$ are continuous on $(0,\infty)$ with $\overline{\Pi}_n^+(0+) = \infty$. On the negative side, let $\overline{\Pi}_n^-(x) = \overline{\Pi}^-(x)$, $x > 0$. Let $(X_t(n))_{t\geq 0}$ be Lévy processes with measures Π_n and other characteristics the same as for X. Define ν_n, A_n, V_n, U_n, $\nu_{n,\pm}$, $V_{n,\pm}$, as in (1.4), (1.5), (1.8) and (3.2), but with Π_n replacing Π. The subscript n functions converge to the original functions at points of continuity of the latter. $X_t(n)$ has characteristic exponent given by (1.1) with Π_n replacing Π, so as $n \to \infty$ we have $X_t(n) \xrightarrow{\mathrm{D}} X_t$ for each $t > 0$. Under assumption (2.1), $\lim_{n\to\infty} P(X_{t_k}(n) > 0) = P(X_{t_k} > 0) > 1 - \delta$ for arbitrary $\delta \in (0, 1/2)$ and k large enough. Thus, $P(X_{t_k}(n) > 0) > 1 - 2\delta$ for $n \geq n_0(k)$ and $k \geq k_0$. So (4.11) holds for the subscript n quantities with probability $1 - 2\delta$. But (4.11) is deterministic so it holds in fact for the subscript n quantities (with probability 1) whenever $n \geq n_0(k)$ and $k \geq k_0$. The proof using continuity of Π_n then shows that (4.17) holds with A, U and $\overline{\Pi}^-$ replaced by A_n, U_n and $\overline{\Pi}_n^-$. Then letting $n \to \infty$ shows that (4.17) itself holds as stated for $k \geq k_0$. Again we get a contradiction, and thus complete the proof that (2.1) implies (2.3) for case (a).

Case (b): Assume that $0 \leq \overline{\Pi}^+(0+) < \infty = \overline{\Pi}^-(0+)$. As in the proof for case (a), we take $\kappa_+ = \kappa_- = \kappa > 2a$, $K_\pm$ to satisfy (3.3), define $d_-(t) > 0$ by (4.9), and write $d_- = d_-(t)$ for $t > 0$. But for d_+ we set $d_+(t) \equiv d_-(t) > 0$. We take $c_+ = 1$ in (3.5) as we may since $t\overline{\Pi}^+(d_+) = t\overline{\Pi}^+(d_-) \leq t\overline{\Pi}^+(0+) \to 0$ as $t \downarrow 0$. With this set-up, (4.11) is true (with d_- replacing d_+) and we can follow the proof of case (a) through to get (4.13) with d and d_+ replaced by d_-; thus,

$$t_k A(d_-) \geq t_k d_-\bigl(\overline{\Pi}^+(d_-) - \overline{\Pi}^-(d_-)\bigr) - K_+d_- + Ld_- + \kappa\sqrt{t_k\bigl(V_+(d_-) + V_-(d_-)\bigr)}. \quad (4.18)$$

Estimating $V_\pm$ along the lines of (4.15) we find the right-hand side of (4.18) is not smaller than

$$\kappa\sqrt{t_k U(d_-)} + (L - K_+ - 1 - \kappa)d_-.$$

Choose L large enough in this to get (4.16), and hence (4.17) with d_- in place of d, hence (2.3) again.

Case (c): Assume that $0 < \overline{\Pi}^-(0+) < \infty = \overline{\Pi}^+(0+)$. Define $d_+(t)$ by (4.7), so we have (4.8). Then (3.7) with $c_+ = 1$ and $\kappa_+ = \kappa = 2a$, together with (2.1), shows that we must have

$$0 \le t_k\big(\gamma - \nu_+(d_+) + \nu_-(0)\big) + K_+d_+ - \kappa\sqrt{t_k V_+(d_+)}, \tag{4.19}$$

for all large k. Here again we write $d_+ = d_+(\lambda t_k)$ for $\lambda > 0$. Inequality (4.19) implies

$$t_k\left(\gamma - \int_{(d_+,1]} y\Pi(\mathrm{d}y) + \int_{(0,1]} y\Pi^{(-)}(\mathrm{d}y)\right) \ge -K_+d_+ + \kappa\sqrt{t_k V_+(d_+)}. \tag{4.20}$$

Subtracting the quantity

$$t_k \int_{(0,d_+]} y\Pi^{(-)}(\mathrm{d}y) \le t_k d_+\big(\overline{\Pi}^-(0+) - \overline{\Pi}^-(d_+)\big)$$

from both sides of (4.20) gives $t_k\nu(d_+)$ on the left, and a quantity no smaller than

$$-t_k d_+\big(\overline{\Pi}^-(0+) - \overline{\Pi}^-(d_+)\big) - K_+d_+ + \kappa\sqrt{t_k V_+(d_+)}$$

on the right. Further adding $t_k d_+(\overline{\Pi}^+(d_+) - \overline{\Pi}^-(d_+))$ to both sides gives (see (1.6))

$$t_k A(d_+) \ge t_k d_+\overline{\Pi}^+(d_+) - t_k d_+\overline{\Pi}^-(0+) - K_+d_+ + \kappa\sqrt{t_k V_+(d_+)}. \tag{4.21}$$

At this stage, as before, assume $\overline{\Pi}^+(x)$ is continuous. It then follows from (4.8) that $t_k\overline{\Pi}^+(d_+(\lambda t_k)) = 1/\lambda$, while $t_k\overline{\Pi}^-(0+) \le 1$ for large k. Thus, from (4.21) we deduce

$$t_k A(d_+) \ge (1/\lambda - K_+ - 1)d_+ + \kappa\sqrt{t_k V_+(d_+)}, \tag{4.22}$$

for large enough k. Further,

$$\begin{aligned} t_k V_+(d_+) &= t_k\big(U(d_+) - V_-(d_+) - d_+^2\overline{\Pi}^+(d_+) - d_+^2\overline{\Pi}^-(d_+)\big) \\ &\ge t_k U(d_+) - t_k d_+^2\big(2\overline{\Pi}^-(0+) + \overline{\Pi}^+(d_+)\big) \\ &\ge t_k U(d_+) - 4d_+^2, \end{aligned}$$

using that $V_-(d_+) \le d_+^2\overline{\Pi}^-(0+)$. So

$$\sqrt{t_k V_+(d_+)} \ge \sqrt{t_k U(d_+)} - 2d_+.$$

Substituting into (4.22), we get

$$t_k A(d_+) \ge \kappa\sqrt{t_k U(d_+)} + (1/\lambda - K_+ - 1 - 2\kappa)d_+,$$

for large k. We can choose λ small enough for the expression in brackets on the right-hand side to be positive. This gives

$$t_k A(d_+) \geq \kappa\sqrt{t_k U(d_+)}, \tag{4.23}$$

which, since $\overline{\Pi}^-(0+) > 0$ is assumed, implies

$$\frac{A(d_+)}{\sqrt{U(d_+)\overline{\Pi}^-(d_+)}} \geq \frac{\kappa}{\sqrt{t_k\overline{\Pi}^-(0+)}} \to \infty \qquad \text{as } k \to \infty,$$

a contradiction with (4.6). We can remove the continuity assumption as before. So (2.3) is proved when $0 < \overline{\Pi}^-(0+) < \infty = \overline{\Pi}^+(0+)$.

Part (ii). Now we will deal with the case when $\overline{\Pi}^-(0+) = 0$ but $\overline{\Pi}^+(0+) = \infty$. Again assume $\overline{\Pi}^+(x)$ is continuous. The working in case (c) is still valid from (4.19) to (4.23). The negative jump process is now absent from X_t and (4.19) gives, with $d_+ = d_+(\lambda t)$,

$$\begin{aligned}
0 &\leq t_k\bigl(\gamma - \nu_+(d_+)\bigr) + K_+ d_+ \\
&= t_k\left(\gamma - \int_{d_+}^1 \overline{\Pi}^+(y)\,\mathrm{d}y - d_+\overline{\Pi}^+(d_+) + \overline{\Pi}^+(1)\right) + K_+ d_+ \\
&\leq t_k\left(\gamma - \int_{d_+}^1 \overline{\Pi}^+(y)\,\mathrm{d}y + \overline{\Pi}^+(1)\right) - (1/\lambda - K_+)d_+,
\end{aligned} \tag{4.24}$$

using $t_k\overline{\Pi}^+(d_+(\lambda t_k)) = 1/\lambda$ in the last inequality (since $\overline{\Pi}^+(x)$ is continuous). Then choosing $\lambda < 1/K_+$ we get

$$\int_{d_+}^1 \overline{\Pi}^+(y)\,\mathrm{d}y \leq \gamma + \overline{\Pi}^+(1).$$

Letting $k \to \infty$ (so $t_k \downarrow 0$ and $d_+ = d_+(\lambda t_k) \downarrow 0$) shows that $\int_0^1 \overline{\Pi}^+(y)\,\mathrm{d}y < \infty$. Since X_t has no negative jumps, from this we deduce that X is of bounded variation with drift $\mathrm{d}_X = A(0+)$ (see Doney and Maller [6], Theorem 2.1 and Remark 1), which is non-negative by (4.23). Thus, X is a subordinator with non-negative drift. It follows that

$$A(x) = \gamma + \overline{\Pi}^+(1) - \int_x^1 \overline{\Pi}^+(y)\,\mathrm{d}y = \mathrm{d}_X + \int_0^x \overline{\Pi}^+(y)\,\mathrm{d}y$$

is non-negative for all $x > 0$. This is proved assuming continuity of Π_n but that assumption can be removed as before. Then $A(x) \geq 0$ together with $\sigma^2 = 0$ implies $\lim_{t\downarrow 0} P(X_t > 0) \to 1$ as $t \downarrow 0$ by Theorem 1.1, hence (2.1).

Part (iii). Finally, suppose $\overline{\Pi}^-(0+) > 0$ and (2.4) holds. Then we can choose $x_k \downarrow 0$ such that

$$\frac{A(x_k)}{\sqrt{U(x_k)\overline{\Pi}^-(x_k)}} \to \infty \quad \text{and} \quad A(x_k) \to \infty,$$

as $k \to \infty$. Following exactly the proof of part (i), we get (4.5), and this implies $X_{t_k}/t_k \xrightarrow{\mathrm{P}} \infty$, since $A(x_k) \to \infty$.

Conversely, suppose $X_{t_k}/t_k \xrightarrow{\mathrm{P}} \infty$ as $k \to \infty$ for a non-stochastic sequence $t_k \downarrow 0$. Then $\lim_{k\to\infty} P(X_{t_k}^M > 0) = 1$ for every $M > 0$, where X_t^M is Lévy with triplet $(\gamma - M, \sigma^2, \Pi)$. Consequently, (2.3) holds with A, U, Π replaced by $A^M(\cdot) = A(\cdot) - M$, $U^M = U$, $\Pi^M = \Pi$, and this modified version implies (2.4). This completes part (iii), and the proof of the theorem.

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