The distribution of the maximal difference between a Brownian bridge and its concave majorant

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We provide a representation of the maximal difference between a standard Brownian bridge and its concave majorant on the unit interval, from which we deduce expressions for the distribution and density functions and moments of this difference. This maximal difference has an application in nonparametric statistics, where it arises in testing monotonicity of a density or regression curve.

Keywords: Brownian bridge; Brownian excursion; concave majorant; Doob's transformation; inverse of Laplace transform; stick-breaking process

1. Introduction

Motivated by applications to the theory of nonparametric statistics, described later in this section, we provide a useful representation of the maximal difference

$$M := \sup_{u\in[0,1]} \left(\left[\overline{C}_{[0,1]} B \right](u) - B(u) \right),$$

where $(B(u), 0 \le u \le 1)$ is a Brownian motion and $\overline{C}_I f$ denotes the (least) concave majorant of a function f defined on an interval I; see Figure 1.

Our representation of M is presented in the following theorem, in terms of the distribution of M_3, the maximum of a standard Brownian excursion, which can also be represented as the maximum of a three-dimensional Bessel bridge, or as

$$M_3 \stackrel{d}{=} \sup_{u\in[0,1]} B^{\mathrm{br}}(u) - \inf_{u\in[0,1]} B^{\mathrm{br}}(u), \tag{1}$$

where B^{br} denotes the *standard Brownian bridge* obtained by conditioning B on $B(1) = 0$; see [5,13] for background and further information on the distribution of M_3, which has been extensively studied.
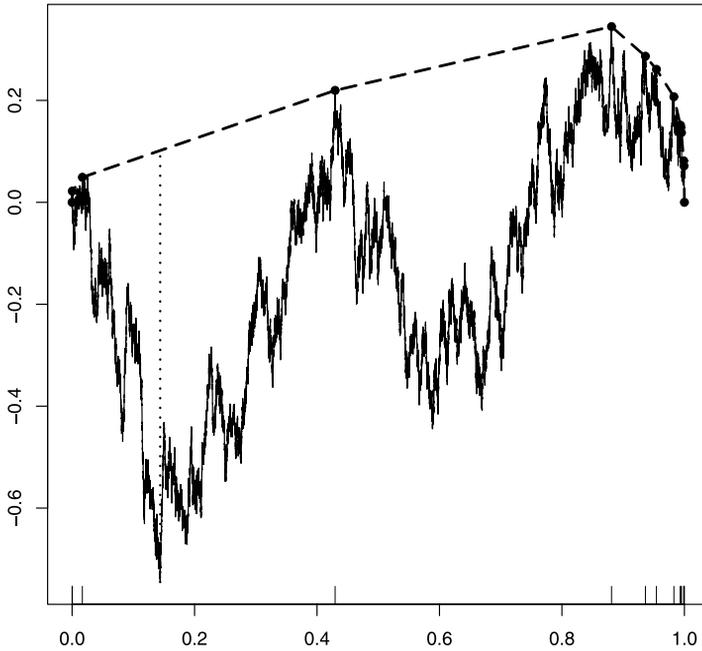
Figure 1. Plot of a Brownian bridge and its concave majorant. The bullets depict the vertices, whose times are indicated by the tick marks on the x-axis. The length of the dotted vertical segment gives the maximum difference between the Brownian bridge and the concave majorant. The Haar approximation was used to generate the Brownian bridge on a discrete partition of $[0, 1]$ with a mesh equal to 2^{-15}.

Theorem 1. *The distribution of the maximal difference M between a Brownian motion B and its concave majorant is determined by the identity in distribution*

$$M \stackrel{d}{=} \max_{j} \sqrt{L_j} M_{3,j}, \tag{2}$$

where

- $(L_j, j = 1, 2, \ldots)$ *is the* uniform stick-breaking process

$$L_1 := W_1, \qquad L_2 := (1 - W_1)W_2, \qquad L_3 := (1 - W_1)(1 - W_2)W_3, \ldots$$

  derived from a sequence $W_1, W_2, \ldots$ of independent uniform $(0, 1)$ variables;
- $(M_{3,j}, j = 1, 2, \ldots)$ *is a sequence of independent random variables, each with the distribution of M_3;*
- *the two sequences (W_j) and $(M_{3,j})$ are independent.*

Moreover, M is independent of $B(1)$, so the identity in distribution (2) also holds if the Brownian motion B on $[0, 1]$ is replaced by the Brownian bridge obtained by conditioning B on $B(1) = b$, for arbitrary real b.

To be more explicit, following Groeneboom [10], we observe that the vertices of the concave majorant $([\overline{C}_{[0,1]}B](u), 0 \leq u \leq 1)$ partition $[0, 1]$ into a countable collection of subintervals, with accumulation of vertices at both 0 and 1, but with only a finite number of vertices almost surely in $(\epsilon, 1 - \epsilon)$ for each $\varepsilon > 0$. The sequence (L_j) represents the lengths of these maximal subintervals, over each of which the concave majorant has a segment with a particular slope. These L_j are arranged in a suitable random order, while $\sqrt{L_j}M_{3,j}$ represents the maximum value of $[\overline{C}_{[0,1]}B](u) - B(u)$ for u in the interval of length L_j. Here, and throughout the paper, the *length* of a segment of a concave majorant refers to the length of the time interval associated with the segment, rather than the length of the segment in a two-dimensional picture such as Figure 1.

For the proof of Theorem 1, we combine two different ingredients:

- Groeneboom's description [10] of the concave majorant of Brownian motion on the infinite interval $[0, \infty)$;
- Suidan's description [24] of the joint law of ranked lengths of intervals in the partition of the time interval $[0, 1]$ generated by vertices of $\overline{C}_{[0,1]}B$.

In principle, the second of these ingredients must be derivable from the first. Following Groeneboom, we use Doob's transformation to map Brownian motion on $[0, \infty)$ to a Brownian bridge on $[0, 1]$ and this mapping determines the law of the interval partition of $[0, 1]$ derived from $\overline{C}_{[0,1]}B^{\mathrm{br}}$. While we provide some details of this in Section 4, we are unable to fully derive the stick-breaking representation of interval lengths in this way. Still, by developing the results of Groeneboom and Suidan and by exploiting the fact that a uniform stick-breaking process is invariant under a size-biased random permutation (see [11,18,19]), we see that the L_j appearing in Theorem 1 can be constructed as follows.

Corollary 2. *Let $U_1, U_2, \ldots$ be a sequence of independent uniform $(0, 1)$ variables, independent of the Brownian motion B, and let $L_1, L_2, \ldots$ be the size-biased random permutation of lengths of segments of $\overline{C}_{[0,1]}B$ defined by*:

- *L_1 is the length of the interval containing U_1;*
- *L_2 is the length of the interval containing the first point U_i that is not in this interval;*
- *L_3 is the length of the interval containing the next point U_j that is not in either of the first two intervals,*

and so on. Then

$$L_1, \qquad L_2/(1 - L_1), \qquad L_3/(1 - L_1 - L_2), \ldots$$

is a sequence of independent uniform $(0, 1)$ random variables and this sequence is independent of $B(1)$.

In particular,

- the length L_1 of the segment of $\overline{C}_{[0,1]}B^{\mathrm{br}}$ covering a uniform random number U_1 is itself uniformly distributed in $[0, 1]$.

We show in the Appendix how this fact can be verified from Groeneboom's [10] joint density for the location of vertices of the concave majorant of Brownian motion on $[0, \infty)$. However, the computations are difficult and we are currently unable to extend this method to obtain the assertions about L_j for $j \geq 2$ from Groeneboom's results. Rather, we rely completely on Suidan's approach for this part of the argument, which is essential for our proof of Theorem 1.

To motivate our study of the distribution of M, we recall that in testing whether a density or a regression function on $[0, 1]$ is decreasing, the supremum distance between the empirical estimator of the function and its concave majorant is used as a test statistic; see Kulikov and Lopuhaä [14] and Durot [8]. From [15], it follows that the supremum distance between the empirical distribution function and its concave majorant attains its maximum at the uniform density on $[0, 1]$. Moreover, it is known that this statistic, when multiplied by $\sqrt{n}$, converges in distribution to the maximal difference M between a Brownian motion B on $[0, 1]$ and its concave majorant. For the regression problem, the distribution of M again appears as the limit of a similar scaling of the supremum distance between the cumulative sum diagram and its concave majorant if the true decreasing regression is constant, which is also known to be the least favorable regression function for the testing problem in question [8]. Durot [8] established continuity of the distribution of M. It is an immediate corollary of Theorem 1 that M, in fact, has a density and we provide formulas for this density and for moments of M in Section 3. Additionally, we give an alternative characterization of M, based on the inverse of the Laplace transform of a function involving modified Bessel functions of the second kind. This other characterization suggests a way of calculating the quantiles of M at any desired precision using some appropriate approximation method for the inverse of a Laplace transform, but we do not pursue this here.

2. The concave majorant of Brownian motion on a finite interval

We need to show that, conditionally given $\overline{C}_{[0,1]}B$, the difference process $\overline{C}_{[0,1]}B - B$ behaves like a succession of independent Brownian excursions between the vertex times of $\overline{C}_{[0,1]}B$. Groeneboom [10] proved a corresponding result for a Brownian motion on $[0, \infty)$, stated as Theorem 3 below. From this result on $[0, \infty)$, we will prove a similar theorem on $[0, 1]$ by using a space–time transformation.

For a fixed $a > 0$, let

$$Z_a := \max_{t \geq 0}\{B(t) - at\}$$

and D_a be the time at which the maximum is attained; see also [7]. The point $(D_a, Z_a + aD_a)$ is one vertex of $\overline{C}_{[0,\infty)}B$ and it follows from [10], Theorem 2.1, that $(D_{1/a}, a > 0)$ is a pure jump process. Let $S_0 > S_1 > S_2 > \cdots$ denote the successive slopes of $\overline{C}_{[0,\infty)}B$ to the right of D_a and $S_{-1} < S_{-2} < \cdots$ the successive slopes of $\overline{C}_{[0,\infty)}B$ to the left of D_a, so $S_0 < a < S_{-1}$ almost surely. For $i \in \mathbb{Z}$, let T_i denote the length of the interval on which the slope of the concave majorant is S_i so that $\{V_i := \sum_{j \leq i} T_j, i \in \mathbb{Z}\}$ is the sequence of times of vertices of $\overline{C}_{[0,\infty)}B$.

Theorem 3 (Groeneboom [10]). *Conditionally given $\overline{C}_{[0,\infty)}B$, the difference process $(\overline{C}_{[0,\infty)}B(t) - B(t), t \geq 0)$ is a succession of independent Brownian excursions of prescribed lengths $T_i = V_i - V_{i-1}, i \in \mathbb{Z}$, between zeros at the times $V_i, i \in \mathbb{Z}$.*

See [4,10,17] for various proofs of this result.

It is important now to distinguish clearly the restriction to [0, 1] of $\overline{C}_{[0,\infty)}B$, the concave majorant of a Brownian motion B on $[0,\infty)$, and $\overline{C}_{[0,1]}B$, the concave majorant of a Brownian motion B on [0, 1]. The former concave majorant has vertices accumulating only at 0, whereas the latter, which is the subject of the following proposition, has vertices accumulating at both 0 and 1. These two concave majorant processes agree on some random interval [0, R] with $0 < R < 1$ almost surely and then differ on $(R, 1]$, where R is the time of the last vertex of $\overline{C}_{[0,\infty)}B$ before time 1, whose distribution is determined by formula (16); see also the remark of Groeneboom [10], page 1022.

Proposition 4. *Let B be a Brownian motion on [0, 1] and $\overline{C}_{[0,1]}B$ its concave majorant.*

  (i) *The random set of times of vertices of $\overline{C}_{[0,1]}B$ is independent of $B(1)$.*
  (ii) *The difference process $(\overline{C}_{[0,1]}B(u) - B(u), 0 \le u \le 1)$ is independent of $B(1)$.*
  (iii) *Consequently, the distribution of both the random set of vertex times and of the difference process is the same for an unconditioned Brownian motion B as it is for a Brownian bridge from $(0, 0)$ to $(1, b)$ for every real value of b.*

Proof. The first two assertions follow easily from the facts that:

  • the set of times of vertices of the concave majorant of a function $(f(u), 0 \le u \le 1)$ is the same for $f(u)$ as it is for $f(u) + cu$ for arbitrary real c;
  • for a Brownian motion B, the process $(B(u) - uB(1), 0 \le u \le 1)$ is a standard Brownian bridge independent of $B(1)$.

The third assertion follows easily from the first two.   □

Theorem 5. *Let X be either an unconditioned Brownian motion or a Brownian bridge from $(0, 0)$ to $(1, b)$ for some real b and let $\{V_i, i \in \mathbb{Z}\}$ denote an indexing of the times of vertices of $\overline{C}_{[0,1]}X$ so that V_i is an increasing function of $i \in \mathbb{Z}$ with $\lim_{i \to -\infty} V_i = 0$ and $\lim_{i \to \infty} V_i = 1$. Moreover, we assume that this indexing depends only on $\overline{C}_{[0,1]}X$.*

Then, conditionally given $V_i = v_i, i \in \mathbb{Z}$, the difference process $(\overline{C}_{[0,1]}X(u) - X(u), 0 \le u \le 1)$ is a concatenation of independent Brownian excursions of prescribed lengths $v_i - v_{i-1}$ between zeros at the times v_i.

Remark 6. Choosing an indexing $\{V_i, i \in \mathbb{Z}\}$ that depends only on $\overline{C}_{[0,1]}B$ is an essential assumption in the theorem above, otherwise the conditional distribution of the difference process between two successive vertices would not necessarily be that of a Brownian excursion.

Proof of Theorem 5. Based on Proposition 4(iii), it suffices to consider the case where B is replaced by a standard Brownian bridge B^{br}. Then, according to Doob's transformation,

$$B^{\text{br}}(u) = (1 - u)\widehat{B}\left(\frac{u}{1-u}\right), \qquad u \in [0, 1],$$

where $\widehat{B}$ is the standard Brownian motion

$$\widehat{B}(t) = (1+t)B^{\text{br}}\left(\frac{t}{1+t}\right), \qquad t \geq 0.$$

It follows that the times V_i of vertices of the concave majorant of B^{br} on $[0, 1]$ may be constructed as $V_i = T_i/(T_i + 1)$, where the T_i's are the times of vertices of the concave majorant of $\widehat{B}$ on $[0, \infty)$. Moreover, the difference between B^{br} and its concave majorant on (V_i, V_{i+1}) is a transformation of the difference between $\widehat{B}$ and its concave majorant on (T_i, T_{i+1}).

We will now show that this transformation maps Brownian excursions to Brownian excursions. Indeed, observe that the transformation between $(u, B^{\text{br}}(u))$ and $(t, \widehat{B}(t))$ is the restriction to a Brownian path of the space–time transformation

$$\mathcal{T}(u, x) = (t, y) := \left(\frac{u}{1-u}, \frac{x}{1-u}\right), \tag{3}$$

where $0 \leq u < 1$, $0 \leq t < \infty$ and x and y both range over all real numbers.

Consider now the conditioning of B^{br} on $B^{\text{br}}(u) = x$ and $B^{\text{br}}(\hat{u}) = \hat{x}$ for some $0 < u < \hat{u} < 1$ and real numbers x and $\hat{x}$. The process

$$X^*(v) := \frac{B^{\text{br}}(u + v(\hat{u} - u)) - x - v(\hat{x} - x)}{\sqrt{\hat{u} - u}}, \qquad 0 \leq v \leq 1,$$

is a standard Brownian bridge, in terms of which the path of B^{br} on $[u, \hat{u}]$ is represented as

$$B^{\text{br}}(u') = x + \sqrt{\hat{u} - u}\, X^*\left(\frac{u' - u}{\hat{u} - u}\right) + \frac{u' - u}{\hat{u} - u}(\hat{x} - x), \qquad u' \in [u, \hat{u}].$$

On the other hand, with $(t, y) = \mathcal{T}(u, x)$ and $(\hat{t}, \hat{y}) = \mathcal{T}(\hat{u}, \hat{x})$, the process

$$Y^*(w) := \frac{\widehat{B}(t + w(\hat{t} - t)) - y - w(\hat{y} - y)}{\sqrt{\hat{t} - t}}, \qquad 0 \leq w \leq 1,$$

is another standard Brownian bridge in terms of which the path of $\widehat{B}$ on $[t, \hat{t}]$ is represented as

$$\widehat{B}(t') = y + \sqrt{\hat{t} - t}\, Y^*\left(\frac{t' - t}{\hat{t} - t}\right) + \frac{t' - t}{\hat{t} - t}(\hat{y} - y), \qquad t' \in [t, \hat{t}].$$

It follows that for an arbitrary choice of $0 \leq u < \hat{u} < 1$ and real values of x and $\hat{x}$, with (t, y) and $(\hat{t}, \hat{y})$ denoting the images of (u, x) and $(\hat{u}, \hat{x})$, respectively, via the space–time transformation $\mathcal{T}$ in (3),

  the image via the space–time transformation of a Brownian bridge from (u, x) to $(\hat{u}, \hat{x})$ is a Brownian bridge from (t, y) to $(\hat{t}, \hat{y})$.

The key observation is that, similarly,

the image via the space–time transformation of the straight line from (u, x) to $(\hat{u}, \hat{x})$ minus a Brownian excursion of length $\hat{u} - u$, shifted to start at time u, is a straight line from (t, y) to $(\hat{t}, \hat{y})$ minus a Brownian excursion of length $\hat{t} - t$, shifted to start at time t.

Intuitively, this is clear from the bridge result, by conditioning each of the bridges to stay above the line joining its end-points. This can be made rigorous by a weak convergence argument, conditioning one of the bridges to go no more than a small distance ϵ above the line, passing to the limit as $\epsilon \to 0$, and appealing to the result of [6]. □

The limiting argument at the end of the previous proof can also be reduced to an invariance of laws of standard Brownian bridges and excursions under a family of space–time transformations indexed by a pair of parameters $0 < u < \hat{u} < 1$. To see this, observe that the relation between B^{br} and $\widehat{B}$ implies that the standard bridge X^* derived from B^{br} on $[u, \hat{u}]$ and the standard bridge Y^* derived from $\widehat{B}$ on $[t, \hat{t}]$ are related by

$$X^*(v) = \frac{1 - u - v(\hat{u} - u)}{\sqrt{(1 - u)(1 - \hat{u})}} Y^* \left(\frac{(1 - u)v}{1 - u - v(\hat{u} - u)} \right), \qquad 0 < v < 1. \tag{4}$$

For the sake of completeness, we give the following lemma, which shows that the mapping from one standard bridge or excursion to another standard bridge or excursion which is implicit in Doob's space–time transformation is non-trivial, but nonetheless easily checked.

Lemma 7. *For each fixed choice of $0 \le u < \hat{u} < 1$, if $(Y^*(v), 0 \le v \le 1)$ is a standard Brownian bridge, then $(X^*(v), 0 \le v \le 1)$ defined by (4) has the same distribution as $(Y^*(v), 0 \le v \le 1)$. Moreover, the same is true with the standard Brownian bridge replaced by a standard Brownian excursion, or by a standard Bessel bridge of any dimension.*

Proof. For the standard Brownian bridge, the result can be derived as above, or checked more directly by observing that Y^* is evidently a centered Gaussian process with continuous paths, so it suffices to check that $\mathbb{E}(X^*(v)X^*(w)) = \mathbb{E}(Y^*(v)Y^*(w)) = v(1 - w)$ for $0 < v < w < 1$ and this is easily done. The result for a Bessel bridge of dimension $d = 1, 2, \ldots$ follows easily from the representation of the square of this process as the sum of squares of d independent standard Brownian bridges. For $d = 3$, this gives the result for the standard excursion, by the well-known identification of the standard excursion with a three-dimensional Bessel bridge due to D. Williams. □

3. The distribution of M

Proof of Theorem 1. Proposition 4 implies that the distribution of the maximal difference M is the same for a standard Brownian motion on $[0, 1]$ as for a Brownian bridge from $(0, 0)$ to $(1, b)$ for an arbitrary real number b. Therefore, it is enough to establish the characterization of M provided in Theorem 1 with a standard Brownian motion B on $[0, 1]$ replaced by a standard Brownian bridge B^{br}.

Let $(V_i, i \in \mathbb{Z})$ denote the sequence of times of vertices of $\overline{C}_{[0,1]}B^{\mathrm{br}}$ indexed in the same way as in Proposition 4 and Theorem 5, and let $(T_i, i \in \mathbb{Z})$ be the corresponding lengths of the segments of $\overline{C}_{[0,1]}B^{\mathrm{br}}$; that is, $T_i = V_i - V_{i-1}$, $i \in \mathbb{Z}$, so the sequence $(T_i, i \in \mathbb{Z})$ is independent of the sequence $(M_{3,i}, i \in \mathbb{Z})$, where the $M_{3,i}$'s are independent random variables identically distributed as M_3, the maximum of a standard Brownian excursion of length 1.

From the definition of M and Theorem 5, we have readily that

$$M \stackrel{d}{=} \max_i \sqrt{T_i} M_{3,i}.$$

Now, let $T^{(1)} \geq T^{(2)} \geq \cdots$ denote the values of T_i put in decreasing order. According to the result of Suidan [24], the distribution of the sequence $(T^{(j)}, j = 1, 2, \ldots)$ is the limiting distribution of ranked lengths of cycles of a uniform random permutation of n elements, with cycle lengths normalized by n, commonly known as the Poisson–Dirichlet distribution with parameters 0 and 1; see also [9,19,22]. Suidan gives this result for the concave majorant of Brownian motion on [0, 1], but it applies just as well to the Brownian bridge, by Proposition 4. It is also known [25] that this asymptotic distribution of cycle lengths is obtained by ranking the terms of a uniform stick-breaking process in decreasing order. Theorem 1 and Corollary 2 follow immediately. □

Theorem 1 provides an easy way of simulating values of M. For the Monte Carlo implementation, we can use the representation (1) of M_3 and the Donsker approximation of B^{br}. It also follows from Theorem 1 that the distribution and density functions of M, denoted hereafter by F_M and f_M, respectively, are given by

$$F_M(x) = E\left[\prod_i F_3\left(\frac{x}{\sqrt{L_i}}\right)\right], \qquad x > 0, \tag{5}$$

and

$$f_M(x) = \sum_i E\left[\frac{1}{\sqrt{L_i}} f_3\left(\frac{x}{\sqrt{L_i}}\right) \prod_{j \neq i} F_3\left(\frac{x}{\sqrt{L_j}}\right)\right], \qquad x > 0, \tag{6}$$

where F_3 and f_3 are the distribution and density functions of M_3, respectively, known to be given by

$$F_3(y) = 1 - 2\sum_{n=1}^{\infty} (4n^2 y^2 - 1) e^{-2n^2 y^2}, \qquad y > 0, \tag{7}$$

and

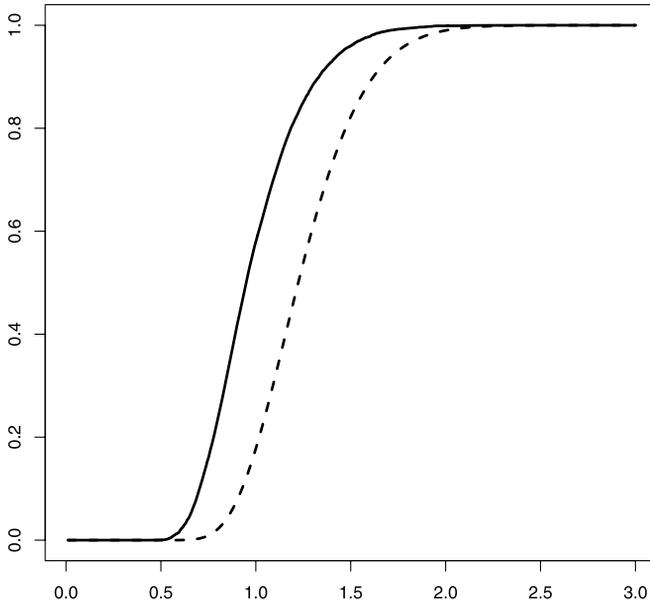$$f_3(y) = 8\sum_{n=1}^{\infty} n^2 y (4n^2 y^2 - 3) e^{-2n^2 y^2}, \qquad y > 0;$$

Figure 2. The solid line is the plot of a Monte Carlo approximation of F_M based on a sample of size 5000. The dashed line is the plot of the distribution function F_3. As the figure suggests, $F_3(u) \leq F_M(u)$ for all u. This is an immediate consequence of (1) and the definition of M.

see, for example, [5,13] for further information about this distribution which has been extensively studied. Monte Carlo approximations of the distribution function F_M and its density f_M based on 5000 independent copies of the uniform stick-breaking process are shown in Figures 2 and 3 along with the distribution and density functions of M_3.

Another technique for exploring features of the distribution of M is to introduce a standard exponential variable γ_1 independent of M and observe that

$$\sqrt{\gamma_1} M \stackrel{d}{=} \max_i \sqrt{\gamma_1 L_i} M_{3,i},$$ (8)

where the $\gamma_1 L_i$ are the points of a Poisson point process with intensity $x^{-1} e^{-x} \, dx$ for $x > 0$, independent of the $M_{3,i}$'s. These are the jumps of a gamma process $(\gamma_s, 0 \leq s \leq 1)$ from which the L_i can be recovered by first normalizing the jumps by γ_1 and then putting the jumps in size-biased random order. This leads to the following proposition.

Proposition 8. *For all $x > 0$,*

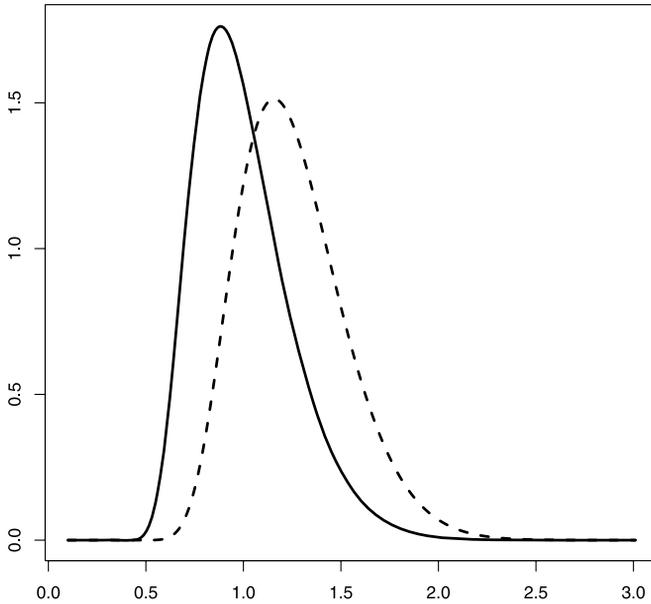$$\mathbb{P}\big(\sqrt{\gamma_1} M \leq x\big) = \exp(-\nu(x, \infty)),$$

Figure 3. The solid line is the plot of a Monte Carlo approximation of f_M based on a sample of size 5000. The dashed line is the plot of the density function f_3.

where

$$v(x, \infty) = \int_0^\infty y^{-1} e^{-y} \left[1 - F_3 \left(\frac{x}{\sqrt{y}} \right) \right] dy. \tag{9}$$

Proof. Note that the event $\{ \sqrt{\gamma_1} M \leq x \}$ occurs if and only if there is no point $\sqrt{\gamma_1 L_i} M_{3,i}$ in the interval (x, ∞). Now, consider the Poisson point process on the product space $\{(z, t) : z > 0, t > 0\}$ such that $\sqrt{\gamma_1 L_i} M_{3,i}$ are the points of this process on the z-axis when $t \in (0, 1)$. The intensity measure v of this process is equal to the image via the map $(x, m) \mapsto \sqrt{xm}$ of the measure

$$x^{-1} e^{-x} \mathbb{P}(M_3 \in dm).$$

It follows that

$$\mathbb{P}\left(\sqrt{\gamma_1} M \leq x \right) = \exp(-v(x, \infty)),$$

with $v(x, \infty)$ given in (9). $\qquad \square$

See also [2,3,20] for similar calculations.

The following theorem gives another representation of F_M and f_M. Let K_m be the modified Bessel function of the second kind and order $m \in [0, \infty)$, and consider the function

$$G(t) = \prod_{n=1}^{\infty} \exp\{-4[2\sqrt{2}tn K_1(2\sqrt{2}tn) - K_0(2\sqrt{2}tn)]\}, \qquad t > 0. \tag{10}$$

Also, let $\mathcal{L}^{-1}$ be the operator of the inverse Laplace transform.

Theorem 9.

(i) *For any real $r > 0$, we have*

$$E(M^r) = \frac{2}{\Gamma(r/2)} \int_0^{\infty} t^{r-1}(1 - G(t)) \, dt. \tag{11}$$

(ii) *For all $x > 0$, the distribution and density functions F_M and f_M are given, respectively, by*

$$F_M(x) = [\mathcal{L}^{-1}(G(\sqrt{t})/t)]\left(\frac{1}{x^2}\right) \tag{12}$$

and

$$f_M(x) = \frac{2}{x^3}[\mathcal{L}^{-1}(1 - G(\sqrt{t}))]\left(\frac{1}{x^2}\right). \tag{13}$$

Proof. It follows from (7) and (9) that

$$v(x, +\infty) = 2 \sum_{n=1}^{\infty} \int_0^{\infty} y^{-1} e^{-y} (4n^2 x^2 y^{-1} - 1) e^{-2n^2 x^2 y^{-1}} \, dy$$

$$= 2 \sum_{n=1}^{\infty} \int_0^{\infty} t^{-1} e^{-2n^2 x^2 t} (2t^{-1} - 1) e^{-t^{-1}} \, dt$$

$$= 2 \sum_{n=1}^{\infty} (2A_n(x) - B_n(x)),$$

where

$$A_n(x) = [\mathcal{L}(e^{-t^{-1}}/t^2)](2n^2 x^2) = 2\sqrt{2}nx K_1(2\sqrt{2}nx) \tag{14}$$

and

$$B_n(x) = [\mathcal{L}(e^{-t^{-1}}/t)](2n^2 x^2) = 2K_0(2\sqrt{2}nx) \tag{15}$$

so that

$$\exp(-\nu(x,\infty)) = \prod_{n=1}^{\infty} \exp\{-4[2\sqrt{2}xn K_1(2\sqrt{2}xn) - K_0(2\sqrt{2}xn)]\} = G(x).$$

The derivation of the expressions in (14) and (15) is deferred to the Appendix. Now, for $r > 0$,

$$E(\gamma_1^{r/2}M^r) = \int_0^{\infty} \left(1 - F_{\gamma_1^{r/2}M^r}(x)\right) dx$$

$$= \int_0^{\infty} \left(1 - \exp(\nu(x^{1/r},\infty))\right) dx$$

$$= \int_0^{\infty} \left(1 - G(x^{1/r})\right) dx = \int_0^{\infty} rx^{r-1}\left(1 - G(x)\right) dx.$$

The claim in (i) now follows from independence of the random variables γ_1 and M.

To show (ii), we again use the independence of γ_1 and M. We can write

$$\mathbb{P}\left(\sqrt{\gamma_1}M \le x\right) = \int_0^{\infty} F_M\left(\frac{x}{\sqrt{s}}\right) e^{-s} ds.$$

Using the change of variable $t = s/\sqrt{x}$, we get

$$\int_0^{\infty} F_M\left(\frac{1}{\sqrt{t}}\right) e^{-tx} dt = \exp(-\nu(\sqrt{x},\infty))/x.$$

Thus, for all $t > 0$,

$$F_M\left(\frac{1}{\sqrt{t}}\right) = \left[\mathcal{L}^{-1}\left(\exp(-\nu(\sqrt{x},\infty))/x\right)\right](t)$$

and the expression of F_M follows. The expression f_M can be obtained immediately by using known properties of the operator $\mathcal{L}$. $\qquad\square$

The identity in (11) gives a way of calculating the rth moment of M via numerical integration. We used this approach to compute $E(M^r)$ for $r = 1, \ldots, 8$ and the values are reported in the first column of Table 1. In the second column are the corresponding empirical estimators for the same moments using 20 000 simulated independent copies of M. On the other hand, the inverse Laplace transforms in the expressions of F_M and f_M given in (12) and (13) can be approximated with very high precision using, for example, the Gaver–Stehfest algorithm. We refer to [1] for a detailed description of this method as well as a nice summary of other classical algorithms used to approximate inverses of Laplace transforms. However, implementation of the Gaver–Stehfest algorithm requires the use of a multiple precision software package or some adequate arbitrary precision library. This will be pursued elsewhere.

Table 1. Approximation of the moments of M of order $r = 1, \ldots, 8$

r	$E(M^r)$	$\overline{X}^r$
1	0.999399	0.997366
2	1.060258	1.056803
3	1.195155	1.190869
4	1.431334	1.427101
5	1.819154	1.816777
6	2.448679	2.452149
7	3.481508	3.499897
8	5.212503	5.266828

4. Complements

Groeneboom's description of $\overline{C}_{[0,\infty)}B$ implies that for each fixed $t > 0$, the joint density of V_t^-, the time of the last vertex of $\overline{C}_{[0,\infty)}B$ before time t, and V_t^+, the time of first vertex of $\overline{C}_{[0,\infty)}B$ after time t, is given by the formula

$$f_{V_t^-, V_t^+}(v_1, v_2) = \frac{2}{(v_2 - v_1)^{3/2}} E\left[Z_+ \left(\frac{X}{\sqrt{v_1}} - \frac{Z}{\sqrt{v_2 - v_1}} \right)_+ \right] 1_{\{0 < v_1 < t < v_2\}},$$

where $y_+ = y 1_{y \geq 0}$ and X and Z are independent standard normal variables. We show in the Appendix (Proposition 11) that this joint density can be presented more explicitly as

$$f_{V_t^-, V_t^+}(v_1, v_2) = \frac{1}{\pi (v_2 - v_1)^2} \left(\sqrt{\frac{v_2 - v_1}{v_1}} - \arctan\left(\sqrt{\frac{v_2 - v_1}{v_1}} \right) \right) 1_{\{0 < v_1 < t < v_2\}}. \qquad (16)$$

Using the time transformation $t \mapsto t/(t+1)$, the joint density of X_u and Y_u, the respective last and first times vertices of the concave majorant of B^{br} occurring respectively before and after $u \in (0, 1)$, is given by

$$f_{X_u, Y_u}(x, y) = \frac{1}{\pi (y - x)^2} \left(\sqrt{\frac{y - x}{x(1 - y)}} - \arctan\left(\sqrt{\frac{y - x}{x(1 - y)}} \right) \right)$$
$$\times 1_{\{0 < x < u < y < 1\}}. \qquad (17)$$

Now, let L_1 be the length of the segment of the concave majorant of a standard Brownian bridge B^{br} on $[0, 1]$ covering a uniform random number U_1 independent of B^{br} and let us verify, using this formula, that the distribution of L_1 is also uniform on $[0, 1]$, as shown already by

Corollary 2. Using the expression in (17), we find, after some algebra, that L_1 has density

$$f_{L_1}(l) = \int_0^{1-l} \frac{1}{\pi l} \left(\sqrt{\frac{l}{x(1-l-x)}} - \arctan\left(\sqrt{\frac{l}{x(1-l-x)}}\right) \right) dx$$

$$= \frac{1-l}{\pi l} \int_0^1 \left(k \frac{1}{\sqrt{l(1-l)}} - \arctan\left(\frac{k}{\sqrt{u(1-u)}}\right) \right) du,$$

using the change of variable $u = x/(1-l)$ and putting $k = \sqrt{l}/(1-l)$.

Now,

$$\int_0^1 \frac{du}{\sqrt{u(1-u)}} = \int_0^{\pi/2} 2\, d\theta = \pi, \qquad \text{by the change of variable } u = \sin^2(\theta).$$

Using the same change of variable, we can write

$$\int_0^1 \arctan\left(\frac{k}{\sqrt{u(1-u)}}\right) du = \int_0^{\pi/2} \arctan\left(\frac{2k}{\sin(2\theta)}\right) \sin(2\theta)\, d\theta$$

$$= \frac{1}{2} \int_0^\pi \arctan\left(\frac{2k}{\sin(t)}\right) \sin(t)\, dt$$

$$= \frac{1}{2} \left[-\cos(t)\arctan\left(\frac{b}{\sin(t)}\right) \right]_0^\pi$$

$$- \frac{1}{2} b \int_0^\pi \frac{\cos^2(t)}{\sin^2(t) + b^2}\, dt \qquad \text{with } b = 2k$$

$$= \frac{\pi}{2} - b \int_0^{\pi/2} \frac{\cos^2(t)}{\sin^2(t) + b^2}\, dt$$

$$= \frac{\pi}{2} - \frac{b}{b^2 + 1} \int_0^{\pi/2} \frac{dt}{\tan^2(t) + b^2/(b^2 + 1)},$$

where the last equality follows from the identity $1/\cos^2(t) = 1 + \tan^2(t)$. Put $c = b^2/(b^2 + 1)$. Using the change of variable $\tan(t) = x$, we get

$$\int_0^{\pi/2} \frac{dt}{\tan^2(t) + c} = \int_0^\infty \frac{dx}{(1+x^2)(x^2 + c)} = \frac{1}{1-c} \int_0^\infty \left(\frac{1}{x^2 + c} - \frac{1}{1 + x^2} \right) dx$$

$$= \frac{1}{1-c} \frac{\pi}{2} \left(\frac{1}{\sqrt{c}} - 1 \right).$$

Now,

$$c = \frac{4l}{2l + 1 + l^2} = \frac{4l}{(l+1)^2}$$

and hence

$$\int_0^1 \arctan\left(\frac{k}{\sqrt{u(1-u)}}\right) du = \frac{\pi}{2}\left[1 - \frac{2\sqrt{l}}{1-l}\left(\frac{l+1}{2\sqrt{l}} - 1\right)\right] = \frac{\pi\sqrt{l}}{1+\sqrt{l}},$$

which implies that $f_{L_1}(l) = 1$ for $0 < l < 1$. Thus, we have an independent verification of the uniform distribution of L_1 asserted by Corollary 2. However, it is far from clear how to derive the further results of Corollary 2 using this approach.

Appendix

In this Appendix, we collect some further identities and computations of an analytic kind which arise in connection with the material in this paper.

Proposition 10. *Let X and Y have the standard bivariate normal distribution with correlation* $E(XY) = \rho$. *We then have*

$$E(X_+Y_+) = \frac{1}{2\pi}\sqrt{1-\rho^2} + \rho P(X > 0, Y > 0), \tag{18}$$

where

$$P(X > 0, Y > 0) = \frac{1}{2\pi}\left(\frac{\pi}{2} + \arctan\left(\rho/\sqrt{1-\rho^2}\right)\right), \tag{19}$$

which, for $\rho < 0$, *can be rewritten as*

$$P(X > 0, Y > 0) = -\frac{1}{2\pi}\arctan\left(\sqrt{1-\rho^2}/\rho\right). \tag{20}$$

Proof. Rosenbaum [21], formula (5), gives an expression for $E(XY1_{\{X>h, Y>k\}})$ in terms of the probability $P(X > h, Y > k)$ and the standard normal density and distribution functions. In the present case, $h = k = 0$ and Rosenbaum's formula simplifies to (18). Formula (19) is well known; see, for instance, [23]. If $\rho < 0$, then the passage from (19) to (20) is made via the trigonometric identity $\arctan(x) + \arctan(1/x) = -\pi/2, \forall x < 0$. □

Proposition 11. *Let* $a, b > 0$ *and let Z and W be independent standard normal variables. We then have*

$$E\left[Z_+\left(\frac{W}{a} - \frac{Z}{b}\right)_+\right] = \frac{1}{2b\pi}\left(\frac{b}{a} - \arctan\left(\frac{b}{a}\right)\right),$$

where $y_+ = y1_{y\geq 0}$.

Proof. Let $t := b/a$, $X := Z$ and $Y = (tW - Z)/\sqrt{1 + t^2}$. Since (X, Y) is a standard bivariate normal with correlation $\rho = -1/\sqrt{t^2 + 1} < 0$, where $t = -\sqrt{1 - \rho^2}/\rho > 0$, it follows from

Proposition 10 that

$$E[Z_+(tW - Z)_+] = -\frac{1}{2\pi\rho}\left(\sqrt{1 - \rho^2} - \rho \arctan\left(\sqrt{1 - \rho^2}/\rho\right)\right)$$

$$= \frac{1}{2\pi}\left(t - \arctan(t)\right)$$

and the conclusion follows after replacing t by b/a. □

Proof of the identities (14) and (15). We start by showing (15). Letting $z = 2\sqrt{2}nx$, we need to show that

$$K_0(z) = \frac{1}{2}\int_0^\infty \frac{e^{-1/t}}{t}e^{-z^2t/4}\,dt.$$

We have that

$$\int_0^\infty \frac{e^{-1/t}}{t}e^{-z^2t/4}\,dt = e^{-z}\int_0^\infty \frac{1}{t}e^{-(zt-2)^2/4t}\,dt$$

and putting $u = (zt - 2)/(2\sqrt{t})$, it follows that $t = (u + \sqrt{u^2 + 2z})^2/z^2$ and $dt = 2t\,du/\sqrt{u^2 + 2z}$. Hence,

$$\int_0^\infty \frac{e^{-1/t}}{t}e^{-z^2t/4}\,dt = 2e^{-z}\int_{-\infty}^\infty \frac{e^{-u^2}}{\sqrt{u^2 + 2z}}\,du.$$

It follows that

$$\frac{1}{2}\int_0^\infty \frac{e^{-1/t}}{t}e^{-z^2t/4}\,dt = e^{-z}\int_{-\infty}^\infty \frac{e^{-u^2}}{\sqrt{u^2 + 2z}}\,du = K_0(z);$$

see, for example, Hunter [12]. To show (14), it is enough to show that

$$\int_0^\infty \frac{e^{-1/t}}{t^2}e^{-z^2t/4}\,dt = zK_1(z).$$

Using the fact that $\lim_{z\to 0} zK_1(z) = 1$ (see, e.g., [16]) and noting that the integral on the left-hand side is equal to 1 for $z = 1$, it is enough to show that

$$-\frac{z}{2}\int_0^\infty \frac{e^{-1/t}}{t}e^{-z^2t/4}\,dt = (zK_1(z))'.$$

The calculations above imply that

$$-\frac{z}{2}\int_0^\infty \frac{e^{-1/t}}{t}e^{-z^2t/4}\,dt = -zK_0(z).$$

We conclude by using the well-known identity $(zK_n(z))' = -zK_{n-1}(z)$, where K_m is the modified Bessel function of the second kind of order m. □

Acknowledgements

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