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# Characterization theorems for the Gneiting class of space–time covariances

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We characterize the Gneiting class of space–time covariance functions and give more relaxed conditions on the functions involved. We then show necessary conditions for the construction of compactly supported functions of the Gneiting type. These conditions are very general since they do not depend on the Euclidean norm.

*Keywords:* compact support; Gneiting's class; positive definite; space–time

## 1. Introduction

The construction of space–time covariance functions is an important subject, the literature for which can be traced back to at least the early 1990s [1,2], where it is emphasized how, under the framework of geostatistical techniques for the study of, for instance, atmospheric and environmental sciences, covariance functions are crucial for estimation and prediction since the best linear predictor depends exclusively on the covariance matrix, which determines the weights of any individual observation in the predictor itself [4].

There are several unsolved problems which are of interest to both the statistical and mathematical communities and this paper provides solutions to two of them.

The first problem is related to the characterization of space–time covariance functions. To the best of our knowledge, there is no literature related to this important problem. In particular, we can find several *permissibility* criteria, that is, sufficient conditions to ensure that a candidate function is positive definite (permissible) on the space–time domain, but no characterization theorem, at least for given classes of covariance functions, is available.

A wide class of covariance functions can be obtained through Gaussian mixtures [4,7,8] for which one can find a large number of contributions having as common origin the Gneiting class of covariance functions [4]: for $(x,t)\in\mathbb{R}^{d+l}$, the function

$$(x,t)\mapsto K(x,t):=h(\|t\|^2)^{-d/2}\varphi\left(\frac{\|x\|^2}{h(\|t\|^2)}\right) \tag{1.1}$$

is positive definite, where $\varphi$ is completely monotone on the positive real line, $h$ is a Bernstein function and $\|\cdot\|$ denotes the Euclidean norm. For $l=1$, the function above is a stationary and non-separable space–time covariance. This function has been persistently used in the literature

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and a Google Scholar search in September 2009 yielded over 90 papers where this covariance has been used for applications to space–time data.

The first result in this paper states necessary and sufficient conditions for the permissibility of the Gneiting class. Also, more general conditions for its permissibility are given.

The second problem confronted in this paper relates to the construction of space–time covariances that are compactly supported in the spatial component. Although such compactly supported covariances are much in demand in the recent literature, there is no single contribution concerning the construction of compactly supported correlations over space and time. This challenge is considerable from a mathematical point of view. A natural perspective is to consider the Gneiting class above and replace the completely monotone function $\varphi(\cdot)$ with a compactly supported one, that is, a function which is identically zero outside a finite range. In particular, the tempting choice $t \mapsto \varphi(t) := (1 - \|t\|^{\alpha})_{+}^{\lambda}$, for positive values of $\alpha$ and $\lambda$ and where $(x)_+$ denotes the positive part of $x$, creates an interesting connection to the celebrated Schoenberg [9] problem, in which the positive definiteness of the function $\varphi$ defined above is related to that of the function $t \mapsto \exp(-t^{\beta})$ for some positive $\beta$. The reader is referred to the survey in [14,15] and the references therein for a thorough review.

In considering this problem, we work in a fairly general framework and let the function $\varphi$ depend on a general seminorm and not on the Euclidean one, as the latter is a restrictive assumption for spatial applications.

The paper is organized as follows. Section 2 completely characterizes the Gneiting class, for which only sufficient conditions have been known. In Section 3, we present necessary conditions for compactly supported covariances of the Gneiting type.

## 2. Characterization of the Gneiting class

In this section, we give a characterization of the Gneiting class. In doing so, we relax the permissibility hypotheses stated in [4]. Two technical lemmas are needed for a more elegant proof of the main result, stated as Theorem 2.1 below.

For a complex-valued function $f : \mathbb{R}^n \to \mathbb{C}$, we write $f \in L(\mathbb{R}^n)$ when $f$ is absolutely integrable on $\mathbb{R}^n$. Similarly, we write $f \in C(\mathbb{R}^n)$ when $f$ is continuous in $\mathbb{R}^n$.

For a real linear space $E$, we denote by $\mathrm{FD}(E)$ the set of all linear finite-dimensional subspaces of $E$.

If $\dim E = n \in \mathbb{N}$ and $e_1, \ldots, e_n$ constitute a basis for $E$, then, by definition, we have

$$\begin{aligned} C(E) &= \{f : E \to \mathbb{C} \mid f(x_1 e_1 + \cdots + x_n e_n) \in C(\mathbb{R}^n)\}, \\ C_0(E) &= \{f \in C(E) \mid f \text{ has compact support}\} \quad \text{and} \\ L(E) &= \{f : E \to \mathbb{C} \mid f(x_1 e_1 + \cdots + x_n e_n) \in L(\mathbb{R}^n)\}. \end{aligned}$$

Obviously, these classes do not depend on the choice of the basis in $E$. Thus, in this case, it is possible to set $E = \mathbb{R}^n$.

If $\dim E = \infty$, then, by definition, we have that $C(E) = \{f : E \to \mathbb{C} \mid f \in C(E_0)$ for all $E_0 \in \mathrm{FD}(E)\}$.

A complex-valued function $f : E \to \mathbb{C}$ is said to be positive definite on $E$ (denoted hereafter $f \in \Phi(E)$) if, for any finite collection of points $\{\xi_i\}_{i=1}^n \in E$, the matrix $(f(\xi_i - \xi_j))_{i,j=1}^n$ is positive definite, that is,

$$\text{for all } a_1, a_2, \dots, a_n \in \mathbb{C} \qquad \sum_{i,j=1}^{n} a_i f(\xi_i - \xi_j)\overline{a}_j \geq 0.$$

Let $E = \mathbb{R}^n$. By Bochner's theorem, the function $f$ is positive definite and continuous in $\mathbb{R}^n$ if and only if $f(x) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}(u,x)}\,\mathrm{d}\mu(u)$, where $(u,x) = u_1x_1 + u_2x_2 + \cdots + u_nx_n$ is a scalar product in $\mathbb{R}^n$ and $\mu$ is a non-negative finite Borel measure on $\mathbb{R}^n$. Additionally, if $f \in C(\mathbb{R}^n) \cap L(\mathbb{R}^n)$, then $f$ is positive definite on $\mathbb{R}^n$ if and only if $\widehat{f}(u) := \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(u,x)} f(x)\,\mathrm{d}x \geq 0$, $u \in \mathbb{R}^n$.

**Lemma 2.1.**

(i) $f \in \Phi(E) \iff f \in \Phi(E_0)\ \forall E_0 \in \mathrm{FD}(E)$.
(ii) *If* $\dim E = n \in \mathbb{N}$, *then* $f \in \Phi(E) \iff fg \in \Phi(E)$ *for all* $g \in \Phi(E) \cap C_0(E)$.

**Proof.** For both parts, the necessity is obvious. For the sufficiency of part (i), for $n \in \mathbb{N}$ and $x_1, \dots, x_n$ in $E$, we have that $x_1, \dots, x_n \in E_0$, where $E_0$ is the linear span of these elements. Obviously, $\dim E_0 \leq n$.

For the sufficiency of part (ii), let $e_1, \dots, e_n$ be a basis in $E$. We then take $g(x_1e_1 + \cdots + x_ne_n) = (1 - \varepsilon|x_1|)_+ \cdots (1 - \varepsilon|x_n|)_+$ and $\varepsilon \downarrow 0$. The proof is thus completed. □

**Lemma 2.2.** *Let the following conditions be satisfied*:

(1) $h, b \in C(E)$ *and* $h(t) > 0$ *for all* $t \in E$;
(2) $\varphi \in C([0,+\infty))$ *and for some* $m \in \mathbb{N}$, *we have* $\int_0^\infty |\varphi(u^2)| u^{m-1}\,\mathrm{d}u < \infty$;
(3) $\rho \in C(\mathbb{R}^m)$, $\rho(tx) = |t|\rho(x)$ *for all* $t \in \mathbb{R}$, $x \in \mathbb{R}^m$ *and* $\rho(x) > 0$, $x \neq 0$.

*Then* $K(x,t) := b(t)\varphi(\frac{\rho^2(x)}{h(t)}) \in \Phi(\mathbb{R}^m \times E) \iff b(t)(h(t))^{m/2} G_m(\sqrt{h(t)}v) \in \Phi(E)$ *for all* $v \in \mathbb{R}^m$ *with* $G_m(\cdot)$ *defined in equation* (2.1) *and*

$$\mathbb{R}^n \ni v \mapsto G_n(v) := \int_{\mathbb{R}^n} \varphi(\rho^2(y))\mathrm{e}^{\mathrm{i}(y,v)}\,\mathrm{d}y. \tag{2.1}$$

**Proof.** Observe that $\varphi(\rho^2(x)) \in L(\mathbb{R}^m)$. We have that

$$\begin{aligned} K(x,t) \in \Phi(\mathbb{R}^m \times E) \quad &\iff \quad K(x,t) \in \Phi(\mathbb{R}^m \times E_0) \qquad \text{for all } E_0 \in \mathrm{FD}(E) \\ &\iff \quad K(x,t)g(t) \in \Phi(\mathbb{R}^m \times E_0) \qquad \text{for all } E_0 \in \mathrm{FD}(E) \text{ and all } g \in \Phi(E_0) \cap C_0(E_0) \\ &\iff \quad \int_{\mathbb{R}^m}\int_{E_0} K(x,t)g(t)\mathrm{e}^{\mathrm{i}(x,v)}\mathrm{e}^{\mathrm{i}(t,u)}\,\mathrm{d}x\,\mathrm{d}t \geq 0 \\ &\qquad \text{for all } E_0 \in \mathrm{FD}(E),\ g \in \Phi(E_0) \cap C_0(E_0) \text{ and } v \in \mathbb{R}^m, u \in E_0. \end{aligned}$$

As for the last integral, a change of variables of the type $x = \sqrt{h(t)}y$ yields that the last inequality is equivalent to

$$\int_{E_0} g(t)b(t)(h(t))^{m/2} G_m\big(\sqrt{h(t)}v\big)\mathrm{e}^{\mathrm{i}(t,u)}\,\mathrm{d}t \geq 0 \qquad \text{for all } v \in \mathbb{R}^m, u \in E_0,$$

which holds if and only if, for all $g \in \Phi(E_0) \cap C_0(E_0)$ and $v \in \mathbb{R}^m$, we have

$$g(t)b(t)(h(t))^{m/2}G_m\big(\sqrt{h(t)}v\big) \in \Phi(E_0) \qquad \text{for all } E_0 \in \mathrm{FD}(E)$$
$$\iff \quad b(t)(h(t))^{m/2}G_m\big(\sqrt{h(t)}v\big) \in \Phi(E) \qquad \text{for all } v \in \mathbb{R}^m.$$

The proof is thus complete. □

A function $f:(0,\infty) \to \mathbb{R}$ is called *completely monotone* if it is arbitrarily often differentiable and $(-1)^n f^{(n)}(x) \geq 0$ for $x > 0$, $n = 0, 1, \ldots$. By the Bernstein–Widder theorem [10], the set $M_{(0,\infty)}$ of completely monotone functions coincides with that of Laplace transforms $\mathcal{L}$ of positive measures $\mu$ on $[0,\infty)$, that is, $f(x) = \mathcal{L}\mu(x) = \int_{[0,\infty)} \mathrm{e}^{-xt}\,\mathrm{d}\mu(t)$, $x > 0$, where we require $\mathrm{e}^{-xt}$ to be $\mu$-integrable for any $x > 0$. By Schoenberg's theorem, the radial function $f(x) = \varphi(\|x\|^2)$, $\varphi \in C([0,+\infty))$ belongs to $\Phi(\mathbb{R}^n)$ for all $n \in \mathbb{N}$ if and only if $\varphi \in M_{(0,\infty)}$.

Theorem 2.1 gives our characterization of the Gneiting class. This has the feature, additional to our introduction of the class in Section 1, that only negative definiteness of the function $h$ is required [8], while Gneiting's assumptions are much more restrictive as it is required that $h'$ is completely monotone on the positive real line. Furthermore, the proof of this result is deferred to the final section for reasons that will become apparent.

**Theorem 2.1.** *Let $h \in C(E)$, $h(t) > 0$ for all $t \in E$. Let $d \in \mathbb{N}$. The following statements are equivalent*:

(1) $K(x,t) := (h(t))^{-d/2}\varphi\big(\frac{\|x\|^2}{h(t)}\big) \in \Phi(\mathbb{R}^d \times E)$ *for all* $\varphi \in C([0,+\infty)) \cap M_{(0,\infty)}$;
(2) $\mathrm{e}^{-\lambda h(t)} \in \Phi(E)$ *for all positive* $\lambda$.

Let us consider examples of functions $h$ for which statement (2) in Theorem 2.1 holds.

***Example 2.1.*** Let $h(t) = \|t\|_p^\alpha + c$, $c > 0$, $0 < p \leq \infty$, $\alpha \geq 0$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, where $\|t\|_p = (\sum_{k=1}^n |t_k|^p)^{1/p}$, $0 < p < \infty$, and $\|t\|_\infty = \sup_{1 \leq k \leq n} |t_k|$. Then $\mathrm{e}^{-\lambda h(t)} \in \Phi(E)$ for all positive $\lambda$ if and only if $0 \leq \alpha \leq \alpha_{n,p}$, where

$$\alpha_{n,p} = \begin{cases} 2 & \text{if } n = 1, 0 < p \leq \infty, \\ p & \text{if } n \geq 2, 0 < p \leq 2, \\ 1 & \text{if } n = 2, 2 < p \leq \infty, \\ 0 & \text{if } n \geq 3, 2 < p \leq \infty. \end{cases} \tag{2.2}$$

The case $0 < p \leq 2$ corresponds to the result of Schoenberg [9]. The other three cases have been investigated by Koldobsky [5] and Zastavnyi [11–13] ($2 < p \leq \infty$, $n \geq 2$). Finally, Misiewicz [6] gave the most recent result ($p = \infty$, $n \geq 3$).

***Example 2.2.*** If $\rho(t)$ is a norm on $\mathbb{R}^2$, then $\mathrm{e}^{-\rho^\alpha(t)} \in \Phi(\mathbb{R}^2)$ for all $0 \leq \alpha \leq 1$ (see, e.g., [14]). Therefore, $\mathrm{e}^{-\lambda h(t)} \in \Phi(\mathbb{R}^2)$ for any $\lambda > 0$, where $h(t) = \rho^\alpha(t) + c$, $0 \leq \alpha \leq 1$, $c > 0$.

***Example 2.3.*** Let $\psi(s) \in \mathbb{R}$, $s > 0$. We then have $\mathrm{e}^{-\lambda\psi} \in M_{(0,\infty)}$ for all $\lambda > 0$ if and only if $\psi' \in M_{(0,\infty)}$. Therefore, if $\psi \in C([0, +\infty))$ and $\psi(s) > 0$ for all $s \geq 0$, and $\psi' \in M_{(0,\infty)}$, then $\mathrm{e}^{-\lambda h(t)} \in \Phi(\mathbb{R}^n)$ for all $\lambda > 0$, $n \in \mathbb{N}$, where $h(t) := \psi(\|t\|^2)$ and, hence (see Theorem 2.1), $K(x, t) := (\psi(\|t\|^2))^{-d/2}\varphi(\frac{\|x\|^2}{\psi(\|t\|^2)}) \in \Phi(\mathbb{R}^d \times \mathbb{R}^n)$ for all $\varphi \in C([0, +\infty)) \cap M_{(0,\infty)}$, $d \in \mathbb{N}$. This result was proven by Gneiting [4].

A complex-valued function $h : E \to \mathbb{C}$ is called *conditionally negative definite on* $E$ (denoted $h \in N(E)$ hereafter) if the inequality $\sum_{k,j=1}^{n} c_k \bar{c}_j h(x_k - x_j) \leq 0$ is satisfied for every positive integer $n$, every collection of points $x_1, \ldots, x_n$ in $E$ and every set of complex numbers $c_1, c_2, \ldots, c_n$, satisfying the condition $\sum_{k=1}^{n} c_k = 0$.

***Example 2.4 (Schoenberg's theorem [9]).*** $\mathrm{e}^{-\lambda h(t)} \in \Phi(E)$ for any $\lambda > 0$ if and only if $h(-t) = \overline{h(t)}$ for all $t \in E$ and $h(t) \in N(E)$.

# 3. Necessary conditions for functions of the Gneiting type

Before presenting the main results contained in this section, some comments are in order. The construction of compactly supported correlation functions is a non-trivial task that has consequences for the estimation of space–time processes for the computational gains that follow. At present, there is no contribution in the literature devoted to non-separable covariances that are compactly supported. Until now, in order to obtain compactly supported correlations, the commonplace approach is to use tapering [3]. We described a more natural approach in the Introduction and the results following subsequently highlight interesting solutions to this problem.

In order to be clear, we will henceforth write $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ for the unit sphere in $\mathbb{R}^d$.

**Theorem 3.1.** *Let the following conditions be satisfied*:

(1) $h \in C(E)$, $h(t) > 0$ *for any* $t \in E$ *and* $h(t) \not\equiv h(0)$ *on* $E$;
(2) $\varphi \in C([0, +\infty))$, $\varphi(0) > 0$;
(3) *for* $d \in \mathbb{N}$, $\rho \in C(\mathbb{R}^d)$, $\rho(tx) = |t|\rho(x)$ $\forall t \in \mathbb{R}, x \in \mathbb{R}^d$ *and* $\rho(x) > 0$, $x \neq 0$;
(4) $K(x, t) := (h(t))^{-d/2}\varphi(\frac{\rho^2(x)}{h(t)}) \in \Phi(\mathbb{R}^d \times E)$.

*Then*:

1. $(h(t))^{-d/2} \in \Phi(E)$ *and* $\varphi(\rho^2(x)) \in \Phi(\mathbb{R}^d)$;
2. *if there exists an integer* $n \in \{1, \ldots, d\}$ *such that* $\int_0^\infty |\varphi(u^2)|u^{n-1}\,\mathrm{d}u < \infty$, *then for all* $m = 1, \ldots, n$ *and* $v \in \mathbb{R}^m$, *the function* $s \mapsto f_{m,v}(s) := s^{m-d}G_m(sv)$, *with* $G_m(\cdot)$ *as defined in* (2.1), *is decreasing on* $(0, \infty)$ *and, furthermore*, $f_{m,v}(\infty) = 0$ *for* $v \neq 0$.
3. *if* $\int_0^\infty |\varphi(u^2)|u^{d-1}\,\mathrm{d}u < \infty$, *then* $G_d(0) > 0$ *and if, in addition*, $G_d$ *is real analytic, then for any* $v \in \mathbb{R}^d$, $v \neq 0$, *the function* $s \mapsto f_{d,v}(s) := G_d(sv)$ *is strictly decreasing on* $[0, +\infty)$ *and* $G_d(v) > 0$ *for all* $v \in \mathbb{R}^d$;
4. *if* $\int_0^\infty |\varphi(u^2)|u^{d+1}\,\mathrm{d}u < \infty$, *then* $\alpha_1(v) := \int_{\mathbb{R}^d} \varphi(\rho^2(y))(y, v)^2\,\mathrm{d}y \geq 0$ *for all* $v \in \mathbb{S}^{d-1}$ *and* $\beta_1 := \int_{\mathbb{R}^d} \varphi(\rho^2(y))\|y\|^2\,\mathrm{d}y \geq 0$ *and, furthermore*, $\alpha_1(v) \equiv 0$ *on* $\mathbb{S}^{d-1}$ *if and only if* $\beta_1 = 0$; *if, in addition*, $\beta_1 > 0$, *then* $\mathrm{e}^{-\lambda h(t)} \in \Phi(E)$ *for any* $\lambda > 0$;

5. *if* $\int_0^\infty |\varphi(u^2)|\mathrm{e}^{\varepsilon u}\,\mathrm{d}u < \infty$ *for some* $\varepsilon > 0$, *then for every* $\lambda > 0$ *and every* $v \in \mathbb{S}^{d-1}$ *we have* $\mathrm{e}^{-\lambda h^p(t)} \in \Phi(E)$, *where*

$$p = p(v) := \min\left\{k \in \mathbb{N} : \alpha_k(v) = \int_{\mathbb{R}^d} \varphi(\rho^2(y))(y, v)^{2k}\,\mathrm{d}y \neq 0\right\}, \qquad v \in \mathbb{S}^{d-1};$$

*the function* $p : \mathbb{S}^{d-1} \to \mathbb{N}$ *is bounded on the unit sphere and*

$$\min_{v \in \mathbb{S}^{d-1}} p(v) = \min\left\{k \in \mathbb{N} : \beta_k = \int_{\mathbb{R}^d} \varphi(\rho^2(y))\|y\|^{2k}\,\mathrm{d}y \neq 0\right\}.$$

**Proof.** Part 1 is obvious.

As for part 2, by Lemma 2.2, we have

$$F_{m,v}(t) := (h(t))^{(m-d)/2} G_m\bigl(\sqrt{h(t)}v\bigr) \in \Phi(E), \qquad m = 1, \dots, n,\ v \in \mathbb{R}^m.$$

Hence, $F_{m,v}(0) = (h(0))^{(m-d)/2} G_m(\sqrt{h(0)}v) \geq 0$ and $|F_{m,v}(t)| \leq F_{m,v}(0)$, $t \in E$. Therefore, $G_m(v) \geq 0$, $v \in \mathbb{R}^m$, and

$$(sh(t))^{(m-d)/2} G_m\bigl(\sqrt{h(t)}sv\bigr) \leq (sh(0))^{(m-d)/2} G_m\bigl(\sqrt{h(0)}sv\bigr)$$

for $m = 1, \dots, n$, $v \in \mathbb{R}^m$, $s > 0$ and for all $t \in E$. The latter inequality is equivalent to

$$f_{m,v}\left(\sqrt{\frac{h(t)}{h(0)}} \cdot s\right) \leq f_{m,v}(s).$$

Since $(h(t))^{-d/2} \in \Phi(E)$, we have $h(t) \geq h(0)$, $t \in E$. Since $h(t) \not\equiv h(0)$ on $E$, there exists a point $t_0 \in E$ such that $q := \sqrt{\frac{h(t_0)}{h(0)}} > 1$. By the mean value theorem, for any $\alpha \in [1, q]$, there exists a $\xi \in E$ such that $\sqrt{\frac{h(\xi)}{h(0)}} = \alpha$. Therefore, $f_{m,v}(\alpha s) \leq f_{m,v}(s)$ for all $s > 0$ and $\alpha \in [1, q]$. Hence, $f_{m,v}(\alpha^2 s) \leq f_{m,v}(\alpha s) \leq f_{m,v}(s)$ for all $s > 0$ and $\alpha \in [1, q]$. Thus, $f_{m,v}(\alpha^p s) \leq f_{m,v}(s)$ for all $s > 0$, $\alpha \in [1, q]$ and $p \in \mathbb{N}$. This implies that the function $f_{m,v}(s)$ decreases in $s \in (0, \infty)$. By the Riemann–Lebesgue theorem, it follows that $G_m(v) \to 0$ as $\|v\| \to \infty$. Hence, $f_{m,v}(\infty) = 0$ for $v \neq 0$.

3.i. From part 2, it follows that for all $v \in \mathbb{R}^d$, $v \neq 0$, the function $G_d(sv)$ decreases in $s \in (0, \infty)$ and, hence, $0 \leq G_d(v) \leq G_d(0)$. Therefore, $G_d(0) > 0$ (otherwise, $G_d(v) \equiv 0$ on $\mathbb{R}^d \Rightarrow \varphi(\rho^2(y)) \equiv 0$ on $\mathbb{R}^d$, which contradicts the condition $\varphi(0) > 0$).

ii. If, in addition, $G_d$ is real analytic, then for all $v \in \mathbb{R}^d$, $v \neq 0$, the function $G_d(sv)$ is strictly decreasing on $[0, \infty)$. This can be proven by contradiction. Let us assume that, for some $v_0 \in \mathbb{R}^d$ and $v_0 \neq 0$, the function $G_d(sv_0)$ is constant on some interval $(\alpha, \beta) \subset (0, \infty)$, $\alpha < \beta$. This would imply that $G_d$ is constant on $[0, \infty)$ and that $G_d(0) = \lim_{s\to\infty} G_d(sv_0) = 0$, which contradicts part i above. Thus, for all $v \in \mathbb{R}^d$, $v \neq 0$, the function $G_d(sv)$ strictly decreases on $[0, \infty)$ and, hence, $G_d(v) > \lim_{s\to\infty} G_d(sv) = 0$.

4. Let $v \in \mathbb{S}^{d-1}$ and define $f_{d,v}(s) := G_d(sv)$. From parts 2 and 3, it follows that the function $f_{d,v}(s)$ decreases on $[0,\infty)$ and that $f_{d,v}(0) > 0$. Obviously, $f_{d,v}(s) \in C^2(\mathbb{R})$ and

$$f_{d,v}(s) = f_{d,v}(0) + \frac{f''_{d,v}(0)}{2}s^2 + \mathrm{o}(s^2), \qquad s \to 0,$$

where $f''_{d,v}(0) = -\alpha_1(v)$. Note that $f''_{d,v}(0) \le 0$, otherwise the function $f_{d,v}(s)$ would be strongly increasing on $[0,c]$ for some $c > 0$, which would contradict part 2. Thus, $\alpha_1(v) \ge 0$ for all $v \in \mathbb{S}^{d-1}$. For $p > 0$, the following integral is constant on $\mathbb{S}^{d-1}$:

$$\int_{\mathbb{S}^{d-1}} |(y,v)|^p \,\mathrm{d}\sigma(v) \equiv c_{d,p} > 0, \qquad y \in \mathbb{S}^{d-1},$$

where $\mathrm{d}\sigma$, if $n \ge 2$, is the surface measure on $\mathbb{S}^{d-1}$ and $\mathrm{d}\sigma(v) = \delta(v-1) + \delta(v+1)$, if $d = 1$ (here, $\delta(v)$ is the Dirac measure with mass 1 concentrated in the point $v = 0$). Therefore,

$$\int_{\mathbb{S}^{d-1}} |(y,v)|^p \,\mathrm{d}\sigma(v) = c_{d,p}\|y\|^p, \qquad y \in \mathbb{R}^d,\ p > 0. \tag{3.1}$$

Hence,

$$\int_{\mathbb{S}^{d-1}} \alpha_1(v)\,\mathrm{d}\sigma(v) = c_{d,2}\beta_1 \ge 0$$

and $\alpha_1(v) \equiv 0$ on $\mathbb{S}^{d-1}$ if and only if $\beta_1 = 0$.

Let, in addition, $\beta_1 > 0$. Then $f''_{d,v_0}(0) = -\alpha_1(v_0) < 0$ for some $v_0 \in \mathbb{S}^{d-1}$ and

$$\psi_n(t) := \left(\frac{G_d(\gamma_n\sqrt{h(t)}v_0)}{G_d(0)}\right)^n = \bigl(1 + g_n(t)\bigr)^n \in \Phi(E) \qquad \forall n \in \mathbb{N}, \gamma_n > 0. \tag{3.2}$$

Now, let us take

$$\gamma_n := \left(-\frac{2f_{d,v_0}(0)}{f''_{d,v_0}(0)} \cdot \frac{\lambda}{n}\right)^{1/2} > 0, \qquad \lambda > 0.$$

Obviously, $\gamma_n \to +0$ and

$$g_n(t) = \frac{f_{d,v_0}(\gamma_n\sqrt{h(t)}) - f_{d,v_0}(0)}{f_{d,v_0}(0)} \sim \frac{f''_{d,v_0}(0)}{2f_{d,v_0}(0)} \cdot \bigl(\gamma_n\sqrt{h(t)}\bigr)^2 = -\frac{\lambda}{n} \cdot h(t) \qquad \text{as } n \to \infty.$$

Therefore, $\psi_n(t) \to \mathrm{e}^{-\lambda h(t)}$ and, hence, $\mathrm{e}^{-\lambda h(t)} \in \Phi(E)$ for all $\lambda > 0$.

5. In this case, $G_d$ is real analytic and

$$\begin{aligned} &f^{(2k)}_{d,v}(0) = (-1)^k\alpha_k(v), \\ &f^{(2k-1)}_{d,v}(0) = 0, \qquad \int_{\mathbb{S}^{d-1}} \alpha_k(v)\,\mathrm{d}\sigma(v) = c_{d,2k}\beta_k, \qquad k \in \mathbb{N}. \end{aligned} \tag{3.3}$$

Therefore, for all $v \in \mathbb{S}^{d-1}$, there exists a natural number $p \in \mathbb{N}$ such that

$$f_{d,v}(s) = f_{d,v}(0) + \frac{f_{d,v}^{(2p)}(0)}{(2p)!} s^{2p} + \mathrm{o}(s^{2p}) \qquad \text{as } s \to 0,$$

where $f_{d,v}^{(2p)}(0) \neq 0$; otherwise, the function $f_{d,v}(0) \equiv f_{d,v}(s) \equiv f_{d,v}(+\infty) = 0$, which would contradict the inequality $G_d(0) > 0$ (see part 3). Hence, $f_{d,v}^{(2p)}(0) < 0$, otherwise the function $f_{d,v}(s)$ would be strongly increasing on $[0, c]$ for some $c > 0$, which would contradict part 2.

Let $v \in \mathbb{S}^{d-1}$ and $p = p(v)$. Take the function in equation (3.2), where $v_0 = v$

$$\gamma_n := \left( -\frac{(2p)! f_{d,v_0}(0)}{f_{d,v_0}^{(2p)}(0)} \cdot \frac{\lambda}{n} \right)^{1/(2p)} > 0, \qquad \lambda > 0.$$

Then $g_n(t) \sim -\frac{\lambda}{n} \cdot h^p(t)$ as $n \to \infty$. Therefore, $\psi_n(t) \to \mathrm{e}^{-\lambda h^p(t)}$ and, hence, $\mathrm{e}^{-\lambda h^p(t)} \in \Phi(E)$ for all $\lambda > 0$.

If $\alpha_k(v_0) \neq 0$ for some $v_0 \in \mathbb{S}^{d-2}$, $k \in \mathbb{N}$, then $\alpha_k(v) \neq 0$ in some neighborhood of a point $v_0$ and, hence, $p(v) \leq p(v_0)$ in this neighborhood. Thus, the function $p(v)$ is locally bounded on $\mathbb{S}^{d-1}$ and, hence, $p(v)$ is bounded there.

Let $m = \min_{v \in \mathbb{S}^{d-1}} p(v) = p(v_0)$ for some $v_0 \in \mathbb{S}^{d-1}$. Then $\alpha_m(v_0) \neq 0$ and, for all $v \in \mathbb{S}^{d-1}$, the equality

$$f_{d,v}(s) = f_{d,v}(0) + \frac{f_{d,v}^{(2m)}(0)}{(2m)!} s^{2m} + \mathrm{o}(s^{2m}) \qquad \text{as } s \to 0$$

holds. Obviously, $(-1)^k \alpha_k(v) = f_{d,v}^{(2k)}(0) = 0$ for all $1 \leq k < m$ (if $m \geq 2$), and $(-1)^m \alpha_m(v) = f_{d,v}^{(2m)}(0) \leq 0$ (otherwise the function $f_{d,v}(s)$ is strongly increasing on $[0, c]$ for some $c > 0$, which would contradict 2). From (3.3), it follows that $\beta_k = 0$ for all $1 \leq k < m$ (if $m \geq 2$) and $(-1)^m \beta_m < 0$. Therefore,

$$m = \min\left\{ k \in \mathbb{N} : \beta_k = \int_{\mathbb{R}^d} \varphi(\rho^2(y)) \|y\|^{2k} \, \mathrm{d}y \neq 0 \right\}$$

and this completes the proof. □

We are now able to give a simple proof of Theorem 2.1.

**Proof of Theorem 2.1.** If $h(t) \equiv h(0) > 0$ on $E$, then the implication (1) $\Rightarrow$ (2) is obvious. If $h(t) \not\equiv h(0)$ on $E$, then this implication follows from statement 4 of Theorem 3.1 for the choice $\varphi(s) = \mathrm{e}^{-s} \in C([0, +\infty)) \cap M_{(0,\infty)}$.

The reverse implication (2) $\Rightarrow$ (1) follows from Lemma 2.2 with the choice $\varphi(s) = \mathrm{e}^{-s}$, from the equality

$$\int_{\mathbb{R}^d} \mathrm{e}^{-1/(2\sigma)\|y\|^2} \mathrm{e}^{\mathrm{i}(y,v)} \, \mathrm{d}y = (2\pi\sigma)^{d/2} \mathrm{e}^{-\sigma/2\|v\|^2}, \qquad v \in \mathbb{R}^d, \sigma > 0,$$

and from the Bernstein–Widder theorem. □

The next theorem is an addition to Theorem 3.1 for the special case $\rho(x) = \|x\|$, that is, when $\rho$ is the Euclidean norm. If $f(x) = \varphi(\|x\|^2)$, $\varphi \in C([0,+\infty))$, $f \in L(\mathbb{R}^n)$, then the Fourier transform above simplifies to the Bessel integral

$$\widehat{f}(u) = (2\pi)^{n/2} g_n(\|u\|), \qquad \text{where } g_n(s) := \int_0^\infty \varphi(u^2) u^{n-1} j_{n/2-1}(su)\,\mathrm{d}u \tag{3.4}$$

and $j_\lambda(u) := u^{-\lambda} J_\lambda(u)$ with $J_\lambda$ a Bessel function of the first kind. In this case, the functions $G_n(\cdot)$ and $g_n(\cdot)$ are related by the known equality $G_n(v) = (2\pi)^{n/2} g_n(\|v\|)$.

**Theorem 3.2.** *Let the following conditions be satisfied*:

(1) $h \in C(E)$, $h(t) > 0$ *for all* $t \in E$ *and* $h(t) \not\equiv h(0)$ *on* $E$;
(2) $\varphi \in C([0,+\infty))$, $\varphi(0) > 0$;
(3) $K(x,t) := (h(t))^{-d/2} \varphi(\frac{\|x\|^2}{h(t)}) \in \Phi(\mathbb{R}^d \times E)$.

*If* $\int_0^\infty |\varphi(u^2)| u^{m-1}\,\mathrm{d}u < \infty$ *for some* $m \in \{1,\dots,d\}$ *and* $g_m$ *is real analytic*, *then the function* $f_m(s) := s^{m-d} g_m(s)$ *is strictly decreasing on* $(0,\infty)$ *and* $g_m(s) > 0$ *for all* $s > 0$.

**Proof.** From Theorem 3.1, we have that $f_m$ decreases on $(0,\infty)$ and $f_m(s) \ge f_m(\infty) = 0$ for $s > 0$. Since $f_m$ is real-analytic on $(0,\infty)$, the function $f_m(s)$ is strictly decreasing on $(0,\infty)$. Otherwise, the function $f_m$ is constant on some open interval $(\alpha,\beta) \subset (0,\infty)$, $\alpha < \beta$, and, hence, it is constant on $(0,\infty)$ and $f_m(s) = f_m(\infty) = 0$, $s > 0$. Therefore, $G_m(v) = (2\pi)^{m/2} g_m(\|v\|) \equiv 0$ on $\mathbb{R}^m$. Hence, $\varphi(\|x\|^2) \equiv 0$ on $\mathbb{R}^m$, which contradicts the condition $\varphi(0) > 0$. Thus, the function $f_m$ is strictly decreasing on $(0,\infty)$ and, hence, $f_m(s) > f_m(\infty) = 0$ for all $s > 0$. □

***Remark 3.1.*** The necessary conditions stated in Theorems 3.1 and 3.2 allow the following hypothesis to be formulated.

*Let the following conditions be satisfied:*

(1) $h \in C(E)$ *and* $h(t) > 0$ *for all* $t \in E$;
(2) $\varphi \in C([0,+\infty))$, $\varphi(0) > 0$ *and* $\varphi$ *has compact support;*
(3) $K(x,t) := (h(t))^{-d/2} \varphi(\frac{\|x\|^2}{h(t)}) \in \Phi(\mathbb{R}^d \times E)$, $d \in \mathbb{N}$.

*We then conjecture that* $h(t) \equiv h(0)$ *on* $E$.

From Theorem 3.2, a weaker version of this hypothesis can be formulated: under the three conditions stated above, and if:

(4) $g_m(s_0) = 0$ for some $m \in \{1,\dots,d\}$ and for some $s_0 > 0$, then we conjecture that $h(t) \equiv h(0)$ on $E$.

Let us assume that $h(t) \not\equiv h(0)$ on $E$. Then (see Theorem 3.2) $g_m(s) > 0$ for all $s > 0$, which contradicts condition (4).

As an example, it is possible to take the function $\varphi(u^2) := (1 - |u|)_+ \in \Phi(\mathbb{R})$, $m = 1$. In this case (see (3.4)), $g_1(s) = \sqrt{\frac{2}{\pi}} \int_0^1 (1-u)\cos(su)\,\mathrm{d}u = \sqrt{\frac{2}{\pi}} \frac{1-\cos s}{s^2}$. Condition (4) is fulfilled for

$m = 1$ and $s_0 = 2\pi$. Therefore, $(h(t))^{-1/2}\varphi(\frac{\|x\|^2}{h(t)}) \in \Phi(\mathbb{R} \times E)$, where $h \in C(E)$ and $h(t) > 0$ for all $t \in E \iff h(t) \equiv h(0)$ on $E$.

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