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Tightness for the interface of the one-dimensional contact process

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We consider a symmetric, finite-range contact process with two types of infection; both have the same (supercritical) infection rate and heal at rate 1, but sites infected by Infection 1 are immune to Infection 2. We take the initial configuration where sites in $(-\infty, 0]$ have Infection 1 and sites in $[1, \infty)$ have Infection 2, then consider the process ρ_t defined as the size of the interface area between the two infections at time t. We show that the distribution of ρ_t is tight, thus proving a conjecture posed by Cox and Durrett in [*Bernoulli* **1** (1995) 343–370].

Keywords: contact process; interfaces

1. Introduction

This paper addresses a conjecture of Cox and Durrett [3] concerning interfaces naturally arising in supercritical contact processes on $\mathbb{Z}^1$.

The contact process on $\mathbb{Z}^d$ is a spin system with operator

$$\Omega f(\eta) = \sum_x \bigl(f(\eta^x) - f(\eta)\bigr)c(x,\eta), \qquad \eta \in \{0,1\}^{\mathbb{Z}^d},$$

where

$$\begin{cases} \eta^x(y) = \eta(y), & \text{if } y \neq x, \\ \eta^x(x) = 1 - \eta(x), & \end{cases}$$

and flip rates $c(x,\eta)$ are given by

$$c(x,\eta) = \begin{cases} 1, & \text{if } \eta(x) = 1, \\ \lambda \sum p(y-x)\eta(y), & \text{if } \eta(x) = 0 \end{cases}$$

for $\lambda > 0$ and probability kernel $p(\cdot)$.

In the following, we take $p(\cdot)$ to have finite range (that is, $\exists M < \infty : p(x) = 0$ for $|x| > M$) and to be symmetric, though this latter hypothesis can be dispensed with via the techniques and results of Bezuidenhout and Gray [1].

Often the contact process is used as a model of the spread of an infection and a configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ represents the state where there is an infection at $x \in \mathbb{Z}^d$ if and only if $\eta(x) = 1$. We will adopt this point of view and speak of a site x being *infected* at time t (for a process $(\eta_t : t \geq 0)$) if $\eta_t(x) = 1$. We will sometimes identify configurations in $\{0,1\}^{\mathbb{Z}^d}$ with their sets of infected sites (that is, we will write ξ instead of $\{x : \xi(x) = 1\}$). As defined above, the contact process is attractive (see [8] for fundamental results associated with this property). Thus, for two configurations ξ_0 and ζ_0 satisfying $\xi_0 \leq \zeta_0$ under the natural partial order, it is possible to construct in a single probability space two processes, $(\xi_t : t \geq 0)$ starting at ξ_0 and $(\zeta_t : t \geq 0)$ starting at ζ_0, satisfying, with probability one, $\xi_t \leq \zeta_t$ for all t.

A consequence is that $\exists \lambda_c^1$ such that for $\lambda > \lambda_c^1$, the invariant limit $\lim_{t\to\infty} \delta_{\mathbb{1}} S(t)$ is a nontrivial measure and for $\lambda < \lambda_c^1$, this limit is $\delta_{\underline{0}}$. There also exists λ_c^2 such that for $\lambda > \lambda_c^2$, $P^{\{0\}}(\tau = \infty) > 0$ for $\tau = \inf\{t : \eta_t \equiv 0\}$, and for $\lambda < \lambda_c^2$, $P^{\{0\}}(\tau = \infty) = 0$. In fact, via duality (see, for example, [5] or [8]), $\lambda_c^1 = \lambda_c^2$, and this critical value will henceforth be denoted by λ_c.

We now introduce some notation. Suppose we are given independent Poisson processes on $[0,\infty)$, $\{D_x\}_{x\in\mathbb{Z}^d}$ of rate 1 and $\{N^{(x,y)}\}_{x,y\in\mathbb{Z}^d}$ of rate $\lambda p(y-x)$. Denote by H a realization of all these independent processes; we say that H is a *Harris construction*. H is thus a Poisson measure on $(\mathbb{Z}^d \cup (\mathbb{Z}^d)^2) \times [0,\infty)$ such that, if $y, z \in \mathbb{Z}^d$ and I is a Borelian subset of $[0,\infty)$, we have $H(\{z\} \times I) = D_z(I)$ and $H(\{(y,z)\} \times I) = N^{(y,z)}(I)$. Given a Harris construction H and $(x,t) \in \mathbb{Z}^d \times [0,\infty)$, denote by $H^{(x,t)}$ the Harris construction obtained by shifting H so that the space origin becomes x and the time origin becomes t. Formally, if $y, z \in \mathbb{Z}^d$ and I is a Borelian subset of $[0,\infty)$, then $H^{(x,t)}(\{z\} \times I) = H(\{z+x\} \times (I+t))$ and $H^{(x,t)}(\{(y,z)\} \times I) = H(\{(y+x, z+x)\} \times (I+t))$.

Given a Harris construction $H = \{(D_x)_{x\in\mathbb{Z}^d}, (N^{(x,y)})_{x,y\in\mathbb{Z}^d}\}$ and $(x,s), (y,t) \in \mathbb{Z}^d \times \mathbb{R}_+$ with $s < t$, we write $(x,s) \leftrightarrow (y,t)$ (in H) if there exists a piecewise constant $\gamma : [s,t] \to \mathbb{Z}^d$ such that:

(i) $\gamma(s) = x, \gamma(t) = y$;
(ii) $\gamma(r) \neq \gamma(r-)$ only if $r \in N^{\gamma(r-),\gamma(r)}$;
(iii) $\nexists s \leq r \leq t$ with $r \in D_{\gamma(r)}$.

Given $A, B, C \subset \mathbb{Z}^d$ and $s, t \in \mathbb{R}_+$, we write $A \times s \leftrightarrow B \times t$ if $(x,s) \leftrightarrow (y,t)$ for some $x \in A$, $y \in B$. Additionally, $A \times \{s\} \leftrightarrow B \times \{t\}$ *inside* C if there exists a path connecting $A \times \{s\}$ and $B \times \{t\}$ and with image contained in C.

Given $\xi_0 \in \{0,1\}^{\mathbb{Z}^d}$ and a Harris construction H, we construct a trajectory $(\eta_t^{\xi_0}(H) : t \geq 0)$ by specifying $\eta_0^{\xi_0}(H) = \xi_0$ and $[\eta_t^{\xi_0}(H)](x) = 1$ if and only if $\xi_0 \times \{0\} \leftrightarrow (x,t)$ in H.

A moment's reflection shows that, under the law of H, $(\eta_t^{\xi_0}(H))_{t\geq 0}$ is a contact process with initial condition ξ_0 and, if $\xi_0 \leq \zeta_0$, then putting $\xi_t = \eta_t^{\xi_0}(H)$ and $\zeta_t = \eta_t^{\zeta_0}(H)$, we obtain the claimed coupling of two processes, one of which is always inferior to the other.

As noted, we will be concerned with one-dimensional contact processes with $\lambda > \lambda_c$. Define $r_t^{\xi_0}(H) = \sup\{x : [\eta_t^{\xi_0}(H)](x) = 1\}$. We will usually omit the dependency on H and when we omit the initial condition and simply write r_t, we take $\xi_0 = I_{(-\infty,0]}$. If ξ_0 is such that $\sum_x \xi_0(x) = \infty$ and $\sup\{x : \xi_0(x) = 1\} < \infty$, then almost surely $\eta_t^{\xi_0} \neq \underline{0}$ and $r_t^{\xi_0} < \infty$ for all t. It is classical that $\frac{r_t}{t} \xrightarrow{t\to\infty} \alpha = \alpha(\lambda) > 0$; see Theorems 2.19 and 2.27 in [8] (even though the process treated there is nearest-neighbor, the proof works for the finite-range case as well).

We consider the following question. Define

$$l_t = l_t(H) = \inf\{x : [\eta_t^{(-\infty,0]}(H)](x) \neq [\eta_t^{\mathbb{1}}(H)](x)\},$$

$$\rho_t = r_t^{(-\infty,0]} - l_t, \qquad \rho_t^+ = \max\{\rho_t, 0\}, \qquad \rho_t^- = \max\{-\rho_t, 0\}.$$

While it is easy to see that $\{r_t < l_t\}$ and $\{l_t < r_t\}$ are events of strictly positive probability, it is reasonable to believe that the two quantities are close. Cox and Durrett conjectured that $\{|\rho_t|\}_{t\geq 0}$ would be a tight collection of random variables. We answer the conjecture affirmatively.

Theorem 1.1. *The law of $\{\rho_t\}_{t\geq 0}$ is tight. That is, for any $\delta > 0$, there exists $L > 0$ such that $\mathbb{P}(|\rho_t| > L) < \delta$ for every $t \geq 0$.*

From the joint process $((\eta_t^{\mathbb{1}}, \eta_t^{(-\infty,0]}) : t \geq 0)$, we can define a process $(\chi_t : t \geq 0)$ on $\{0, 1, 2\}^{\mathbb{Z}}$ by

$$\chi_t(x) = \begin{cases} 0, & \text{if } \eta_t^{(-\infty,0]}(x) = \eta_t^{\mathbb{1}}(x) = 0, \\ 1, & \text{if } \eta_t^{(-\infty,0]}(x) = 1, \\ 2, & \text{if } \eta_t^{(-\infty,0]}(x) = 0, \eta_t^{\mathbb{1}}(x) = 1. \end{cases}$$

It is not difficult to see that χ_t is a realization of a process taking values in $\{0, 1, 2\}^{\mathbb{Z}}$ with initial configuration equal to $I_{(-\infty,0]} + 2 \cdot I_{(0,\infty)}$ and the following rates:

$$\begin{aligned}
&0 \to 1 && \text{at rate } \lambda \sum p(y-x) I_{\chi(y)=1}; \\
&0 \to 2 && \text{at rate } \lambda \sum p(y-x) I_{\chi(y)=2}; \\
&2 \to 0 && \text{at rate } 1; \\
&1 \to 0 && \text{at rate } 1; \\
&2 \to 1 && \text{at rate } \lambda \sum p(y-x) I_{\chi(y)=1}.
\end{aligned}$$

The particle system with the above transition rates is a model for hierarchical competition considered in [6] and [7]; the following interpretation is provided. Sites in state 0 are said to contain grass, in state 1 to contain trees and in state 2 to contain bushes. When trees attempt to occupy new territory, they are able to displace bushes, but bushes cannot displace trees. Since, in our case, we take the initial configuration $I_{(-\infty,0]} + 2 \cdot I_{(0,\infty)}$, we expect the area taken by trees to grow to the right towards the area originally taken by bushes. However, since we allow for non-nearest-neighbor interactions, we may observe a mixed area where the two coexist. With the above notation, this area appears when $\rho_t > 0$. Alternatively, it may happen that there is no mixed area and a gap of grass appears between the two homogeneous zones (in the case $\rho_t < 0$). Theorem 1.1 states that with large probability, and uniformly in time, neither the mixed nor the intermediate grass area is too large.

The proof is divided into two parts. The first part, namely the proof of tightness of $\{\rho_t^+\}$, is given at the end of Section 2. The key ingredients are the celebrated result of Bezuidenhout and Grimmett [2], the renormalization arguments employed by, among others, Durrett (see [5]) and the construction carried out in [10]. These permit us to argue that from a single (x,t) with $\eta_t^{(-\infty,0]}(x)=1$, there will be positive probability that inside a cone $C_{x,t}=\{(y,s):|y-x|\le\beta(s-t)\}$, $\eta^{\mathbb{1}}$ and $\eta^{(-\infty,0]}$ are equal. In Section 3, a much simpler argument is employed to establish tightness of $\{\rho_t^-\}$.

2. Tightness of $\{\rho_t^+\}$

2.1. Right edge speed

Given $\gamma>0$, we say that $(0,0)\in\mathbb{Z}\times[0,\infty)$ is *γ-slow up to time T* if $r_t\le\gamma t\,\forall t\le T$. If this is satisfied for all T, then we say that $(0,0)$ is *γ-slow*.

Lemma 2.1. (i) *For any $\varepsilon>0$, there exists $\gamma>0$ such that $\mathbb{P}((0,0)$ is γ-slow$)>1-\varepsilon$.*
(ii) *For any $\gamma>\alpha$, we have*

$$\mathbb{P}((0,0)\text{ is }\gamma\text{-slow})>0 \tag{2.1}$$

and there exist $c,C>0$ such that

$$\mathbb{P}((0,0)\text{ is }\gamma\text{-slow up to time }T\text{ but not }\gamma\text{-slow})\le C\mathrm{e}^{-cT}. \tag{2.2}$$

Proof. Almost surely, $t\mapsto r_t$ is right-continuous with left limits, identically zero in a neighborhood of 0 and satisfies $r_t/t\to\alpha$. It follows that almost surely, $\{r_t/t:t\ge0\}$ is bounded, hence we have (i). It also follows that, given $\gamma>\alpha$, we can obtain $R>0$ such that $\mathbb{P}(r_t/t<R/t+\gamma\,\forall t)>0$. Now,

$$\mathbb{P}((0,0)\text{ is }\gamma\text{-slow})\ge\mathbb{P}\big(r_t\le0\,\forall t\in[0,R/\gamma],r_s^{(-\infty,0]}\big(H^{(r_{R/\gamma},R/\gamma)}\big)<R+\gamma s\,\forall s\ge0\big).$$

The first event on the above probability depends only on the Harris construction H on $[0,R/\gamma]$, whereas the second depends only on H on $[R/\gamma,+\infty)$, so they are independent. Also noting that $\mathbb{P}(r_s^{(-\infty,0]}(H^{(r_{R/\gamma},R/\gamma)})<R+\gamma s\,\forall s\ge0)=\mathbb{P}(r_s<R+\gamma s\,\forall s\ge0)$, we get, by translation invariance,

$$\mathbb{P}((0,0)\text{ is }\gamma\text{-slow})\ge\mathbb{P}(r_t\le0\,\forall t\in[0,R/\gamma])\cdot\mathbb{P}(r_s<R+\gamma s\,\forall s\ge0).$$

The second probability above is positive by our choice of R. The first one is also positive because it contains the event $\{(-\infty,0]\times[0,R/\gamma]\nleftrightarrow(0,+\infty)\times[0,R/\gamma]\}$, which has positive probability since it corresponds to a finite number of Poisson processes having no arrivals in a finite time interval. We thus have (2.1).

To establish (2.2), fix $\gamma'\in(\alpha,\gamma)$ and note that

$$\begin{aligned}&\mathbb{P}(r_t\le\gamma t\text{ for all }t\in[0,T]\text{ but not for all }t\ge0)\\&\quad\le\mathbb{P}(\exists t>T:r_t>\gamma t)\le\mathbb{P}(\exists t>T:r_t>\gamma t,r_T\le\gamma'T)+\mathbb{P}(r_T>\gamma'T).\end{aligned}$$

By Lemma 2 in [10] (a large deviations result for r_t), $\gamma' > \alpha$ implies that the second term in the sum decays exponentially fast in T and, by translation invariance, the first term is less than $\mathbb{P}(\exists s > 0 : r_s > (\gamma - \gamma')T + \gamma s)$. It will therefore suffice to prove that $\mathbb{P}(\exists s > 0 : r_s > k + \gamma s)$ decays exponentially fast as k tends to infinity. Indeed, put $\theta = \mathbb{P}(\exists t > 0 : r_t \geq M + \gamma t)$ (remember that M is the range of the process) and $T_N = \inf\{t \geq 0 : r_t \geq 2MN + \gamma t\}$ for $N \geq 1$. We have $\theta < 1$ by (2.1) and

$$\begin{aligned}\mathbb{P}(T_{N+1} < \infty) &= \mathbb{P}\big(\exists t > 0 : r_t \geq 2M(N+1) + \gamma t\big) \\ &\leq \mathbb{P}\big(T_N < \infty, \exists s > 0 : r_s^{(-\infty,0]}\big(H^{(r_{T_N}, T_N)}\big) \geq M + \gamma s\big) \\ &= \mathbb{P}(T_N < \infty) \cdot \mathbb{P}(\exists s > 0 : r_s > M + \gamma s) = \mathbb{P}(T_N < \infty) \cdot \theta.\end{aligned}$$

Thus $\mathbb{P}(T_N < \infty) \leq \theta^N$. Now, if $k \geq 1$, then

$$\mathbb{P}(\exists s > 0 : r_s > k + \gamma s) \leq \mathbb{P}(\exists s > 0 : r_s \geq 2M\sigma + \gamma s) \leq \mathbb{P}(T_\sigma < \infty) \leq \theta^\sigma,$$

where σ denotes the largest integer strictly smaller than $k/2M$. □

2.2. Descendancy barriers

In this section, we define an event called the *formation of a descendancy barrier*. This will mean that, inside a certain area delimited by a vertical cone that grows upward from the origin, all infected sites will be connected to the origin. Additionally, no infection from one side of the cone will be able to pass to the other side without being connected to the origin. These barriers, which appear with positive probability, as we will show, are the essential structure in our proof of tightness of $\{\rho_t^+\}$.

We first give a brief exposition of oriented percolation and state a result that will be needed later. For a detailed treatment of the subject, see the survey [4].

Let $\Lambda = \{(m,n) \in \mathbb{Z} \times \mathbb{Z}_+ : m + n \text{ is even}\}$, $\Omega = \{0,1\}^\Lambda$ and $\mathcal{F}$ be the σ-algebra generated by cylinder sets of Ω. Points of Ω will be denoted by Ψ, with $\Psi(m,n) \in \{0,1\}$ for $(m,n) \in \Lambda$. $\mathbb{P}_p$ will denote the product measure $(p\delta_1 + (1-p)\delta_0)^{\otimes\Lambda}$. The vertical axis of Λ will be interpreted as time.

Given $k \geq 1$, $\varepsilon > 0$ and a probability $\mathbb{P}$ on $\mathcal{F}$, we say that $(\Omega, \mathcal{F}, \mathbb{P})$ is a k-dependent oriented percolation system with closure below ε if

$$\mathbb{P}\big(\Psi(m_i, n) = 0, 1 \leq i \leq r \mid \{\Psi(m,s) : 1 \leq s < n, (m,s) \in \Lambda\}\big) < \varepsilon^r, \tag{2.3}$$

where $r \geq 1$, $(m_i, n) \in \Lambda\ \forall i$ and $|m_{i_1} - m_{i_2}| > 2k$ when $i_1 \neq i_2$.

Given $\Psi \in \Omega$, we say that two points $(x,m), (y,n) \in \Lambda$ with $m < n$ are *connected by an open path* if there exists a sequence $x_0 = x, x_1, \ldots, x_{n-m} = y$ in $\mathbb{Z}$ such that $|x_{i+1} - x_i| = 1\ \forall i \in \{0, \ldots, n-m-1\}$ and $\Psi(x_i, m+i) = 1\ \forall i \in \{0, \ldots, n-m\}$. We say that (x,m) *percolates up to time n* when it is connected by an open path to a point at height n. Finally, we say that (x,m) *percolates* when there is an infinite open path starting from it.

In [4], it is proved that if p is sufficiently large, then the origin percolates with positive probability in $(\Omega, \mathcal{F}, \mathbb{P}_p)$. Moreover, the rightmost particle connected to the origin at time n, denoted R_n, almost surely satisfies $\lim R_n/n = \tilde{\alpha}(p) > 0$ as $n \to \infty$. To obtain similar results for k-dependent systems, we use the following particular case of Theorem 0.0 in [9].

Lemma 2.2. *Fix $k \in \mathbb{N}$ and $0 < p < 1$. There exists $\varepsilon > 0$ such that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a k-dependent oriented percolation system with closure below ε, then $\mathbb{P}$ stochastically dominates $\mathbb{P}_p$.*

Using these facts and an argument similar to the one used in Lemma 2.1, we can prove the following lemma.

Lemma 2.3. *Fix an arbitrary $0 < \beta < 1$ and define the events*

$$\Gamma(i) = \left\{ \begin{array}{c} \textit{There exist two open paths, one starting at } (-2,0), \textit{ the other at } (2,0) \\ \textit{and both reaching time } i. \textit{ Neither of them intersects } \{(m,n) : -\beta n \le m \le \beta n\} \end{array} \right\},$$
$$\Gamma = \left\{ \begin{array}{c} \textit{There exist two infinite open paths, one starting at } (-2,0) \\ \textit{and the other at } (2,0). \textit{ Neither of them intersects } \{(m,n) : -\beta n \le m \le \beta n\} \end{array} \right\}. \tag{2.4}$$

For any k and $\bar{\delta} > 0$, there exists $\varepsilon > 0$ such that if $(\Omega, \mathcal{F}, \mathbb{P})$ is a k-dependent percolation system with closure below ε, then:

(i) $\mathbb{P}(\Gamma) > 1 - \bar{\delta}$;
(ii) $\mathbb{P}(\Gamma(i) \backslash \Gamma) \le D\mathrm{e}^{-di}$ *for some $d, D > 0$.*

We now construct a mapping $H \mapsto \Psi_H$ of Harris constructions into points of Ω; this is essentially a repetition of the mapping developed in [10]. The construction will depend on large integers K and N (in particular, much larger than the range M) whose choice will be described in Proposition 2.4. Given $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$, define

$$I_m = \left(\frac{mN}{2} - \frac{N}{2}, \frac{mN}{2} + \frac{N}{2} \right] \cap \mathbb{Z},$$
$$J_{(m,n)} = \left[\frac{mN}{2} - M, \frac{mN}{2} + M \right] \times [KNn, KN(n+1)] \cap \mathbb{Z} \times [0, +\infty). \tag{2.5}$$

We start defining an auxiliary $\Phi_H \in \{0,1,2\}^{\Lambda}$. Given $(m,0) \in \Lambda$, put $\Phi_H(m,0) = 1$ if H and the trajectory $\eta^{\mathbb{1}}(H)$ satisfy the following conditions:

$$\text{there is no vacant interval at time } KN \text{ of length } N^{1/2} \text{ inside } I_{m-1} \cup I_{m+1}; \tag{2.6}$$

$$\text{every occupied site in } I_{m-1} \cup I_{m+1} \text{ at time } KN \text{ is a descendant of } I_m \times \{0\}; \tag{2.7}$$

$$\begin{array}{l} \text{there does not exist } (z,s) \in J_{(m,0)} \text{ such that} \\ I_m \times \{0\} \not\leftrightarrow (z,s) \text{ and } (I_m^C \times [0,s]) \leftrightarrow (z,s); \end{array} \tag{2.8}$$

put $\Phi_H(m,0)=0$ otherwise. Given $(m,n)\in\Lambda$ with $n\geq 1$, put $\Phi_H(m,n)=1$ if

$$1\in\{\Phi_H(m-1,n-1),\Phi_H(m+1,n-1)\};\tag{2.9}$$

$$\begin{aligned}&\text{there is no vacant interval at time } KN(n+1)\\ &\text{of length } N^{1/2} \text{ inside } I_{m-1}\cup I_{m+1};\end{aligned}\tag{2.10}$$

$$\begin{aligned}&\text{every occupied site in } I_{m-1}\cup I_{m+1} \text{ at time } KN(n+1)\\ &\text{is a descendant of } (I_m\cap\eta^{\mathbb{1}}_{KNn})\times KNn;\end{aligned}\tag{2.11}$$

$$\begin{aligned}&\text{there does not exist } (z,s)\in J_{(m,n)} \text{ such that}\\ &\big((I_m\cap\eta^{\mathbb{1}}_{KNn})\times KNn\big)\not\leftrightarrow(z,s) \text{ and } (I_m^C\times[KNn,s])\leftrightarrow(z,s).\end{aligned}\tag{2.12}$$

If (2.9) fails, put $\Phi_H(m,n)=2$, and in every other case, put $\Phi_H(m,n)=0$. Finally, set

$$\Psi_H(m,n)=\begin{cases}0, & \text{if } \Phi_H(m,n)=0,\\ 1, & \text{otherwise.}\end{cases}$$

Note that, with this construction, if there is an infinite open path $\{(m_i,n_i)\}_{i\geq 0}$ leaving the origin in Ψ_H, we must have $\Phi_H(m_i,n_i)=1$ for every i.

We now have the following proposition.

Proposition 2.4 (Mountford and Sweet [10]). *There exist k, K – depending only on the parameter λ of the contact process – with the following property: for any $\varepsilon>0$, there exists N such that Ψ_H defined from K and N is a k-dependent percolation system with closure below ε.*

Remark 2.5. Conditions (2.6) and (2.10) are only necessary to establish Proposition 2.4 and will not be used in the sequel. Also, N in Proposition 2.4 can be chosen as large as we want; in particular, as already mentioned, we take both K and N to be larger than the range M.

In what follows, the oriented percolation dependency parameter k, the constant β and associated events $\Gamma,\Gamma(i)$, the renormalization constants N, K and the closure density ε will be fixed in the following way:

- K and k are functions of λ, as explained in the last proposition above;
- β will be any fixed number in $(0,1)$;
- Γ and $\Gamma(i)$ will be defined from β, as in (2.4);
- $\delta>0$ will be given during the proof of Theorem 1.1;
- ε will be chosen corresponding to $\bar{\delta}=\delta/6$, k, β, as in Lemma 2.3;
- N will be chosen corresponding to ε, as in Proposition 2.4.

Introducing some more terminology, we call the origin *β-expanding* when:

$$\text{If } x \in \mathbb{Z},\ y \in I_{-2} \cup I_0 \cup I_2,\ x \neq y,\ t \leq 1 \text{ and } (x,0) \leftrightarrow (y,t), \text{ then } (0,0) \leftrightarrow (y,t); \tag{2.13}$$

$$D_0 \cap [0,1] = \varnothing; \tag{2.14}$$

$$(0,0) \leftrightarrow (z,1)\ \forall z \in I_{-2} \cup I_0 \cup I_2; \tag{2.15}$$

$$\Psi_{H^{(0,1)}} \in \Gamma. \tag{2.16}$$

Condition (2.13) means that whenever an infection is transmitted to a site in $I_{-2} \cup I_0 \cup I_2$ before time 1, there must exist an earlier/simultaneous (possibly indirect) transmission from $(0,0)$ to the same site. Condition (2.14) means that there is no healing at $\{0\} \times [0,1]$. Condition (2.15) means that at time 1, every site in $I_{-2} \cup I_0 \cup I_2$ carries an infection that descends from the origin. Condition (2.16) states that the percolation structure defined after placing the origin at $(0,1)$ has the properties defined in (2.4). The β dependency is in the third event since Γ depends on β, and also in the choice of the parameters of the renormalization.

We say that $(0,0)$ is *β-expanding up to a time $T > 1$* when (2.13)–(2.15) are satisfied and $\Psi_{H^{(0,1)}} \in \Gamma(i)$, where i satisfies $T \in (1 + KN(i-1), 1 + KNi]$. We then have the following lemma.

Lemma 2.6. (i) $\mathbb{P}((0,0)$ *is β-expanding*$) > 0$.

(ii) $\mathbb{P}((0,0)$ *is β-expanding up to time T, but not β-expanding*$) \leq \bar{D}\mathrm{e}^{-\bar{d}T}$ *for some* $\bar{d}, \bar{D} > 0$.

Proof. It is clear that with positive probability, (2.13)–(2.15) happen simultaneously. Also, they are independent of (2.16), which, in turn, has positive probability, by Lemma 2.3, since Ψ_H is supercritical. Hence, the origin has positive probability of being β-expanding, proving (i). Now, note that

$$\begin{aligned} &\{(0,0) \text{ is } \beta\text{-expanding up to time } T, \text{ but not } \beta\text{-expanding}\} \\ &\quad \subset \bigl\{\Psi_{H^{(0,1)}} \in \Gamma\bigl(\lfloor (T-1)/KN \rfloor\bigr) \backslash \Gamma\bigr\}, \end{aligned}$$

where $\lfloor x \rfloor$ denotes the integer part of x. The probability of the last event in the above expression is bounded by $D\mathrm{e}^{-d(\lfloor (T-1)/KN \rfloor)}$, by Lemma 2.3, so we have (ii). □

Let us now present the properties that motivated this construction. We start defining, for $\rho > 0$,

$$V(\rho) = \{(z,s) \in \mathbb{Z} \times [0,+\infty) : -\rho s \leq z \leq \rho s\}.$$

We then have the following proposition.

Proposition 2.7. *Suppose that the origin is β-expanding. There then exists a* (*deterministic*) $0 < \bar{\beta} < 1$ *with the following three properties*:

(i) *if* $x, z \in \mathbb{Z}$, $(x,0) \leftrightarrow (z,s)$ *and* $(z,s) \in V(\bar{\beta})$, *then* $(0,0) \leftrightarrow (z,s)$;

(ii) $r_s^0 \geq \begin{cases} \bar{\beta}s, \textit{ if } s \geq 1 \\ 0, \textit{ if } s < 1 \end{cases} \geq \max\{0, \bar{\beta}s - 1\}\ \forall s \geq 0;$

(iii) *if* $x, z \in \mathbb{Z}$ *have different signs and* $(x, 0) \leftrightarrow (z, s)$, *then* $(0, 0) \leftrightarrow (z, s)$.

Proof. If $s \leq 1$ in parts (i), (ii) or (iii), then the statements hold for any $\bar{\beta} < 1$, by (2.13) and (2.14). Hence, from now on, we assume that $s > 1$ in all three parts. Suppose that the origin is β-expanding. Since $\Psi_{H^{(0,1)}} \in \Gamma$, there exist sequences $\{m_n^r\}_{n\geq 0}$, $\{m_n^l\}_{n\geq 0}$ in $\mathbb{Z}$ such that

$$\begin{aligned} m_0^l = -2, \qquad m_0^r = 2, \\ |m_{n+1}^l - m_n^l| = |m_{n+1}^r - m_n^r| = 1, \\ \Psi_{H^{(0,1)}}(m_n^l, n) = \Psi_{H^{(0,1)}}(m_n^r, n) = 1, \\ m_n^l < -\beta n < \beta n < m_n^r, \qquad n \geq 0. \end{aligned} \tag{2.17}$$

Define

$$B^l = \bigcup_{n=0}^{\infty} \bigl[(I_{m_n^l} \times KNn) \cup J_{(m_n^l,n)}\bigr], \qquad B^r = \bigcup_{n=0}^{\infty} \bigl[(I_{m_n^r} \times KNn) \cup J_{(m_n^r,n)}\bigr],$$
$$B = B^l \cup B^r \cup (I_0 \times \{0\}).$$

B^l is a union of horizontal lines (the "$I_m \times KNn$"'s), one for each height level KNn, and rectangles of base $2M$ and height KN (the "$J_{(m,n)}$"'s); each rectangle connects a pair of horizontal lines. B^l is thus a connected subset of $\mathbb{R} \times [0, +\infty)$. The same can be said about B^r. So, B is also connected and its complement in $\mathbb{R} \times [0, +\infty)$ has two connected components, which will be referred to as "above" and "below" B. Also, note that since $N > 2M$, $\forall (x, t) \in B$, we either have $[x - M, x] \times \{t\} \subset B$ or $[x, x + M] \times \{t\} \subset B$. In other words, the three sets whose union defines B (B^l, B^r and $I_0 \times \{0\}$) have width larger than M at any time level.

Putting together (2.7), (2.8), (2.11), (2.12) and the three first conditions in (2.17), we can conclude that in the trajectory $\eta^{\mathbb{1}}(H^{(0,1)})$, every infected site in $(0, 1) + B := \{(z, 1 + s) : (z, s) \in B\}$ descends from $(I_{-2} \cup I_0 \cup I_2) \times \{1\}$. Then, because of (2.15), in the trajectory $\eta^{\mathbb{1}}(H)$, every infected site in $(0, 1) + B$ descends from $(0, 0)$.

It follows from the last condition of (2.17) that there exists $0 < \bar{\beta} < 1$ such that $V(\bar{\beta})$ is contained in the union of $(I_{-2} \cup I_0 \cup I_2) \times [0, 1]$ and the area above $(0, 1) + B$.

Now, take x and z as in (i). Since $s > 1$ and $(z, s) \in V(\bar{\beta})$, (z, s) must be above $(0, 1) + B$. So, any path starting from $(x, 0)$ and reaching (z, s) must have a point $(y, t) \in (0, 1) + B$ and thus, as we have seen, it must be the case that $(0, 0) \leftrightarrow (y, t) \leftrightarrow (z, s)$.

Part (ii) follows from the facts that for any $s > 1$, $(\bar{\beta}s, s)$ is to the left of $(0, 1) + B^r$, and that $\eta_s^0 \cap \{x : (x, s) \in B^r\} \neq \varnothing$.

Finally, take x, z as in (iii) and let ζ be the path linking $(x, 0)$ and (z, s). We separately consider the two cases: there exist $y \neq x$ and $t < 1$ such that $(y, t) \in \zeta$ or not. In the first case, (iii) follows from (2.13). In the second case, noting that x and z have different signs and ζ has horizontal displacements of size at most M, and using our remarks about B being connected and its width being larger than M at any time level, we conclude that $(\gamma(t), t) \in (0, 1) + B$ for some $t \in [0, s]$. (iii) then follows from the fact that any infection in $(0, 1) + B$ descends from $(0, 0)$. $\square$

2.3. Proof of tightness of $\{\rho_t^+\}$

Call the origin (β,γ)*-good up to time* T (resp., (β,γ)*-good*) when it is both β-expanding and γ-slow up to time T (resp., β-expanding and γ-slow). Additionally, call a point (x,t) β*-expanding*, γ*-slow* or (β,γ)*-good* when $(0,0)$ has the corresponding property on $H^{(x,t)}$.

Lemma 2.8. *For $\gamma > 0$ sufficiently large, we have*:

(i) $\mathbb{P}((0,0)$ *is* (β,γ)*-good*$) > 0$;

(ii) $\mathbb{P}((0,0)$ *is* (β,γ)*-good up to time* T *but not* (β,γ)*-good*$) \leq F\mathrm{e}^{-fT}$ *for some* $f, F > 0$;

(iii) *given* $0 \leq a < b$, $\mathbb{P}((r_t,t)$ *is not* (β,γ)*-good for any* $t \in [a,b]) \leq G\mathrm{e}^{-g\sqrt{b-a}}$ *for some* $g, G > 0$ *not depending on* a, b.

Proof. The only point that does not follow directly from Lemmas 2.1 and 2.6 is (iii). We start proving the result when $a = 0$. Given a Harris construction H, define $\mu(H) = \sup\{t \geq 0 : (0,0)$ is (β,γ)-good up to time t in $H\}$,

$$\sigma_1(H) = \begin{cases} 1, & \text{if } \mu(H) < 1, \\ KN(n+1)+1, & \text{if } \mu(H) \in \bigl[KNn+1, KN(n+1)+1\bigr), \\ +\infty, & \text{if } \mu(H) = +\infty, \end{cases}$$

$\sigma_0(H) = 0$ and $\sigma_{i+1}(H) = \sigma_i(H) + \sigma(H^{(r_{\sigma_i(H)},\sigma_i(H))})$ if $i \geq 1$ and $\sigma_i(H) < +\infty$. (The r_{σ_i} that appears is defined with respect to the original trajectory $\eta^{(-\infty,0]}(H)$, with no change of coordinates.) Each σ_i is a stopping time for the process $t \mapsto H_t$. It follows from the strong Markov property and translation invariance of the law of H that the law of $H^{(r_{\sigma_i},\sigma_i)}$ conditioned to $\{\sigma_i < +\infty\}$ is the same as that of H. In particular, conditioned on $\{\sigma_i < +\infty\}$, $\sigma_{i+1} - \sigma_i$ has the law of σ_1, which satisfies:

- $\mathbb{P}(\sigma_1 = +\infty) \equiv \theta > 0$, by (i);
- $\mathbb{P}(T < \sigma_1 < +\infty) < \bar{F}\mathrm{e}^{-\bar{f}T}$ for some $\bar{f}, \bar{F} > 0$, by (ii).

Let $\tau = \inf\{s : (r_s, s)$ is (β,γ)-good$\}$. Now, if i_0 is the first i such that $\sigma_{i+1} = +\infty$, we have $\tau \leq \sigma_{i_0}$ and

$$\begin{aligned} \mathbb{P}(\tau > b) &\leq \mathbb{P}(\sigma_{i_0} > b) \leq \mathbb{P}\bigl(i_0 > \sqrt{b}\bigr) + \mathbb{P}\bigl(i_0 \leq \sqrt{b}, \sigma_{i_0} > b\bigr) \\ &\leq (1-\theta)^{\sqrt{b}} + \mathbb{P}\bigl(i_0 \leq \sqrt{b}, \sigma_{j+1} - \sigma_j > \sqrt{b} \text{ for some } 1 \leq j \leq i_0\bigr) \\ &\leq (1-\theta)^{\sqrt{b}} + \sqrt{b} \cdot \bar{F}\mathrm{e}^{-\bar{f}\sqrt{b}} \leq G\mathrm{e}^{-g\sqrt{b}} \end{aligned}$$

for some suitably chosen g, G.

For $a > 0$, repeat the proof starting from (r_a, a) instead of $(0,0)$ and note that the constants $\bar{f}$ and $\bar{F}$ do not depend on a. $\square$

Proof of Theorem 1.1 (First part). Fix $\delta > 0$. This is the δ that takes part in our renormalization construction, as mentioned in the paragraph after Proposition 2.4. We want to prove that for any T, $\rho_T < L$ with probability larger than $1-\delta$. To this end, we will proceed in two steps. First, we will define a "good event" depending on T, $\mathcal{H}(T)$, with $\mathbb{P}(\mathcal{H}(T)) > 1-\delta$. We will then choose $L > 0$ and see that in $\mathcal{H}(T)$, every infection in $\eta_T^{\mathbb{1}}$ that is to the left of $r_T - L$ must descend from $(-\infty, 0] \times 0$.

(A) *Choice of the good event.* By Lemma 2.1(i), we can choose $\gamma > 0$ such that the event

$$\mathcal{H}_1 = \{(0,0) \text{ is } \gamma\text{-slow}\}$$

has probability larger than $1-\delta/3$. We can also assume that γ satisfies (iii) in Lemma 2.8.

We can choose $S > 0$ such that

$$\big\{\exists x \in [-S, 0] \text{ such that } H^{(x,0)} \text{ satisfies (2.13)–(2.15)}\big\}$$

has probability larger than $1-\delta/6$; note that this event depends only on the Harris construction on the time interval $[0,1]$. Also, for any x, we have $\mathbb{P}(\Psi_{H^{(x,1)}} \in \Gamma) = \mathbb{P}(\Psi_H \in \Gamma) > 1-\delta/6$, by our choice of ε (see the remark after Proposition 2.4); for any x, this event depends only on the Harris construction on the time interval $[1, +\infty)$ and is thus independent of the former event. Therefore, putting

$$\mathcal{H}_2 = \{\text{there exists } x \in [-S, 0] \text{ such that } (x, 0) \text{ is } \beta\text{-expanding}\},$$

we have $\mathbb{P}(\mathcal{H}_2) > 1-\delta/3$.

Now, choose $R > 0$ such that $\sum_{n=1}^{\infty} G\mathrm{e}^{-g\sqrt{R+n}} < \delta/3$, where g, G are defined in Lemma 2.8(iii). Given $\bar{R} > 0$, define the time intervals

$$I_0 = [0, \bar{R}], \qquad I_n = (\sup I_{n-1}, \sup I_{n-1} + R + n] \qquad \text{for } n \geq 1,$$

so that $I_n = (\bar{R} + (n-1)R + \frac{n(n-1)}{2}, \bar{R} + nR + \frac{n(n+1)}{2}]$, $|I_n| = R + n$ for $n \geq 1$. We now choose $\bar{R}$ large enough so that

$$\forall n \geq 2, \forall t \in I_{n-1} \qquad \frac{2\bar{\beta}t - S}{\bar{\beta} + \gamma} > |I_{n-1} \cup I_n|. \tag{2.18}$$

Given $T > 0$, define $\bar{n}(T) = \sup\{n \geq 1 : I_n \subset [0, T]\}$; if $I_0 \cup I_1 \nsubseteq [0, T]$, put $\bar{n}(T) = -\infty$. The idea is that, given the time interval $[0, T]$, we will place the intervals I_n from top to bottom, that is, $T - I_0, T - I_1, \ldots,$ up to the last one that fits, which will be $I_{\bar{n}(T)}$. Now, define the event

$$\begin{aligned}\mathcal{H}_3 = \mathcal{H}_3(T) = \{&\text{for each } n \in [1, \bar{n}(T)], \text{ there exists } t \in T - I_n \\ &\text{such that } (r_t, t) \text{ is } (\beta, \gamma)\text{-good}\};\end{aligned}$$

if $\bar{n}(T) = -\infty$, simply take $\mathcal{H}_3$ to be the whole space. Now, as a consequence of Lemma 2.8(iii), we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{H}_3(T)) &\geq 1 - \sum_{n=1}^{\bar{n}(T)} \mathbb{P}\big((t, r_t) \text{ is never } (\beta, \gamma)\text{-good when } t \in T - I_n\big) \\ &\geq 1 - \sum_{n=1}^{\infty} G\mathrm{e}^{-g\sqrt{|I_n|}} = 1 - \sum_{n=1}^{\infty} G\mathrm{e}^{-g\sqrt{R+n}} > 1 - \delta/3.\end{aligned}$$

In conclusion, if $\mathcal{H}(T) = \mathcal{H}_1 \cap \mathcal{H}_2 \cap \mathcal{H}_3(T)$, then $\mathbb{P}(\mathcal{H}(T)) > 1 - \delta$ for any T.

(B) *Choice of L and proof that the interface area is smaller than L in the good event.* Let $L = \gamma(R + \bar{R} + 1) + S$; note that L does not depend on T. We first treat the case $T \le \bar{R} + R + 1$. We might omit it: since $\sup_{t \le T} |\rho_t| < \infty$ almost surely, it suffices to prove its tightness in $[T, +\infty)$ for sufficiently large T. However, we find that this case illustrates the main idea of the proof without the technical complications that appear in the general picture.

Let $V = V(\bar{\beta}) = \{(z, s) \in \mathbb{Z} \times [0, +\infty) : -\bar{\beta}s \le z \le \bar{\beta}s\}$, where $\bar{\beta} > 0$ is such that the conclusion of part (i) of Proposition 2.7 holds. Given $A \subset \mathbb{Z} \times [0, +\infty)$ and $t \ge 0$, define $\Pi_t(A) = \{z : (z, t) \in A\}$.

Fix $H \in \mathcal{H}(T)$. Since $H \in \mathcal{H}_2$, we can take $x \in [-S, 0]$ such that $(x, 0)$ is β-expanding. Also, since $H \in \mathcal{H}_1$, $(0, 0)$ is γ-slow and, in particular, $r_T < \gamma T$. Thus,

$$\begin{aligned} r_T - L < \gamma T - L &\le \gamma(\bar{R} + R + 1) - \gamma(\bar{R} + R + 1) - S \\ &= -S < x + \bar{\beta}T < \sup \Pi_T\big((x, 0) + V\big) + 1; \end{aligned}$$

the $+1$ is required because $x + \bar{\beta}T$ may not be an integer. Assume that for $y > 0$ and w satisfying $r_T - w > L$, we have $(y, 0) \leftrightarrow (w, T)$. Note that $w < r_T - L \le \sup \Pi_T((x, 0) + V)$. If $w \in \Pi_T((x, 0) + V)$, then it follows from Proposition 2.7(i) and translation invariance that $(x, 0) \leftrightarrow (w, T)$. If $w < \inf \Pi_T((x, 0) + V)$, then w and y are in opposite sides of x and it follows from Proposition 2.7(ii) and translation invariance that $(x, 0) \leftrightarrow (w, T)$. This shows that any infection in $(-\infty, r_T - L] \times T$ that descends from $[1, +\infty) \times 0$ must also descend from $(-\infty, 0] \times 0$, completing the proof of this case.

Before starting the other case, we make some trivial remarks. Suppose $(a, s), (b, t) \in \mathbb{Z} \times [0, +\infty)$ are such that $a \le b$ and $s < t$. Let ζ^* be the smallest value of ζ at which $\Pi_\zeta((a, s) + V) \cap \Pi_\zeta((b, t) + V) \ne \varnothing$. ζ^* is either t (in the case $(b, t) \in (a, s) + V$) or the time of intersection of the lines $\zeta \mapsto a + \bar{\beta}(\zeta - s)$ and $\zeta \mapsto b - \bar{\beta}(\zeta - t)$, that is,

$$\zeta^*((a, s), (b, t)) = \max\left\{t, \frac{b - a + \bar{\beta}(t + s)}{2\bar{\beta}}\right\}. \tag{2.19}$$

Also,

$$\zeta > \zeta^*((a, s), (b, t)) \quad \Longrightarrow \quad \Pi_\zeta\big((a, s) + V\big) \cup \Pi_\zeta\big((b, t) + V\big) \text{ is an interval.} \tag{2.20}$$

Now, take $T > \bar{R} + R + 1$ and $H \in \mathcal{H}(T)$. Again, $(0, 0)$ is γ-slow and there exists $x \in [-S, 0]$ such that $(x, 0)$ is β-expanding. Also, since $H \in \mathcal{H}_3(T)$, there exist $t_1 \in T - I_{\bar{n}}, t_2 \in T - I_{\bar{n}-1}, \ldots, t_{\bar{n}} \in T - I_1$ such that (r_{t_i}, t_i) is (β, γ)-good for $i = 1, \ldots, \bar{n}$. Note that since $(0, 0)$ and each (r_{t_i}, t_i) is γ-slow, we have

$$\begin{aligned} r_{t_1} &\le \gamma t_1, \\ r_{t_{n+1}} &\le r_{t_n} + \gamma(t_{n+1} - t_n), \qquad n = 1, \ldots, \bar{n} - 1, \\ r_T &\le r_{t_{\bar{n}}} + \gamma(T - t_{\bar{n}}), \end{aligned} \tag{2.21}$$

and by Proposition 2.7(ii) and translation invariance, we have

$$
\begin{aligned}
&r_{t_1} \geq x, \\
&r_{t_{n+1}} \geq r_{t_n}, \qquad n = 1, \ldots, \bar{n} - 1.
\end{aligned} \tag{2.22}
$$

We claim that the cones $(r_{t_i}, t_i) + V$ each overlap with their neighbors before time T, that is,

$$
\begin{aligned}
&\zeta^*((x, 0), (r_{t_1}, t_1)) < T, \\
&\zeta^*((r_{t_i}, t_i), (r_{t_{i+1}}, t_{i+1})) < T, \qquad i = 1, \ldots, \bar{n} - 1.
\end{aligned} \tag{2.23}
$$

Let us prove the first expression in (2.23). If $(r_{t_1}, t_1) \in (x, 0) + V$, then $\zeta^*((x, 0), (r_{t_1}, t_1)) = t_1 < T$. Assume that $(r_{t_1}, t_1) \notin (x, 0) + V$. Since $-S \leq x < r_{t_1} \leq \gamma t_1$, we have $r_{t_1} - x \leq \gamma t_1 + S$. Also,

$$
0 \in T - I_{\bar{n}+1} \quad \Longrightarrow \quad T \in I_{\bar{n}+1} \quad \overset{(2.18)}{\Longrightarrow} \quad \frac{2\bar{\beta}T - S}{\bar{\beta} + \gamma} > |I_{\bar{n}+1} \cup I_{\bar{n}+2}| > |I_{\bar{n}} \cup I_{\bar{n}+1}|
$$

and since we also have that $t_1 \in T - I_{\bar{n}}$, we obtain $t_1 = t_1 - 0 < |I_{\bar{n}} \cup I_{\bar{n}+1}| < \frac{2\bar{\beta}T - S}{\bar{\beta}+\gamma}$. Putting these inequalities together and using (2.19), we get

$$
\begin{aligned}
\zeta^*((x, 0), (r_{t_1}, t_1)) &= \frac{r_{t_1} - x + \bar{\beta} t_1}{2\bar{\beta}} \leq \frac{\gamma t_1 + S + \bar{\beta} t_1}{2\bar{\beta}} \\
&< \frac{S}{2\bar{\beta}} + \frac{2\bar{\beta}T - S}{\bar{\beta} + \gamma} \cdot \frac{\gamma + \bar{\beta}}{2\bar{\beta}} = T.
\end{aligned}
$$

For the second expression in (2.23), if $(r_{t_{i+1}}, t_{i+1}) \in (r_{t_i}, t_i) + V$, then $\zeta^*((r_{t_i}, t_i), (r_{t_{i+1}}, t_{i+1})) = t_{i+1} < T$. Assume that $(r_{t_{i+1}}, t_{i+1}) \notin (r_{t_i}, t_i) + V$ and write

$$
\begin{aligned}
\zeta^*((r_{t_i}, t_i), (r_{t_{i+1}}, t_{i+1})) &= \frac{r_{t_{i+1}} - r_{t_i} + \bar{\beta}(t_{i+1} + t_i)}{2\bar{\beta}} \leq \frac{\gamma(t_{i+1} - t_i) + \bar{\beta}(t_{i+1} + t_i)}{2\bar{\beta}} \\
&= \frac{\gamma(t_{i+1} - t_i)}{2\bar{\beta}} + \frac{t_{i+1} + t_i}{2}.
\end{aligned}
$$

Since $t_i \in T - I_{\bar{n}-i+1}$ and $t_{i+1} \in T - I_{\bar{n}-i}$, we have $t_{i+1} - t_i \leq |I_{\bar{n}-i} \cup I_{\bar{n}-i+1}| \leq \frac{2\bar{\beta}t - S}{\bar{\beta}+\gamma}$ for any $t \in I_{\bar{n}-i}$, by (2.18). In particular, this holds for $t = T - t_{i+1}$. Therefore,

$$
\begin{aligned}
\zeta^*((r_{t_i}, t_i), (r_{t_{i+1}}, t_{i+1})) &\leq \frac{\gamma}{2\bar{\beta}} \cdot \frac{2\bar{\beta}(T - t_{i+1}) - S}{\bar{\beta} + \gamma} \leq \frac{2\bar{\beta}(T - t_{i+1}) - S}{2\bar{\beta}} + \frac{t_{i+1} + t_i}{2} \\
&\leq T - t_{i+1} + \frac{t_{i+1} + t_i}{2} \leq T.
\end{aligned}
$$

Now, define the union of cones $U = [\bigcup_{n=1}^{\bar{n}}((r_{t_n}, t_n) + V)] \cup [(x, 0) + V]$. Using (2.20) and (2.23), we conclude that $\Pi_T(U)$ is an interval.

Since $t_{\bar{n}} \in T - I_1$, we have $T - t_{\bar{n}} \leq \bar{R} + R + 1$. Also, using the last inequality in (2.21), we obtain

$$r_T < r_{t_{\bar{n}}} + \gamma(T - t_{\bar{n}}) < r_{t_{\bar{n}}} + \gamma(\bar{R} + R + 1),$$

so, using $L = \gamma(\bar{R} + R + 1) + S$, we have

$$r_T - L < r_{t_{\bar{n}}} + \gamma(\bar{R} + R + 1) - \gamma(\bar{R} + R + 1) - S < r_{t_{\bar{n}}} < \sup \Pi_T(U). \tag{2.24}$$

As before, take $y > 0$ and w satisfying $r_T - w > L$ and $(y, 0) \leftrightarrow (w, T)$. Since $w < r_T - L < \sup \Pi_T(U)$ and $\Pi_T(U)$ is an interval, there are two possibilities:

(a) $w \in \Pi_T(U)$
In this case, by the definition of U, we either have $w \in \Pi_T((x, 0) + V)$ (hence $(x, 0) \leftrightarrow (w, T)$, as we already saw) or $w \in \Pi_T((r_{t_i}, t_i) + V)$ for some i. In this last case, there exists z such that $(z, t_i) \leftrightarrow (w, T)$ and hence, by part (i) of Proposition (2.7), $(r_{t_i}, t_i) \leftrightarrow (w, T)$, which implies that $(-\infty, 0] \times \{0\} \leftrightarrow (w, T)$.

(b) $w < \inf \Pi_T(U)$
By the same argument that was used in the case $T < \bar{R} + R + 1$, we have $(x, 0) \leftrightarrow (w, T)$.

In conclusion, in any case, we have $(-\infty, 0] \times 0 \leftrightarrow (w, T)$. We have thus shown that any point (w, T) that is connected to $[1, +\infty)$ but not to $(-\infty, 0]$ must be to the right of $(r_T - L, T)$, that is, that $\rho_T < L$, as required. □

3. Tightness of $\{\rho_t^-\}$

In the following lemma, we will reuse the renormalization structure built in the last section. We fix an arbitrary $\beta \in (0, 1)$ and k, K as in Proposition 2.4, then choose a closure density ε such that the event Γ of Lemma 2.3 has positive probability. Finally, we choose N such that Ψ_H has closure density below ε (again as in Proposition 2.4).

Lemma 3.1. *For any $\sigma > 0$, there exists $L > 0$ such that for any $T > 0$,*

$$\mathbb{P}(\textit{there exists } t \leq T \textit{ such that } r_t > r_T + L) < \sigma. \tag{3.1}$$

Proof. As in the proof of Theorem 1.1, we will define an event $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3(T)$ such that $\mathbb{P}(\mathcal{G}) > 1 - \sigma$ and choose an appropriate $L > 0$; we will then show that in $\mathcal{G}$, we have

$$r_t < r_T + L \qquad \forall t \leq T. \tag{3.2}$$

The first event is the same as before: $\mathcal{G}_1 = \{(0, 0) \text{ is } \gamma\text{-slow}\}$, with γ chosen so that this has probability $> 1 - \sigma/3$ (see Lemma 2.1). Put $\mathcal{G}_2 = \{r_t > -S\, \forall t \geq 0\}$ with $S > 0$ chosen such that this has probability greater than $1 - \sigma/3$; this is possible because $\inf\{r_t : t \geq 0\} > -\infty$ almost surely.

Increasing γ so that the conclusions of Lemma 2.8 hold, we may choose $R > 0$ such that $\sum_{n=0}^{\infty} G\mathrm{e}^{-g\sqrt{R+n}} < \sigma/3$, where g, G are as in part (iii) of Lemma 2.8. We then put

$$I_0 = [0, R), \qquad I_n = [\sup I_{n-1}, \sup I_{n-1} + R + n) \qquad \text{for } n \geq 1,$$

so that $I_n = [nR + \frac{(n-1)n}{2}, (n+1)R + \frac{(n+1)n}{2})$, $|I_n| = R + n$ when $n \geq 0$. We also put $\bar{n}(T) = \sup\{n \geq 0 : I_n \subset [0, T]\}$; if $I_0 \not\subseteq [0, T]$, then put $\bar{n}(T) = -\infty$. Next, define

$$\mathcal{G}_3(T) = \{\text{for each } n \in [0, \bar{n}(T)], \text{ there exists } t \in T - I_n \text{ such that } (r_t, t) \text{ is } (\beta, \gamma)\text{-good}\};$$

if $\bar{n}(T) = -\infty$, take $\mathcal{G}_3$ to be the whole space. By the choice of R and Lemma 2.8(iii), $\mathbb{P}(\mathcal{G}_3(T)) > 1 - \sigma/3$. Thus, $\mathbb{P}(\mathcal{G}) > 1 - \sigma$, as required.

Let us recall that

$$(r_s, s) \text{ is } (\beta, \gamma)\text{-good}, \qquad s' > s \quad \Longrightarrow \quad r_s + \bar{\beta}(s' - s) - 1 \leq r_{s'} \leq r_s + \gamma(s' - s), \tag{3.3}$$

where $\bar{\beta}$ is defined in Proposition 2.7. Choose L such that

$$L \geq \gamma(2R + 1) + S \quad \text{and} \tag{3.4}$$

$$L \geq S + \gamma(2R + 2n + 1) - \bar{\beta}\left(nR + \frac{n(n-1)}{2}\right) + 1 \qquad \forall n \geq 0. \tag{3.5}$$

We proceed to prove that (3.2) is satisfied in $\mathcal{G}$. Fix $0 < t < T$. We deal with three cases:

- $t < T \leq 2R + 1$. Since the origin is γ-slow, we have $r_t \leq \gamma t \leq \gamma(2R + 1)$. Since we are in $\mathcal{G}_2$, we have $r_T > -S$. Therefore, $r_T + L > -S + L \overset{(3.4)}{\geq} \gamma(2R + 1) \geq r_t$.
- $T > 2R + 1$, $t \in (T - I_{\bar{n}}) \cup (T - I_{\bar{n}+1})$ (the point being that t is close to zero, so there does not necessarily exist a (β, γ)-good point below (r_t, t)). By the definition of $\bar{n}$, we have $0 \in I_{\bar{n}+1}$, so $t < |I_{\bar{n}} \cup I_{\bar{n}+1}| = 2R + 2\bar{n} + 1$ and $r_t < \gamma t < \gamma(2R + 2\bar{n} + 1)$. Also, by the definition of $\mathcal{G}_3$, there exists $t^* \in T - I_{\bar{n}}$ such that (r_{t^*}, t^*) is (β, γ)-good. $t^* \in T - I_{\bar{n}}$ implies that $T - t^* \geq \inf I_{\bar{n}} = \bar{n}R + \frac{(\bar{n}-1)\bar{n}}{2}$. We then have

$$\begin{aligned} r_T + L &\overset{(3.3)}{\geq} r_{t^*} + \bar{\beta}(T - t^*) + L - 1 > -S + \bar{\beta}\left(\bar{n}R + \frac{(\bar{n} - 1)\bar{n}}{2}\right) + L - 1 \\ &\overset{(3.5)}{\geq} \gamma(2R + 2\bar{n} + 1) \geq r_t. \end{aligned}$$

- $T > 2R + 1$, $t \in T - I_n$ with $n < \bar{n}$. Here, $n + 1 \leq \bar{n}$, so there exists $t^* \in T - I_{n+1}$ such that (r_{t^*}, t^*) is (β, γ)-good. Note that $t > t^*$, $t - t^* < |I_n \cup I_{n+1}| = 2R + 2n + 1$, so (3.3) gives

$$r_t \leq r_{t^*} + \gamma(t - t^*) \leq r_{t^*} + \gamma(2R + 2n + 1). \tag{3.6}$$

On the other hand, $T - t^* \geq |I_0 \cup \cdots \cup I_n| = (n + 1)R + \frac{(n+1)n}{2}$, so

$$\begin{aligned} r_T + L &\overset{(3.3)}{\geq} r_{t^*} + \bar{\beta}(T - t^*) + L - 1 \geq r_{t^*} + \bar{\beta}\left(nR + \frac{(n + 1)n}{2}\right) + L - 1 \\ &\overset{(3.5)}{\geq} r_{t^*} + \gamma(2R + 2n + 1) \overset{(3.6)}{\geq} r_t. \end{aligned}$$

$\square$

Lemma 3.2. *For any $\sigma > 0$, there exists $L > 0$ such that for any $T > 0$,*

$$\mathbb{P}\big([0, +\infty) \times 0 \leftrightarrow [0, L] \times T \text{ inside } (0, +\infty)\big) > 1 - \sigma. \tag{3.7}$$

This follows from the fact that r_t has positive asymptotic speed and a simple duality argument; we omit the proof.

For $T > 0$, define $q_T = \max\{r_t : 0 \le t \le T\}$. We now proceed to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 (Second part). Fix $\delta > 0$. By Lemmas 3.1 and 3.2, we can obtain $L_1, L_2 > 0$ such that

$$\begin{aligned} \mathbb{P}(q_T \le r_T + L_1) &> \sqrt{1-\delta}, \\ \mathbb{P}\big([0, +\infty) \times 0 \leftrightarrow [0, L_2] \times T \text{ inside } \{(x, t) : x \ge 0\}\big) &> \sqrt{1-\delta}. \end{aligned}$$

Put $L = L_1 + L_2 + M$. For any $T > 0$, we have

$$\begin{aligned} \mathbb{P}(\rho_t \ge -L) &= \mathbb{P}\big((0, +\infty) \times 0 \leftrightarrow (r_T, r_T + L] \times T\big) \\ &\ge \mathbb{P}\big(q_T \le r_T + L_1, [r_T + L_1 + M + 1, +\infty) \times 0 \\ &\qquad \leftrightarrow [r_T + L_1 + M + 1, r_T + L] \times T \text{ inside } [r_T + L_1 + M + 1, +\infty)\big) \\ &= \sum_{x=-L_1}^{+\infty} \mathbb{P}\big(r_T = x, q_T \le x + L_1, [x + L_1 + M + 1, +\infty) \times 0 \\ &\qquad \leftrightarrow [x + L_1 + M + 1, x + L] \times T \text{ inside } [x + L_1 + M + 1, +\infty)\big). \end{aligned}$$

(The sum starts at $-L_1$ because $q_T \ge 0$, so we can only have $q_T \le r_T + L_1$ when $r_T \ge -L_1$.) Now, in each of the above probabilities, the first two events depend on the Harris construction on the set $(-\infty, x + L_1 + M] \times [0, +\infty)$, whereas the third event depends on the Harris construction on $[x + L_1 + M + 1, +\infty) \times [0, +\infty)$. They are thus independent and the sum becomes

$$\begin{aligned} &\sum_{x=-L_1}^{+\infty} \mathbb{P}(r_T = x, q_T \le x + L_1) \cdot \mathbb{P}\big([x + L_1 + M + 1, +\infty) \times 0 \\ &\qquad \leftrightarrow [x + L_1 + M + 1, x + L] \times T \\ &\qquad \text{inside } [x + L_1 + M + 1, +\infty)\big) \\ &= \mathbb{P}\big([0, +\infty) \times 0 \leftrightarrow [0, L_2] \times T \text{ inside } [0, +\infty)\big) \cdot \sum_{x=-L_1}^{+\infty} \mathbb{P}(r_T = x, q_T \le x + L_1) \\ &= \mathbb{P}\big([0, +\infty) \times 0 \leftrightarrow [0, L_2] \times T \text{ inside } [0, +\infty)\big) \cdot \mathbb{P}(q_T \le r_T + L_1) > 1 - \delta, \end{aligned}$$

completing the proof. □

References

[1] Bezuidenhout, C. and Gray, L. (1994). Critical attractive spin systems. *Ann. Probab.* **22** 1160–1194. MR1303641

[2] Bezuidenhout, C. and Grimmett, G. (1990). The critical contact process dies out. *Ann. Probab.* **18** 1462–1486. MR1071804

[3] Cox, J. and Durrett, R. (1995). Hybrid zones and voter model interfaces. *Bernoulli* **1** 343–370. MR1369166

[4] Durrett, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999–1040. MR0757768

[5] Durrett, R. (1988). *Lecture Notes on Particle Systems and Percolation*. Belmont, CA: Wadsworth. MR0940469

[6] Durrett, R. and Møller, A. (1991). Complete convergence theorem for a competitive model. *Probab. Theory Related Fields* **88** 121–136. MR1094080

[7] Durrett, R. and Swindle, G. (1991). Are there bushes in a forest? *Stochastic Process. Appl.* **37** 19–31. MR1091691

[8] Liggett, T. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* **276**. New York: Springer. MR0776231

[9] Liggett, T., Schonmann, R. and Stacy, A. (1997). Domination by product measures. *Ann. Probab.* **25** 71–95. MR1428500

[10] Mountford, T. and Sweet, T. (2000). An extension of Kuczek's argument to nonnearest neighbor contact process. *J. Theoret. Probab.* **13** 1061–1081. MR1820502

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