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Strong approximations of BSDEs in a domain

BRUNO BOUCHARD[1] and STÉPHANE MENOZZI[2]

[1]*CEREMADE, Université Paris 9, place du Maréchal de Lattre de Tassigny, 75016 Paris, France.*
E-mail: bouchard@ceremade.dauphine.fr; url: www.ceremade.dauphine.fr/~bouchard
[2]*LPMA, Université Paris 7, 175 rue du Chevaleret, 75013 Paris, France.*
E-mail: menozzi@math.jussieu.fr; url: www.proba.jussieu.fr/~menozzi/

We study the strong approximation of a backward SDE with finite stopping time horizon, namely the first exit time of a forward SDE from a cylindrical domain. We use the Euler scheme approach of (*Stochastic Process. Appl.* **111** (2004) 175–206 and *Ann. Appl. Probab.* **14** (2004) 459–488). When the domain is piecewise smooth and under a non-characteristic boundary condition, we show that the associated strong error is at most of order $h^{1/4-\varepsilon}$, where h denotes the time step and ε is any positive parameter. This rate corresponds to the strong exit time approximation. It is improved to $h^{1/2-\varepsilon}$ when the exit time can be exactly simulated or for a weaker form of the approximation error. Importantly, these results are obtained without uniform ellipticity condition.

Keywords: backward SDEs; discrete-time approximation; first boundary value problem

1. Introduction

Let $T>0$ be a finite time horizon and $(\Omega,\mathcal{F},\mathbb{P})$ be a stochastic basis supporting a d-dimensional Brownian motion W. We assume that the filtration $\mathbb{F}=(\mathcal{F}_t)_{t\le T}$ generated by W satisfies the usual assumptions and that $\mathcal{F}_T=\mathcal{F}$.

Let (X,Y,Z) be the solution of the decoupled Brownian forward-backward SDE,

$$X_t = X_0 + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s, \tag{1.1}$$

$$Y_t = g(\tau, X_\tau) + \int_t^T \mathbf{1}_{s<\tau} f(X_s,Y_s,Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \qquad t\in[0,T], \tag{1.2}$$

where τ is the first exit time of $(t,X_t)_{t\le T}$ from a cylindrical domain $D=[0,T)\times\mathcal{O}$ for some open piecewise smooth connected set $\mathcal{O}\subset\mathbb{R}^d$ and b, σ, f and g satisfy the usual Lipschitz continuity assumption.

This kind of system appears in many applications. In particular, it is well known that it is related to the solution of the semi-linear Cauchy–Dirichlet problem

$$-\mathcal{L}u - f(\cdot,u,Du\sigma)=0 \qquad \text{on } D, \qquad u=g \qquad \text{on } \partial_p D, \tag{1.3}$$

where $\mathcal{L}$ is the (parabolic) Dynkin operator associated with X, that is, for $\psi\in C^{1,2}$,

$$\mathcal{L}\psi := \partial_t\psi + \langle b, D\psi\rangle + \tfrac12 \operatorname{Tr}[aD^2\psi], \qquad a:=\sigma\sigma^*,$$

and $\partial_p D := ([0,T) \times \partial \mathcal{O}) \cup (\{T\} \times \bar{\mathcal{O}})$ is the parabolic boundary of D. More precisely, if the solution u of (1.3) is sufficiently smooth, then $Y = u(\cdot, X)$ and $Z = Du\sigma(\cdot, X)$. Thus, in the regular frame, solving (1.2) is essentially equivalent to solving (1.3).

In this paper, we study an Euler scheme type approximation of (1.1)–(1.2) similar to the one introduced in [5,29]; see also [2,3,24]. We first consider the Euler scheme approximation $\bar{X}$ of X on some grid $\pi := \{t_i = ih, i \le n\}$ with modulus $h := T/n$, $n \in \mathbb{N}^*$. The exit time τ is approximated by the first discrete exit time $\bar{\tau}$ of $(t_i, \bar{X}_{t_i})_{t_i \in \pi}$ from D. Then, the backward Euler scheme of (Y, Z) is defined for $i = n-1, \dots, 0$ as

$$\bar{Y}_{t_i} := \mathbb{E}[\bar{Y}_{t_{i+1}} | \mathcal{F}_{t_i}] + 1_{t_i < \bar{\tau}} h f(\bar{X}_{t_i}, \bar{Y}_{t_i}, \bar{Z}_{t_i}), \qquad \bar{Z}_{t_i} := h^{-1} \mathbb{E}[\bar{Y}_{t_{i+1}} (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}]$$

with the terminal condition $\bar{Y}_T = g(\bar{\tau}, \bar{X}_{\bar{\tau}})$. Here, g is a suitable extension of the boundary condition to the whole space $[0,T] \times \mathbb{R}^d$.

The main purpose of this paper is to provide bounds for the (square of the) discrete-time approximation error up to a stopping time $\theta \le T$ $\mathbb{P}$-a.s. defined as

$$\operatorname{Err}(h)_\theta^2 := \max_{i<n} \mathbb{E}\Big[\sup_{t \in [t_i, t_{i+1}]} \mathbf{1}_{t \le \theta} |Y_t - \bar{Y}_{t_i}|^2 \Big] + \mathbb{E}\bigg[\int_0^\theta \big\| Z_t - \bar{Z}_{\phi(t)} \big\|^2 \, \mathrm{d}t \bigg], \tag{1.4}$$

where $\phi(t) := \sup\{s \in \pi : s \le t\}$.

We are interested in two important cases: $\theta = T$ and $\theta = \tau \wedge \bar{\tau}$. The quantity $\operatorname{Err}(h)_T$ coincides with the usual strong approximation error computed up to T. The term $\operatorname{Err}(h)_{\tau \wedge \bar{\tau}}$ should really be considered as a weak approximation error since the length of the random time interval $[0, \tau \wedge \bar{\tau}]$ cannot be controlled sharply in practice. It essentially provides a bound for $Y_0 - \bar{Y}_0$, or, equivalently in terms of (1.3), $u(0, X_0) - \bar{Y}_0$. We should point out that a precise analysis of the weak error has been carried out by Gobet and Labart in [15] in the uniformly elliptic case with $\mathcal{O} = \mathbb{R}^d$.

As in [5,23] and [29], who also considered the limit case $\mathcal{O} = \mathbb{R}^d$ (i.e., $\tau = T$), the approximation error can be naturally related to the error due to the approximation of X by $\bar{X}_\phi$ and the regularity of the solution (Y, Z) of (1.2) through the quantities

$$\mathcal{R}(Y)^\pi_{\mathcal{S}^2} := \max_{i<n} \mathbb{E}\Big[\sup_{t \in [t_i, t_{i+1}]} |Y_t - Y_{t_i}|^2 \Big] \quad \text{and} \quad \mathcal{R}(Z)^\pi_{\mathcal{H}^2} := \mathbb{E}\bigg[\int_0^T \big\| Z_t - \hat{Z}_{\phi(t)} \big\|^2 \, \mathrm{d}t \bigg],$$

where

$$\hat{Z}_{t_i} := h^{-1} \mathbb{E}\bigg[\int_{t_i}^{t_{i+1}} Z_s \, \mathrm{d}s \,\Big|\, \mathcal{F}_{t_i} \bigg] \qquad \text{for } i < n. \tag{1.5}$$

In the case $f = 0$, Y is a martingale and Y_{t_i} is the best L^2-approximation of Y_t on the time interval $[t_i, t_{i+1}]$ by an $\mathcal{F}_{t_i}$-measurable random variable. In this case, Doob's inequalities imply that $\mathbb{E}[\sup_{t \in [t_i, t_{i+1}]} |Y_t - \bar{Y}_{t_i}|^2] \ge \mathbb{E}[|Y_{t_{i+1}} - Y_{t_i}|^2] \ge c\mathbb{E}[\sup_{t \in [t_i, t_{i+1}]} |Y_t - Y_{t_i}|^2]$ for some universal constant $c > 0$.

Moreover, the definition (1.5) implies that $\hat{Z}_\phi$ is the best approximation in $L^2([0,T]\times\Omega, \mathrm{d}t\otimes \mathrm{d}\mathbb{P})$ of Z by a process which is constant on each time interval $[t_i, t_{i+1})$. Thus, $\mathcal{R}(Z)^\pi_{\mathcal{H}^2} \le \mathbb{E}[\int_0^T \|Z_t - \bar{Z}_{\phi(t)}\|^2\,\mathrm{d}t]$.

This justifies why $\mathcal{R}(Y)^\pi_{\mathcal{S}^2}$ and $\mathcal{R}(Z)^\pi_{\mathcal{H}^2}$ should play a crucial role in the convergence rate of $\mathrm{Err}(h)$ to 0 as $h\to 0$.

Bounds for similar quantities have previously been studied in [5,29] in the case $\mathcal{O}=\mathbb{R}^d$ and in [2,24] in the case of reflected BSDEs. All of these articles use a Malliavin calculus approach to derive a particular representation of Z. Due to the exit time, these techniques fail in our setting. We propose a different approach that relies on mixed analytic/probabilistic arguments. Namely, we first adapt some barrier techniques from the PDE literature (see, e.g., Chapter 14 in [12] and Section 6.2 below) to provide a bound for the modulus of continuity of u on the boundary, and then some stochastic flows and martingale arguments to obtain an interior control on this modulus. Under the standing assumptions of Section 2, we are able to derive that $\mathcal{R}(Y)^\pi_{\mathcal{S}^2} + \mathcal{R}(Z)^\pi_{\mathcal{H}^2} = \mathrm{O}(h)$ and that u is 1/2-Hölder in time and Lipschitz continuous in space.

To derive our final error bound on $\mathrm{Err}(h)_\theta$, we must additionally take into consideration the error coming from the approximation of τ by $\bar\tau$. We show that $\mathbb{E}[|\tau-\bar\tau|] = \mathrm{O}(h^{1/2-\varepsilon})$ for all $\varepsilon>0$. Combined with the previous controls on $\mathcal{R}(Y)^\pi_{\mathcal{S}^2}$ and $\mathcal{R}(Z)^\pi_{\mathcal{H}^2}$, this allows us to show that $\mathrm{Err}(h)_T = \mathrm{O}(h^{1/4-\varepsilon})$. Exploiting an additional control on a weaker form of error on $\tau-\bar\tau$, we also derive that $\mathrm{Err}(h)_{\tau\wedge\bar\tau} = \mathrm{O}(h^{1/2-\varepsilon})$. As a matter of fact, the global error is driven by the approximation error of the exit time which propagates backward due to the Lipschitz continuity of u.

Importantly, we do not assume specific non-degeneracies of the diffusion coefficient, but only a uniform non-characteristic boundary condition and uniform ellipticity close to the corners – recall that $\mathcal{O}$ is piecewise smooth. Using the transformation proposed in [20], these results could be extended to drivers with quadratic growth (for a bounded boundary condition g). Also, without major difficulties, our results could be extended to time-dependent domains and coefficients (b, σ and f) under natural assumptions on the time regularity. We restrict our attention here to the homogeneous cylindrical case for simplicity.

We note that the numerical implementation of the above scheme requires the approximation of the conditional expectations involved. It can be performed by nonparametric regression techniques (see, e.g., [16] and [22]) or a quantization approach (see, e.g., [1] and [8,9]). In both cases, the additional error is analyzed in the above papers and can be extended to our framework. We note that the Malliavin approach of [5] cannot be directly applied here due to the presence of the exit time. Concerning a direct computable algorithm, we mention the work of Milstein and Tretyakov [25] who use a simple random walk approximation of Brownian motion. However, their results require strong smoothness assumptions on the solution of (1.3), as well as a uniform ellipticity condition.

The rest of the paper is organized as follows. We start with some notation and assumptions in Section 2. Our main results are presented in Section 3. In Section 4, we provide a first bound on the error: it involves the error due to the discrete-time approximation of τ by $\bar\tau$ and the regularity of the solution (Y,Z) of (1.2). The discrete approximation of τ is specifically studied in Section 5. Finally, Section 6 is devoted to the analysis of the regularity of (1.3) and (1.2) under our current assumptions.

2. Notation and assumptions

Any element $x \in \mathbb{R}^d$, $d \geq 1$, will be identified with a line vector with ith component x^i and Euclidean norm $\|x\|$. The scalar product on $\mathbb{R}^d$ is denoted by $\langle x, y\rangle$. The open ball with center x and radius r is denoted by $B(x,r)$, $\bar{B}(x,r)$ being its closure. Given a non-empty set $A \subset \mathbb{R}^d$, we similarly denote by $B(A,r)$ and $\bar{B}(A,r)$ the sets $\{x \in \mathbb{R}^d : d(x,A) < r\}$ and $\{x \in \mathbb{R}^d : d(x,A) \leq r\}$, where $d(x,A)$ stands for the Euclidean distance of x to A. For a $(m \times d)$-dimensional matrix M, we denote by M^* its transpose and we write $M \in \mathbb{M}^d$ if $m = d$. For a smooth function $f(t,x)$, Df and $D^2 f$ stand for, respectively, its gradient (as a line vector) and Hessian matrix with respect to its second component. If it depends on some extra components, we denote by $\partial_t f(t,x,y,z)$, $\partial_x f(t,x,y,z), \ldots$ its partial gradients.

2.1. Euler scheme approximation of BSDEs

From now on, we assume that the coefficients of (1.1)–(1.2) satisfy the following.

(HL) There is a constant $L > 0$ such that for all $(t,x,y,z,t',x',y',z') \in ([0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d)^2$,

$$\|(b,\sigma,g,f)(t,x,y,z) - (b,\sigma,g,f)(t',x',y',z')\| \leq L\|(t,x,y,z) - (t',x',y',z')\|,$$
$$\|(b,\sigma,g,f)(t,x,y,z)\| \leq L\big(1 + \|(x,y,z)\|\big).$$

Under this assumption, it is well known (see, e.g., [27,28]) that we have existence and uniqueness of a solution (X,Y,Z) in $\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$, where we denote by $\mathcal{S}^2$ the set of real-valued adapted continuous processes ξ satisfying $\|\xi\|_{\mathcal{S}^2} := \mathbb{E}[\sup_{t\leq T} |\xi_t|^2]^{1/2} < \infty$, and by $\mathcal{H}^2$, the set of progressively measurable $\mathbb{R}^d$-valued processes ζ for which $\|\zeta\|_{\mathcal{H}^2} := \mathbb{E}[\int_0^T |\zeta_t|^2\,\mathrm{d}t]^{1/2} < \infty$.

As usual, we shall approximate the solution of (1.1) by its Euler scheme $\bar{X}$ associated to a grid

$$\pi := \{t_i = ih, i \leq n\}, \qquad h := T/n,\ n \in \mathbb{N}^*,$$

defined by

$$\bar{X}_t = X_0 + \int_0^t b\big(\bar{X}_{\phi(s)}\big)\,\mathrm{d}s + \int_0^t \sigma\big(\bar{X}_{\phi(s)}\big)\,\mathrm{d}W_s, \qquad t \geq 0, \tag{2.1}$$

where we recall that $\phi(s) := \arg\max\{t_i, i \leq n : t_i \leq s\}$ for $s \geq 0$.

Regarding the approximation of (1.2), we adapt the approach of [29] and [5]. First, we approximate the exit time τ by the first exit time of the Euler scheme $(t, \bar{X}_t)_{t\in\pi}$ from D on the grid π:

$$\bar{\tau} := \inf\{t \in \pi : \bar{X}_t \notin \mathcal{O}\} \wedge T.$$

Remark 2.1. Note that one could also approximate τ by $\tilde{\tau} := \inf\{t \in [0,T] : \bar{X}_t \notin \mathcal{O}\} \wedge T$, the first exit time of the "continuous version" of the Euler scheme $(t, \bar{X}_t)_{t\in[0,T]}$, as is done for linear problems, that is f is independent of (Y,Z) (see, e.g., [14]). However, in the case where $\mathcal{O}$

is not a half-space, this requires additional local approximations of the boundary by tangent hyperplanes and will not allow improvement of our strong approximation error (compare Corollary 2.3.2. in [13] with Theorem 3.1 below).

We then define the discrete-time process $(\bar{Y}, \bar{Z})$ on π by

$$\bar{Y}_{t_i} := \mathbb{E}[\bar{Y}_{t_{i+1}} | \mathcal{F}_{t_i}] + \mathbf{1}_{t_i < \bar{\tau}} h f(\bar{X}_{t_i}, \bar{Y}_{t_i}, \bar{Z}_{t_i}), \tag{2.2}$$

$$\bar{Z}_{t_i} := h^{-1} \mathbb{E}[\bar{Y}_{t_{i+1}} (W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}], \qquad i < n, \tag{2.3}$$

with the terminal condition

$$\bar{Y}_T = g(\bar{\tau}, \bar{X}_{\bar{\tau}}). \tag{2.4}$$

Observe that $\bar{Y}_{t_i} \mathbf{1}_{t_i \geq \bar{\tau}} = g(\bar{\tau}, \bar{X}_{\bar{\tau}}) \mathbf{1}_{t_i \geq \bar{\tau}}$ and $\bar{Z}_{t_i} \mathbf{1}_{t_i \geq \bar{\tau}} = 0$.

One can easily check that $(\bar{Y}_{t_i}, \bar{Z}_{t_i}) \in L^2$ for all $i \leq n$ under **(HL)**. It then follows from the martingale representation theorem that we can find $\tilde{Z} \in \mathcal{H}^2$ such that

$$\bar{Y}_{t_{i+1}} - \mathbb{E}[\bar{Y}_{t_{i+1}} | \mathcal{F}_{t_i}] = \int_{t_i}^{t_{i+1}} \tilde{Z}_s \, \mathrm{d}W_s \qquad \text{for all } i < n. \tag{2.5}$$

This allows us to consider a continuous-time extension of $\bar{Y}$ in $\mathcal{S}^2$ defined on $[0, T]$ by

$$\bar{Y}_t = g(\bar{\tau}, \bar{X}_{\bar{\tau}}) + \int_t^T \mathbf{1}_{s < \bar{\tau}} f\big(\bar{X}_{\phi(s)}, \bar{Y}_{\phi(s)}, \bar{Z}_{\phi(s)}\big) \, \mathrm{d}s - \int_t^T \tilde{Z}_s \, \mathrm{d}W_s. \tag{2.6}$$

Remark 2.2. Observe that $Z = 0$ on $]\tau, T]$ and $\tilde{Z} = 0$ on $]\bar{\tau}, T]$. For later use, note also that the Itô isometry and (2.5) together imply that

$$\bar{Z}_{t_i} = h^{-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \tilde{Z}_s \, \mathrm{d}s \,\middle|\, \mathcal{F}_{t_i}\right], \qquad i < n. \tag{2.7}$$

2.2. Assumptions on $\mathcal{O}$, σ and g

Our main result holds under some additional assumptions on $\mathcal{O}$, σ and g. Without loss of generality, we can specify them in terms of the constant L which appears in **(HL)**.

We first assume that the domain $\mathcal{O}$ is a finite intersection of smooth domains with compact boundaries:

(D1) We have $\mathcal{O} := \bigcap_{\ell=1}^m \mathcal{O}^\ell$, where $m \in \mathbb{N}^*$ and $\mathcal{O}^\ell$ is a C^2 domain of $\mathbb{R}^d$ for each $1 \leq \ell \leq m$. Moreover, $\mathcal{O}^\ell$ has a compact boundary, $\sup\{\|x\| : x \in \partial\mathcal{O}^\ell\} \leq L$, for each $1 \leq \ell \leq m$.

It follows from Appendix 14.6 of [12] that there is a function d which coincides with the algebraic distance to $\partial\mathcal{O}$, in particular, $\mathcal{O} := \{x \in \mathbb{R}^d : d(x) > 0\}$, and is C^2 outside a neighborhood

$B(\mathcal{C}, L^{-1})$ of the set of corners

$$\mathcal{C} := \bigcap_{\ell \neq k=1}^{m} \partial \mathcal{O}^{\ell} \cap \partial \mathcal{O}^{k}.$$

We also assume that the domain satisfies a uniform exterior sphere condition as well as a uniform truncated interior cone condition:

(D2) For all $x \in \partial\mathcal{O}$, there exist $y(x) \in \mathcal{O}^c$, $r(x) \in [L^{-1}, L]$ and $\delta(x) \in B(0,1)$ such that

$$\bar{B}(y(x), r(x)) \cap \bar{\mathcal{O}} = \{x\} \quad \text{and}$$
$$\{x' \in B(x, L^{-1}) : \langle x' - x, \delta(x)\rangle \geq (1 - L^{-1})\|x' - x\|\} \subset \bar{\mathcal{O}}.$$

In view of **(D1)**, these last assumptions are actually automatically satisfied outside a neighborhood of the set of corners (see, e.g., Appendix 14.6 of [12]).

In order to ensure that the associated first boundary value problem is well posed in the (unconstrained) viscosity sense, we shall also assume that

$$a := \sigma\sigma^*$$

satisfies a non-characteristic boundary condition outside the set of corners $\mathcal{C}$ and a uniform ellipticity condition on a neighborhood of $\mathcal{C}$:

(C) We have

$$\inf\{n(x)a(x)n(x)^* : x \in \partial\mathcal{O} \setminus B(\mathcal{C}, L^{-1})\} \geq L^{-1}, \qquad \text{where } n(x) := Dd(x),$$

and

$$\inf\{\xi a(x)\xi^* : \xi \in \partial B(0,1), x \in \bar{\mathcal{O}} \cap B(\mathcal{C}, L^{-1})\} \geq L^{-1}.$$

In particular, it guarantees that the process X is non-adherent to the boundary.

Observe that n coincides with the inner normal unit on $\partial\mathcal{O}$ outside the set of corners. By abuse of notation, we write $n(x)$ for $Dd(x)$ whenever this quantity is well defined, even if $x \notin \partial\mathcal{O}$.

Importantly, we do not assume that σ is non-degenerate in the whole domain.

Finally, we assume that g is sufficiently smooth:

(Hg) $g \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and $\|\partial_t g\| + \|Dg\| + \|D^2 g\| \leq L$ on $[0,T] \times \mathbb{R}^d$.

Clearly, this smoothness assumption could be imposed only on a neighborhood of $\partial\mathcal{O}$. Since it is compact and Y depends on g only on $\partial\mathcal{O}$, we can always construct a suitable extension of g on $\mathbb{R}^d$ which satisfies the above condition. Actually, one could assume only that g is Lipschitz in (t,x) and has a Lipschitz continuous derivative in x. With this slightly weaker condition, all our arguments would remain valid after possibly replacing g by a sequence of regularized versions and then passing to the limit; see Section 6.4 for similar kinds of arguments.

3. Main results

We first provide a general control on the quantities in (1.4) in terms of $\mathcal{R}(Y)^{\pi}_{\mathcal{S}^2}$, $\mathcal{R}(Z)^{\pi}_{\mathcal{H}^2}$ and $|\tau - \bar{\tau}|$. We should mention that this type of result is now rather standard when $\mathcal{O} = \mathbb{R}^d$ (see, e.g., [5]) and requires only the Lipschitz continuity assumptions of **(HL)**.

Proposition 3.1. *Assume that* **(HL)** *and* **(Hg)** *hold. There then exist $C_L > 0$ and a positive random variable ξ_L satisfying $\mathbb{E}[(\xi_L)^p] \le C_L^p$ for all $p \ge 2$ such that*

$$\mathrm{Err}(h)^2_T \le C_L\left(h + \mathcal{R}(Y)^{\pi}_{\mathcal{S}^2} + \mathcal{R}(Z)^{\pi}_{\mathcal{H}^2} + \mathbb{E}\left[\xi_L|\tau - \bar{\tau}| + \mathbf{1}_{\bar{\tau}<\tau}\int_{\bar{\tau}}^{\tau} \|Z_s\|^2\,\mathrm{d}s\right]\right) \tag{3.1}$$

and

$$\begin{aligned}\mathrm{Err}(h)^2_{\tau\wedge\bar{\tau}} \le \mathrm{Err}(h)^2_{\tau_+\wedge\bar{\tau}} &\le C_L\big(h + \mathcal{R}(Y)^{\pi}_{\mathcal{S}^2} + \mathcal{R}(Z)^{\pi}_{\mathcal{H}^2}\big) + \mathbb{E}\big[\mathbb{E}[\xi_L|\tau - \bar{\tau}||\mathcal{F}_{\tau_+\wedge\bar{\tau}}]^2\big] \\ &\quad + C_L\mathbb{E}\left[\mathbf{1}_{\bar{\tau}<\tau}\mathbb{E}\left[\int_{\bar{\tau}}^{\tau} \|Z_s\|\,\mathrm{d}s\Big|\mathcal{F}_{\bar{\tau}}\right]^2\right],\end{aligned} \tag{3.2}$$

where τ_+ is the next time after τ in the grid π: $\tau_+ := \inf\{t \in \pi : \tau \le t\}$.

The proof will be provided in Section 4 below. Note that we shall control $\mathrm{Err}(h)^2_{\tau\wedge\bar{\tau}}$ through the slightly stronger term $\mathrm{Err}(h)^2_{\tau_+\wedge\bar{\tau}}$; see (3.2). This will allow us to work with stopping times with values in the grid π, which will be technically easier; see Remark 4.2 below.

In order to provide a convergence rate for $\mathrm{Err}(h)^2_T$ and $\mathrm{Err}(h)^2_{\tau_+\wedge\bar{\tau}}$, it remains to control the quantities $\mathcal{R}(Y)^{\pi}_{\mathcal{S}^2}$, $\mathcal{R}(Z)^{\pi}_{\mathcal{H}^2}$ and the terms involving the difference between τ and $\bar{\tau}$.

The error due to the approximation of τ by $\bar{\tau}$ is controlled by the following estimate that extends to the non-uniformly elliptic case previous results obtained in [13]; see its Corollary 2.3.2. The proof of this theorem is provided in Section 5 below.

Theorem 3.1. *Assume that b and σ satisfy* **(HL)** *and that* **(D1)** *and* **(C)** *hold. Then, for $\varepsilon \in (0, 1)$ and each positive random variable ξ satisfying $\mathbb{E}[(\xi)^p] \le C_L^p$ for all $p \ge 1$, there exists $C_L^{\varepsilon} > 0$ such that*

$$\mathbb{E}\big[\mathbb{E}[\xi|\tau - \bar{\tau}||\mathcal{F}_{\tau_+\wedge\bar{\tau}}]^2\big] \le C_L^{\varepsilon} h^{1-\varepsilon}.$$

In particular, for each $\varepsilon \in (0, 1/2)$, there exists $C_L^{\varepsilon} > 0$ such that

$$\mathbb{E}[|\tau - \bar{\tau}|] \le C_L^{\varepsilon} h^{1/2-\varepsilon}.$$

In [13], the last bound is derived under a uniform ellipticity condition on σ and cannot be exploited in our setting – recall that we only assume **(C)**. Up to the ε term, it cannot be improved. Indeed, in the special case of a uniformly elliptic diffusion in a smooth bounded domain, it has been shown in [17] that $\mathbb{E}[\tau - \bar{\tau}] = Ch^{1/2} + \mathrm{o}(h^{1/2})$ for some $C > 0$; see Theorem 2.3 of this reference.

Our next result concerns the regularity of (Y, Z) and is an extension to our framework of similar results obtained in [3,5,23] and [2] in different contexts.

Theorem 3.2. *Let the conditions* **(HL)**, **(D1)**, **(D2)**, **(C)** *and* **(Hg)** *hold. Then*

$$\mathcal{R}(Y)^{\pi}_{\mathcal{S}^2} + \mathcal{R}(Z)^{\pi}_{\mathcal{H}^2} \le C_L h. \tag{3.3}$$

Moreover, for all stopping times θ, ϑ satisfying $\theta \le \vartheta \le T$ $\mathbb{P}$-a.s., we have

$$\mathbb{E}\Big[\sup_{\theta \le s \le \vartheta} |Y_s - Y_\theta|^{2p}\Big] \le \mathbb{E}[\xi^p_L |\vartheta - \theta|^p], \qquad p \ge 1, \tag{3.4}$$

and

$$\mathbb{E}\bigg[\int_\theta^\vartheta \|Z_s\|^p \, \mathrm{d}s \bigg| \mathcal{F}_\theta\bigg] \le \mathbb{E}[\xi^p_L |\vartheta - \theta| | \mathcal{F}_\theta], \qquad p = 1, 2, \tag{3.5}$$

where ξ^p_L is a positive random variable which satisfies $\mathbb{E}[|\xi^p_L|^q] < \infty$ for all $q \ge 1$.

In addition, the unique continuous viscosity solution u of (1.3), in the class of continuous solutions with polynomial growth, is uniformly $1/2$-Hölder continuous in time and Lipschitz continuous in space, that is,

$$|u(t, x) - u(t', x')| \le C_L(|t - t'|^{1/2} + \|x - x'\|) \qquad \text{for all } (t, x) \text{ and } (t', x') \in \bar{D}. \tag{3.6}$$

The proof is provided in Section 6 below. The bound (3.5) can be interpreted as a weak bound on the gradient, whenever it is well defined, of the viscosity solution of (1.3). It implies that Y is $1/2$-Hölder continuous in L^2-norm. This result is rather standard under our Lipschitz continuity assumption in the case where $\mathcal{O} = \mathbb{R}^d$, that is, $\tau = T$, but seems to be new in our context and under our assumptions. The bound $\mathcal{R}(Z)^{\pi}_{\mathcal{H}^2} \le C_L h$ can be seen as a weak regularity result on this gradient. It would be straightforward if one could show that $Du\sigma$ is uniformly $1/2$-Hölder in time and Lipschitz in space, which is not true in general.

Combining the above estimates, we finally obtain our main result, which provides an upper bound for the convergence rate of $\mathrm{Err}(h)^2_{\tau_+ \wedge \bar{\tau}}$ (and thus for $\mathrm{Err}(h)^2_{\tau \wedge \bar{\tau}}$) and $\mathrm{Err}(h)^2_T$.

Theorem 3.3. *Let the conditions* **(HL)**, **(D1)**, **(D2)**, **(C)** *and* **(Hg)** *hold. Then, for each $\varepsilon \in (0, \frac{1}{2})$, there exists $C^\varepsilon_L > 0$ such that*

$$\mathrm{Err}(h)^2_{\tau_+ \wedge \bar{\tau}} \le C^\varepsilon_L h^{1-\varepsilon} \quad \textit{and} \quad \mathrm{Err}(h)^2_T \le C^\varepsilon_L h^{1/2-\varepsilon}.$$

This extends the results of [2,3,29] which obtained similar bounds in different contexts.

Remark 3.1. When τ can be exactly simulated, we can replace $\bar{\tau}$ by τ in the scheme (2.2)–(2.3). In this case, the two last terms in the right-hand sides of (3.1) and (3.2) cancel and we retrieve the convergence rate of the case $\mathcal{O} = \mathbb{R}^d$ (see, e.g., [5]).

Remark 3.2. Note that the Lipschitz continuity assumption with respect to the x variable on g and f is only used to control at the right order the error term coming from the approximation of X by $\bar{X}$ in g and f. If one is only interested in the convergence of $\operatorname{Err}(h)_T$, this assumption can be weakened. Indeed, if we only assume that

($\mathbf{HL}_1'$): b, σ satisfy **(HL)**, $\sup\{|f(\cdot, y, z)|,\ (y, z) \in \mathbb{R} \times \mathbb{R}^d\}$ and g have polynomial growth and $f(x, \cdot)$ is uniformly Lipschitz continuous, uniformly in $x \in \mathbb{R}^d$,

a weak version of (3.1) can still be established up to an obvious modification of the proof of Proposition 4.2 below. Namely, there exists $C > 0$ and a positive random variable ξ satisfying $\mathbb{E}[(\xi)^p] \le C_L^p$ for all $p \ge 2$ for which

$$
\begin{aligned}
\operatorname{Err}(h)_T^2 \le C\biggl(& h + \mathbb{E}\biggl[\int_0^T |Y_s - Y_{\phi(s)}|^2 \,\mathrm{d}s\biggr] + \mathcal{R}(Z)^\pi_{\mathcal{H}^2} \\
&+ \mathbb{E}\biggl[\xi|\tau - \bar{\tau}| + \int_0^T \mathbf{1}_{\bar{\tau}<\tau} \int_{\bar{\tau}}^{\tau} \|Z_s\|^2 \,\mathrm{d}s\biggr]\biggr) \\
&+ C\mathbb{E}\biggl[|g(\tau, X_\tau) - g(\bar{\tau}, \bar{X}_{\bar{\tau}})|^2 + \int_0^T \bigl|f(X_s, Y_s, Z_s) - f\bigl(\bar{X}_{\phi(s)}, Y_s, Z_s\bigr)\bigr|^2 \,\mathrm{d}s\biggr].
\end{aligned}
\tag{3.7}
$$

The terms $\mathbb{E}[\int_0^T |Y_s - Y_{\phi(s)}|^2 \,\mathrm{d}s]$ and $\mathcal{R}(Z)^\pi_{\mathcal{H}^2}$ are easily seen to go 0 with h (see, e.g., the proof of Proposition 2.1 in [3] for details). As for the other terms in the first line, it suffices to appeal to Theorem 3.1, which implies that $\mathbb{E}[\xi|\tau - \bar{\tau}|] \to 0$ and that $\bar{\tau} \to \tau$ in probability under **(D1)** and **(C)**. Note that the last assertion implies that $\mathbb{E}[\int_0^T \mathbf{1}_{\bar{\tau}<\tau} \int_{\bar{\tau}}^{\tau} \|Z_s\|^2 \,\mathrm{d}s] \to 0$ and $X_\tau - \bar{X}_{\bar{\tau}} \to 0$ in probability. Hence, under the additional continuity assumption

($\mathbf{HL}_2'$): g and $f(\cdot, y, z)$ are continuous for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, we deduce that the two last terms in the second line also go to 0.

4. Euler scheme approximation error: proof of Proposition 3.1

In this section, we provide the proof of Proposition 3.1. We first recall some standard controls on X, (Y, Z) and $\bar{X}$ which hold under **(HL)**.

From now on, C_L^η denotes a generic constant whose value may change from line to line, but which depends only on X_0, L and some extra parameter η (we simply write C_L if it depends only on X_0 and L). Similarly, ξ_L^η denotes a generic non-negative random variable such that $\mathbb{E}[|\xi_L^\eta|^p] \le C_L^{\eta,p}$ for all $p \ge 1$ (we simply write ξ_L if it does not depend on the extra parameter η).

Proposition 4.1. *Let* **(HL)** *hold. Fix* $p \ge 2$. *Let* ϑ *be a stopping time with values in* $[0, T]$. *Then,*

$$
\mathbb{E}\biggl[\sup_{t\in[\vartheta,T]} \|Y_t\|^p + \biggl(\int_\vartheta^T \|Z_t\|^2 \,\mathrm{d}t\biggr)^{p/2} \bigg| \mathcal{F}_\vartheta\biggr] \le C_L^p(1 + \|X_\vartheta\|^p)
$$

and

$$
\mathbb{E}\Bigl[\sup_{t\in[\vartheta,T]} (\|X_t\|^p + \|\bar{X}_t\|^p) \big| \mathcal{F}_\vartheta\Bigr] \le \xi_L^p.
$$

Moreover,

$$\max_{i<n}\mathbb{E}\Big[\sup_{t\in[t_i,t_{i+1}]}(\|X_t-X_{t_i}\|^p+\|\bar{X}_t-\bar{X}_{t_i}\|^p)\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\|X_t-\bar{X}_t\|^p\Big]\le C_L^p h^{p/2},$$

$$\mathbb{P}\Big[\sup_{t\le T}\|\bar{X}_t-\bar{X}_{\phi(t)}\|>r\Big]\le C_L r^{-4}h,\qquad r>0,$$

and, if θ is a stopping time with values in $[0,T]$ such that $\vartheta\le\theta\le\vartheta+h$ $\mathbb{P}$-a.s., then

$$\mathbb{E}[\|\bar{X}_\theta-\bar{X}_\vartheta\|^p+\|X_\theta-X_\vartheta\|^p|\mathcal{F}_\vartheta]\le \xi_L^p h^{p/2}.$$

Remark 4.1. For later use, observe that the Lipschitz continuity assumptions **(HL)** ensure that

$$\mathbb{E}\Big[\sup_{t\in[\vartheta,T]}\|\bar{Y}_t\|^p+\Big(\int_\vartheta^T\|\tilde{Z}_t\|^2\,\mathrm{d}t\Big)^{p/2}\Big|\mathcal{F}_\vartheta\Big]<\infty\qquad\text{for all } p\ge 2.$$

In order to avoid the repetition of similar arguments depending on whether we consider $\mathrm{Err}(h)^2_\theta$ with $\theta=T$ or $\theta=\tau_+\wedge\bar{\tau}$, we first state an abstract version of Proposition 3.1 for some stopping time θ with values in π.

Proposition 4.2. *Assume that b,σ and f satisfy* **(HL)**. *For all stopping times θ with values in π, we then have*

$$\begin{aligned}\mathrm{Err}(h)^2_\theta\le C_L\Big(&h+\mathbb{E}[|Y_\theta-\bar{Y}_\theta|^2]+\mathcal{R}(Y)^\pi_{\mathcal{S}^2}+\mathcal{R}(Z)^\pi_{\mathcal{H}^2}\\&+\mathbb{E}\Big[\int_{\bar{\tau}\wedge\tau\wedge\theta}^{(\bar{\tau}\vee\tau)\wedge\theta}(\xi_L+\mathbf{1}_{\bar{\tau}<\tau}\|Z_s\|^2)\,\mathrm{d}s\Big]\Big).\end{aligned}$$

Let us first make the following remark which will be of important use below.

Remark 4.2. Let $\vartheta\le\theta$ $\mathbb{P}$-a.s. be two stopping times with values in π and H be some adapted process in $\mathcal{S}^2$. Then, recalling that $t_{i+1}-t_i=h$, it follows from (2.7) and Jensen's inequality that

$$\begin{aligned}\mathbb{E}\Big[\int_\vartheta^\theta H_{\phi(s)}\|\bar{Z}_{\phi(s)}\|^2\,\mathrm{d}s\Big]&=\sum_{i<n}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}\mathbf{1}_{\vartheta\le t_i<\theta}H_{t_i}\Big\|\mathbb{E}\Big[h^{-1}\int_{t_i}^{t_{i+1}}\tilde{Z}_u\,\mathrm{d}u\Big|\mathcal{F}_{t_i}\Big]\Big\|^2\,\mathrm{d}s\Big]\\&\le\sum_{i<n}\mathbb{E}\Big[\int_{t_i}^{t_{i+1}}\mathbf{1}_{\vartheta\le t_i<\theta}H_{t_i}h^{-1}\int_{t_i}^{t_{i+1}}\|\tilde{Z}_u\|^2\,\mathrm{d}u\,\mathrm{d}s\Big]\\&\le\mathbb{E}\Big[\int_\vartheta^\theta H_{\phi(s)}\|\tilde{Z}_s\|^2\,\mathrm{d}s\Big].\end{aligned}$$

By definition of $\hat{Z}$ (see (1.5)), the same inequality holds with $(\hat{Z}, Z)$ or $(\hat{Z} - \bar{Z}, Z - \tilde{Z})$ in place of $(\bar{Z}, \tilde{Z})$. This remark will allow us to control $\|Z - \bar{Z}_\phi\|$ through $\|Z - \tilde{Z}\|$ and $\|Z - \hat{Z}_\phi\|$ (see (4.3) below), which is a key argument in the proof of Proposition 4.2. Observe that the above inequality does not apply if ϑ and θ do not take values in π. This explains why it is easier to work with τ_+ instead of τ, that is, work on $\mathrm{Err}(h)^2_{\tau_+\wedge\bar{\tau}}$ instead of $\mathrm{Err}(h)^2_{\tau\wedge\bar{\tau}}$.

Proof of Proposition 4.2. We adapt the arguments used in the proof of Theorem 3.1 in [5] to our setting. By applying Itô's lemma to $(Y - \bar{Y})^2$ on $[t \wedge \theta, t_{i+1} \wedge \theta]$ for $t \in [t_i, t_{i+1}]$ and $i < n$, we first deduce from (1.2) and (2.6) that

$$\begin{aligned}\Delta^\theta_{t,t_{i+1}} &:= \mathbb{E}\left[|Y_{t\wedge\theta} - \bar{Y}_{t\wedge\theta}|^2 + \int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \|Z_s - \tilde{Z}_s\|^2 \, \mathrm{d}s\right] \\ &= \mathbb{E}[|Y_{t_{i+1}\wedge\theta} - \bar{Y}_{t_{i+1}\wedge\theta}|^2] + \mathbb{E}\left[2\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (Y_s - \bar{Y}_s)\big(\mathbf{1}_{s<\tau} f(\Theta_s) - \mathbf{1}_{s<\bar{\tau}} f\big(\bar{\Theta}_{\phi(s)}\big)\big)\, \mathrm{d}s\right],\end{aligned}$$

where the martingale terms cancel due to Proposition 4.1 and Remark 4.1, and where $\Theta := (X, Y, Z)$ and $\bar{\Theta} := (\bar{X}, \bar{Y}, \bar{Z})$. Using the inequality $2ab \le a^2 + b^2$, we then deduce that for $\alpha > 0$ to be chosen later,

$$\begin{aligned}\Delta^\theta_{t,t_{i+1}} \le{}& \mathbb{E}[|Y_{t_{i+1}\wedge\theta} - \bar{Y}_{t_{i+1}\wedge\theta}|^2] + \alpha\mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} |Y_s - \bar{Y}_s|^2 \, \mathrm{d}s\right] \\ &+ 2\alpha^{-1}\mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{s<\bar{\tau}}\big(f(\Theta_s) - f\big(\bar{\Theta}_{\phi(s)}\big)\big)^2 \, \mathrm{d}s + \int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\bar{\tau}\le s<\tau}(f(\Theta_s))^2 \, \mathrm{d}s\right] \\ &+ 2\alpha^{-1}\mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \mathbf{1}_{\tau\le s<\bar{\tau}}(f(\Theta_s))^2 \, \mathrm{d}s\right].\end{aligned}$$

Recall from Remark 2.2 that $Z = 0$ on $]\tau, T]$. Since $Y_t = g(\tau, X_\tau)$ on $\{t \ge \tau\}$, we then deduce from **(HL)** and Proposition 4.1 that

$$\begin{aligned}\Delta^\theta_{t,t_{i+1}} \le{}& \mathbb{E}[|Y_{t_{i+1}\wedge\theta} - \bar{Y}_{t_{i+1}\wedge\theta}|^2] + \alpha\mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} |Y_s - \bar{Y}_s|^2 \, \mathrm{d}s\right] \\ &+ C_L\alpha^{-1}\mathbb{E}\left[h|Y_{t_i\wedge\theta} - \bar{Y}_{t_i\wedge\theta}|^2 + \int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \big|Y_s - Y_{\phi(s)}\big|^2 \, \mathrm{d}s\right] \\ &+ C_L\alpha^{-1}\mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} \big(h + \big\|Z_s - \hat{Z}_{\phi(s)}\big\|^2 + \big\|\hat{Z}_{\phi(s)} - \bar{Z}_{\phi(s)}\big\|^2\big)\, \mathrm{d}s\right] \\ &+ C_L\alpha^{-1}\mathbb{E}\left[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta} (\xi_L \mathbf{1}_{\tau\wedge\bar{\tau}\le s\le\tau\vee\bar{\tau}} + \mathbf{1}_{\bar{\tau}\le s<\tau}\|Z_s\|^2)\, \mathrm{d}s\right].\end{aligned} \tag{4.1}$$

It then follows from Gronwall's lemma that

$$
\begin{aligned}
&\mathbb{E}[|Y_{t\wedge\theta}-\bar{Y}_{t\wedge\theta}|^2]\\
&\quad\le (1+C_L^\alpha h)\mathbb{E}[|Y_{t_{i+1}\wedge\theta}-\bar{Y}_{t_{i+1}\wedge\theta}|^2]\\
&\qquad+(C_L\alpha^{-1}+C_L^\alpha h)\mathbb{E}\biggl[h|Y_{t_i\wedge\theta}-\bar{Y}_{t_i\wedge\theta}|^2+\int_{t\wedge\theta}^{t_{i+1}\wedge\theta}\bigl|Y_s-Y_{\phi(s)}\bigr|^2\,\mathrm{d}s\biggr]\\
&\qquad+(C_L\alpha^{-1}+C_L^\alpha h)\mathbb{E}\biggl[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta}\bigl(h+\|Z_s-\hat{Z}_{\phi(s)}\|^2+\|\hat{Z}_{\phi(s)}-\bar{Z}_{\phi(s)}\|^2\bigr)\,\mathrm{d}s\biggr]\\
&\qquad+(C_L\alpha^{-1}+C_L^\alpha h)\mathbb{E}\biggl[\int_{t\wedge\theta}^{t_{i+1}\wedge\theta}(\xi_L\mathbf{1}_{\tau\wedge\bar{\tau}\le s\le\tau\vee\bar{\tau}}+\mathbf{1}_{\bar{\tau}\le s<\tau}\|Z_s\|^2)\,\mathrm{d}s\biggr].
\end{aligned}
\tag{4.2}
$$

Substituting (4.2) into (4.1) applied with $t=t_i$, using Remark 4.2, taking $\alpha>0$ sufficiently large, depending on the constants C_L, and h small leads to

$$
\begin{aligned}
\Delta^\theta_{t_i,t_{i+1}}&\le(1+C_Lh)\mathbb{E}[|Y_{t_{i+1}\wedge\theta}-\bar{Y}_{t_{i+1}\wedge\theta}|^2]\\
&\quad+C_L\mathbb{E}\biggl[\int_{t_i\wedge\theta}^{t_{i+1}\wedge\theta}\bigl(h+\bigl|Y_s-Y_{\phi(s)}\bigr|^2+\|Z_s-\hat{Z}_{\phi(s)}\|^2\bigr)\,\mathrm{d}s\biggr]\\
&\quad+C_L\mathbb{E}\biggl[\int_{t_i\wedge\theta}^{t_{i+1}\wedge\theta}(\xi_L\mathbf{1}_{\tau\wedge\bar{\tau}\le s\le\tau\vee\bar{\tau}}+\mathbf{1}_{\bar{\tau}\le s<\tau}\|Z_s\|^2)\,\mathrm{d}s\biggr].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\Delta^\theta:=\max_{i<n}\mathbb{E}[|Y_{t_i\wedge\theta}-\bar{Y}_{t_i\wedge\theta}|^2]+\mathbb{E}\biggl[\int_0^\theta\|Z_s-\tilde{Z}_s\|^2\,\mathrm{d}s\biggr]&\\
\le C_L\bigl(\mathbb{E}[|Y_\theta-\bar{Y}_\theta|^2]+h+\mathcal{R}(Y)^\pi_{\mathcal{S}^2}+\mathcal{R}(Z)^\pi_{\mathcal{H}^2}\bigr)&\\
+C_L\mathbb{E}\biggl[\xi_L|\bar{\tau}\wedge\theta-\tau\wedge\theta|+\int_0^\theta\mathbf{1}_{\bar{\tau}\le s<\tau}\|Z_s\|^2\,\mathrm{d}s\biggr].&
\end{aligned}
$$

We conclude the proof by again using Remark 4.2 to obtain

$$
\begin{aligned}
&\mathbb{E}\biggl[\int_0^\theta\|Z_s-\bar{Z}_{\phi(s)}\|^2\biggr]\\
&\quad\le C_L\biggl(\mathbb{E}\biggl[\int_0^\theta\|\hat{Z}_{\phi(s)}-\bar{Z}_{\phi(s)}\|^2\,\mathrm{d}s\biggr]+\mathbb{E}\biggl[\int_0^T\|Z_s-\hat{Z}_{\phi(s)}\|^2\,\mathrm{d}s\biggr]\biggr)\\
&\quad\le C_L\biggl(\mathbb{E}\biggl[\int_0^\theta\|Z_s-\tilde{Z}_s\|^2\,\mathrm{d}s\biggr]+\mathbb{E}\biggl[\int_0^T\|Z_s-\hat{Z}_{\phi(s)}\|^2\,\mathrm{d}s\biggr]\biggr)
\end{aligned}
\tag{4.3}
$$

which implies the required result, by the definition of $\mathrm{Err}(h)^2_\theta$ in (1.4). □

The above result implies the first estimate of Proposition 3.1.

Proof of (3.1) of Proposition 3.1. It suffices to apply Proposition 4.2 for $\theta = T$ and observe that the Lipschitz continuity of g implies that

$$\begin{aligned}&\mathbb{E}[|g(\tau, X_\tau) - g(\bar\tau, \bar X_{\bar\tau})|^2]\\&\quad\le C_L \mathbb{E}\left[|\tau - \bar\tau|^2 + \|X_{\bar\tau} - \bar X_{\bar\tau}\|^2 + \left\|\int_{\tau\wedge\bar\tau}^{\tau\vee\bar\tau} b(X_s)\,\mathrm{d}s + \int_{\tau\wedge\bar\tau}^{\tau\vee\bar\tau} \sigma(X_s)\,\mathrm{d}W_s\right\|^2\right],\end{aligned}$$

where $|\tau - \bar\tau|^2 \le T|\tau - \bar\tau|$, $\mathbb{E}[\|X_{\bar\tau} - \bar X_{\bar\tau}\|^2] \le C_L h$ by Proposition 4.1 and

$$\mathbb{E}\left[\left\|\int_{\tau\wedge\bar\tau}^{\tau\vee\bar\tau} b(X_s)\,\mathrm{d}s + \int_{\tau\wedge\bar\tau}^{\tau\vee\bar\tau} \sigma(X_s)\,\mathrm{d}W_s\right\|^2\right] \le \mathbb{E}[\xi_L|\tau - \bar\tau|]$$

by Doob's inequality, **(HL)** and Proposition 4.1 again. □

In order to prove (3.2) of Proposition 3.1, we need the following easy lemma whose proof is standard see [4] for details.

Lemma 4.1. *Let* **(HL)** *hold. Then,*

$$\max_{i<n}\left(\|\bar Y_{t_i}\| + \sqrt{h}\|\bar Z_{t_i}\|\right) \le \xi_L \quad \textit{and} \quad \|\bar Y\|_{\mathcal{S}^2} + \|\bar Z_\phi\|_{\mathcal{H}^2} + \|\tilde Z\|_{\mathcal{H}^2} \le C_L. \tag{4.4}$$

Proof of (3.2) of Proposition 3.1. Applying Proposition 4.2 to $\theta := \tau_+ \wedge \bar\tau$ and recalling Remark 2.2 leads to

$$\mathrm{Err}(h)^2_{\tau_+\wedge\bar\tau} \le C_L\left(h + \mathbb{E}[|Y_{\tau_+\wedge\bar\tau} - \bar Y_{\tau_+\wedge\bar\tau}|^2] + \mathcal{R}(Y)^\pi_{\mathcal{S}^2} + \mathcal{R}(Z)^\pi_{\mathcal{H}^2}\right).$$

It remains to show that

$$\begin{aligned}&\mathbb{E}[|\bar Y_{\tau_+\wedge\bar\tau} - Y_{\tau_+\wedge\bar\tau}|^2]\\&\quad\le C_L\left(h + \mathbb{E}\big[\mathbb{E}[\xi_L|\tau - \bar\tau||\mathcal{F}_{\tau_+\wedge\bar\tau}]^2\big] + \mathbb{E}\left[\mathbf{1}_{\bar\tau<\tau}\mathbb{E}\left[\int_{\bar\tau}^{\tau}\|Z_s\|\,\mathrm{d}s|\mathcal{F}_{\bar\tau}\right]^2\right]\right).\end{aligned}\tag{4.5}$$

Since f is L-Lipschitz continuous under **(HL)**, we can find an $\mathbb{R}^d$-valued adapted process χ which is bounded by L and satisfies

$$f\left(\bar X_{\phi(s)}, \bar Y_{\phi(s)}, \bar Z_{\phi(s)}\right) = f\left(\bar X_{\phi(s)}, \bar Y_{\phi(s)}, 0\right) + \left\langle \chi_{\phi(s)}, \bar Z_{\phi(s)}\right\rangle \tag{4.6}$$

on $[0, T]$. Set

$$H_t := \mathcal{E}\left(\int_0^t \mathbf{1}_{\tau_+\le s<\bar\tau}\chi_{\phi(s)}\,\mathrm{d}W_s\right), \qquad t \le T,$$

where $\mathcal{E}$ stands for the usual Doléans–Dade exponential martingale, and define $\mathbb{Q} \sim \mathbb{P}$ by

$\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} = H_T$. It follows from Girsanov's theorem that

$$W^{\mathbb{Q}} = W - \int_0^{\cdot} \mathbf{1}_{\tau_+ \le s < \bar{\tau}}\, \chi_{\phi(s)}\, \mathrm{d}s$$

is a $\mathbb{Q}$-Brownian motion. Now, observe that, by (4.6) and (2.6),

$$Y_t = g(\tau, X_\tau) + \int_{t\wedge\tau}^{\tau} f(X_s, Y_s, Z_s)\,\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s\,\mathrm{d}W_s^{\mathbb{Q}}, \tag{4.7}$$

$$\bar{Y}_t = g(\bar{\tau}, \bar{X}_{\bar{\tau}}) + \int_{t\wedge\bar{\tau}}^{\bar{\tau}} \big(f\big(\bar{X}_{\phi(s)}, \bar{Y}_{\phi(s)}, \bar{Z}_{\phi(s)}\big) - \mathbf{1}_{\tau_+\le s}\langle \chi_{\phi(s)}, \tilde{Z}_s\rangle\big)\,\mathrm{d}s - \int_{t\wedge\bar{\tau}}^{\bar{\tau}} \tilde{Z}_s\,\mathrm{d}W_s^{\mathbb{Q}}. \tag{4.8}$$

In view of (4.6)–(4.8), it then suffices to show that

$$\mathbb{E}\big[\mathbb{E}^{\mathbb{Q}}[g(\bar{\tau}, \bar{X}_{\bar{\tau}}) - g(\tau, X_\tau)|\mathcal{F}_{\tau_+\wedge\bar{\tau}}]^2\big] \le C_L\big(h + \mathbb{E}\big[\mathbb{E}[\xi_L|\tau-\bar{\tau}||\mathcal{F}_{\tau_+\wedge\bar{\tau}}]^2\big]\big), \tag{4.9}$$

$$\mathbb{E}\bigg[\mathbf{1}_{\tau_+<\bar{\tau}}\mathbb{E}^{\mathbb{Q}}\bigg[\int_{\tau_+}^{\bar{\tau}} f\big(\bar{X}_{\phi(s)}, \bar{Y}_{\phi(s)}, 0\big)\,\mathrm{d}s\bigg|\mathcal{F}_{\tau_+}\bigg]^2\bigg] \le \mathbb{E}\big[\mathbb{E}[\xi_L(|\tau-\bar{\tau}|+h)|\mathcal{F}_{\tau_+\wedge\bar{\tau}}]^2\big], \tag{4.10}$$

$$\mathbb{E}\bigg[\mathbf{1}_{\tau_+<\bar{\tau}}\mathbb{E}^{\mathbb{Q}}\bigg[\int_{\tau_+}^{\bar{\tau}} \langle \chi_{\phi(s)}, \bar{Z}_{\phi(s)} - \tilde{Z}_s\rangle\,\mathrm{d}s\bigg|\mathcal{F}_{\tau_+}\bigg]^2\bigg] \le C_L h, \tag{4.11}$$

$$\begin{aligned}\mathbb{E}\bigg[\mathbf{1}_{\bar{\tau}<\tau_+}\mathbb{E}^{\mathbb{Q}}\bigg[\int_{\bar{\tau}}^{\tau} f(X_s, Y_s, Z_s)\,\mathrm{d}s\bigg|\mathcal{F}_{\bar{\tau}}\bigg]^2\bigg] &\le C_L\big(h + \mathbb{E}\big[\mathbb{E}[\xi_L|\tau-\bar{\tau}||\mathcal{F}_{\tau_+\wedge\bar{\tau}}]^2\big]\big) \\ &\quad + C_L\mathbb{E}\bigg[\mathbf{1}_{\bar{\tau}<\tau}\mathbb{E}\bigg[\int_{\bar{\tau}}^{\tau} \|Z_s\|\,\mathrm{d}s|\mathcal{F}_{\bar{\tau}}\bigg]^2\bigg].\end{aligned} \tag{4.12}$$

We start with the first term. By using **(HL)**, applying Itô's lemma to $(g(t, X_t))_{t\ge 0}$ between $\bar{\tau}$ and τ, using Proposition 4.1 and the bound on χ, as well as standard estimates (recall **(Hg)** and Proposition 4.1), we easily check that on $\{\tau_+ > \bar{\tau}\} \subset \{\tau > \bar{\tau}\}$,

$$\begin{aligned}&\big|\mathbb{E}^{\mathbb{Q}}[g(\tau, X_\tau) - g(\bar{\tau}, \bar{X}_{\bar{\tau}})|\mathcal{F}_{\bar{\tau}}]\big| \\ &\quad \le C_L\|X_{\bar{\tau}} - \bar{X}_{\bar{\tau}}\| + \bigg|\mathbb{E}^{\mathbb{Q}}\bigg[\int_{\bar{\tau}}^{\tau} \big(\mathbf{1}_{\tau_+\le s<\bar{\tau}}\langle \chi_{\phi(s)}\sigma^*, Dg\rangle + \mathcal{L}g\big)(s, X_s)\,\mathrm{d}s\bigg|\mathcal{F}_{\bar{\tau}}\bigg]\bigg| \\ &\quad \le C_L\|X_{\bar{\tau}} - \bar{X}_{\bar{\tau}}\| + \mathbb{E}[\xi_L|\tau_+ - \bar{\tau}||\mathcal{F}_{\bar{\tau}}].\end{aligned}$$

Similarly, on $\{\tau_+ < \bar{\tau}\}$,

$$\big|\mathbb{E}^{\mathbb{Q}}[g(\tau_+, X_{\tau_+}) - g(\bar{\tau}, \bar{X}_{\bar{\tau}})|\mathcal{F}_{\tau_+}]\big| \le C_L\|X_{\tau_+} - \bar{X}_{\tau_+}\| + \mathbb{E}[\xi_L|\tau_+ - \bar{\tau}||\mathcal{F}_{\tau_+}].$$

We then conclude the proof of (4.9) by appealing to **(HL)** and Proposition 4.1 to obtain

$$\mathbb{E}[\|X_{\tau_+} - \bar{X}_{\tau_+}\|^2 + \|X_{\bar{\tau}} - \bar{X}_{\bar{\tau}}\|^2] + \mathbb{E}[|g(\tau_+, X_{\tau_+}) - g(\tau, X_\tau)|^2] \le C_L h;$$

recall that $0 \le \tau_+ - \tau \le h$.

The second term (4.10) is controlled by appealing to **(HL)**, Lemma 4.1 and Proposition 4.1; recall that $\tau_+ - \tau \le h$. Concerning the third term (4.11), we observe that $\{\tau_+ \le s\} = \{\tau \le \phi(s)\} \in \mathcal{F}_{\phi(s)}$ and $\{\bar\tau > s\} = \{\bar\tau > \phi(s)\} \in \mathcal{F}_{\phi(s)}$. It then follows from (2.7) that on $\{\tau_+ < \bar\tau\}$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\int_{\tau_+}^{\bar\tau} \langle \chi_{\phi(s)}, \bar Z_{\phi(s)} - \tilde Z_s\rangle \,\mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_+\wedge\bar\tau}\right] \\
&= \mathbb{E}\left[\int_{\tau_+}^{\bar\tau} H_s \langle \chi_{\phi(s)}, \bar Z_{\phi(s)} - \tilde Z_s\rangle \,\mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_+}\right] \\
&= \mathbb{E}\left[\int_{\tau_+}^{\bar\tau} H_{\phi(s)} \left\langle \chi_{\phi(s)}, h^{-1}\int_{\phi(s)}^{\phi(s)+h} \tilde Z_u \,\mathrm{d}u - \tilde Z_s\right\rangle \mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_+}\right] \\
&\quad + \mathbb{E}\left[\int_{\tau_+}^{\bar\tau} \left(H_s - H_{\phi(s)}\right) \langle \chi_{\phi(s)}, \bar Z_{\phi(s)} - \tilde Z_s\rangle \,\mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_+}\right]
\end{aligned}$$

and, since $\bar\tau$ and τ_+ take values in π,

$$\int_{\tau_+}^{\bar\tau} H_{\phi(s)} \left\langle \chi_{\phi(s)}, h^{-1}\int_{\phi(s)}^{\phi(s)+h} \tilde Z_u \,\mathrm{d}u - \tilde Z_s\right\rangle \mathrm{d}s = 0.$$

On the other hand, the Cauchy–Schwarz inequality and the boundedness of χ imply that

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_{\tau_+}^{\bar\tau} \left(H_s - H_{\phi(s)}\right) \langle \chi_{\phi(s)}, \bar Z_{\phi(s)} - \tilde Z_s\rangle \,\mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_+\wedge\bar\tau}\right]\right| \\
&\le C_L \left|\mathbb{E}\left[\int_{\tau_+}^{\bar\tau} \left(H_s - H_{\phi(s)}\right)^2 \mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_+\wedge\bar\tau}\right]\right|^{1/2} \left|\mathbb{E}\left[\int_{\tau_+}^{\bar\tau} \left\| \bar Z_{\phi(s)} - \tilde Z_s\right\|^2 \mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_+\wedge\bar\tau}\right]\right|^{1/2} \\
&\le \xi_L h^{1/2} \left|\mathbb{E}\left[\int_{\tau_+}^{\bar\tau} \left\| \bar Z_{\phi(s)} - \tilde Z_s\right\|^2 \mathrm{d}s \,\middle|\, \mathcal{F}_{\tau_+\wedge\bar\tau}\right]\right|^{1/2}.
\end{aligned}$$

Recalling Lemma 4.1 and combining the above inequalities leads to (4.11).

The last term (4.12) is easily controlled by using **(HL)**, Remark 2.2 and Proposition 4.1. □

5. Exit time approximation error: proof of Theorem 3.1

In this section, we provide the proof of Theorem 3.1. We start with a partial argument which essentially allows us to reduce to the case where $m = 1$, that is, $\mathcal{O}$ has no corners, by working separately on the exit times of the different domains $\mathcal{O}^\ell$,

$$\tau_+^\ell := \inf\{t \in \pi : \exists s \le t \text{ s.t. } X_s \notin \mathcal{O}^\ell\} \wedge T \quad \text{and} \quad \bar\tau^\ell := \inf\{t \in \pi : \bar X_t \notin \mathcal{O}^\ell\} \wedge T.$$

Below, we shall prove the following Proposition.

Proposition 5.1. *Assume that* **(HL)**, **(D1)** *and* **(C)** *hold. Then, for each* $\varepsilon > 0$,

$$\mathbb{E}\big[\mathbb{E}[|\tau_+^\ell - \bar\tau^\ell||\mathcal{F}_{\tau_+^\ell \wedge \bar\tau^\ell}]^2\big] \le C_L^\varepsilon h^{1-\varepsilon} \qquad \forall 1 \le \ell \le m. \tag{5.1}$$

This implies the statements of Theorem 3.1.

Proof of Theorem 3.1. Since $\tau_+ = \min_{\ell \le m} \tau_+^\ell$ and $\bar\tau = \min_{\ell \le m} \bar\tau^\ell$, we have

$$\mathbb{E}[|\tau_+ - \bar\tau||\mathcal{F}_{\tau_+ \wedge \bar\tau}] \le \sum_{\ell=1}^m \mathbb{E}[|\tau_+^\ell - \bar\tau^\ell||\mathcal{F}_{\tau_+^\ell \wedge \bar\tau^\ell}](\mathbf{1}_{\tau_+ = \tau_+^\ell < \bar\tau} + \mathbf{1}_{\bar\tau = \bar\tau^\ell \le \tau_+}),$$

which, combined with (5.1), leads to

$$\mathbb{E}\big[\mathbb{E}[|\tau - \bar\tau||\mathcal{F}_{\tau_+ \wedge \bar\tau}]^2\big] \le C_L^\varepsilon h^{1-\varepsilon} \tag{5.2}$$

since $|\tau_+ - \tau| \le h$. This leads to the second assertion of Theorem 3.1. On the other hand, given a positive random variable ξ satisfying $\mathbb{E}[\xi^p] \le C_L^p$ for all $p \ge 1$, we deduce from Hölder's inequality that

$$\mathbb{E}[\xi|\tau - \bar\tau||\mathcal{F}_{\tau_+ \wedge \bar\tau}]^2 \le \xi_L^\varepsilon \mathbb{E}\big[|\tau - \bar\tau|^{1/(1-\varepsilon)}|\mathcal{F}_{\tau_+ \wedge \bar\tau}\big]^{2(1-\varepsilon)} \le \xi_L^\varepsilon T^{2\varepsilon} \mathbb{E}[|\tau - \bar\tau||\mathcal{F}_{\tau_+ \wedge \bar\tau}]^{2(1-\varepsilon)}$$

and

$$\mathbb{E}\big[\xi \mathbb{E}[\xi|\tau - \bar\tau||\mathcal{F}_{\tau_+ \wedge \bar\tau}]^2\big] \le C_L^\varepsilon \mathbb{E}\big[\mathbb{E}[|\tau - \bar\tau||\mathcal{F}_{\tau_+ \wedge \bar\tau}]^2\big]^{1-\varepsilon}.$$

In view of (5.2), this leads to the first assertion of Theorem 3.1, after possibly changing ε. □

The rest of this section is devoted to the proof of (5.1) for some fixed ℓ. We first provide an a priori control on the difference between τ_+^ℓ and $\bar\tau^\ell$. We use the standard idea of introducing a test function on which we can apply Itô's lemma between τ_+^ℓ and $\bar\tau^\ell$ so that the Lebesgue integral term provides an upper bound for the difference between these two times; see, for example, Lemma 3.1, Chapter 3 in [10] for an application to the construction of upper bounds for the moments of the first exit time of a uniformly elliptic diffusion from a bounded domain.

To this end, we introduce the family of test functions

$$F_\ell := d_\ell^2/\gamma, \qquad 1 \le \ell \le m,$$

for some $\gamma > 0$ to be fixed below. Here, d_ℓ is a $C^2(\mathbb{R}^d)$ function which coincides with the algebraic distance to $\partial\mathcal{O}^\ell$ on a neighborhood of $\partial\mathcal{O}^\ell$ and such that

$$\mathcal{O}^\ell := \{x \in \mathbb{R}^d : d_\ell(x) > 0\} \quad \text{and} \quad \partial\mathcal{O}^\ell := \{x \in \mathbb{R}^d : d_\ell(x) = 0\}.$$

The existence of such a map is guaranteed by the smoothness assumption **(D1)** (see, e.g., [12]). Observe that after possibly changing L and considering a suitable extension of d_ℓ outside a neighbourhood of the compact boundary $\partial\mathcal{O}^\ell$, we can assume that

$$\|d_\ell\| + \|Dd_\ell\| + \|D^2 d_\ell\| \le L \qquad \text{on } \mathbb{R}^d. \tag{5.3}$$

Observe that

$$\mathcal{L}F_\ell = \frac{1}{\gamma}\big[(2\langle b, n_\ell\rangle + \mathrm{Tr}[aD^2 d_\ell])d_\ell + \mathrm{Tr}[a(n_\ell)^* n_\ell]\big], \tag{5.4}$$

where $n_\ell := Dd_\ell$ coincides with the unit inward normal for $x \in \partial\mathcal{O}^\ell$; recall **(D1)**.

In view of **(HL)**, **(D1)**, (5.3) and **(C)**, there is some $C_L > 0$ such that, for each $1 \le \ell \le m$,

$$\mathcal{L}F_\ell \ge \frac{1}{\gamma}\big(-C_L d_\ell + n_\ell a(n_\ell)^*\big) \ge 1 \quad \text{and} \quad n_\ell a(n_\ell)^* \ge L^{-1}/2 \qquad \text{on } B(\partial\mathcal{O}^\ell, r) \tag{5.5}$$

if we choose $r > 0$ and $\gamma > 0$ small enough, but depending only on L. For later use, also observe that, after possibly changing r, one can actually choose it such that

$$n_\ell(x)a(y)n_\ell(x)^* \ge L^{-1}/2 \qquad \text{for all } x, y \in B(\partial\mathcal{O}^\ell, r) \text{ s.t. } \|x-y\| \le r. \tag{5.6}$$

We now fix $r, \gamma > 0$ such that (5.5) and (5.6) hold, and define the sets

$$A_\ell := \{X_s \in B(\partial\mathcal{O}^\ell, r), \forall s \in [\bar\tau^\ell, \tau_+^\ell]\}, \qquad B_\ell := \{|d_\ell(X_{\tau_+^\ell})| \le h^{1/2-\eta}\},$$
$$\bar A_\ell := \{\bar X_s \in B(\partial\mathcal{O}^\ell, r), \forall s \in [\tau_+^\ell, \bar\tau^\ell]\}, \qquad \bar B_\ell := \{|d_\ell(\bar X_{\bar\tau^\ell})| \le h^{1/2-\eta}\}$$

for some $\eta \in (0, 1/4)$ to be chosen later. Observe that A_ℓ (resp. $\bar A_\ell$) is well defined on $\{\bar\tau^\ell \le \tau_+^\ell\}$ (resp. $\{\tau_+^\ell \le \bar\tau^\ell\}$).

We can now provide our first control on $|\tau_+^\ell - \bar\tau^\ell|$. Recall that ξ_L^ε (ξ_L if it does not depend on some extra parameter ε) denotes a positive random variable whose value may change from line to line, but satisfies $\mathbb{E}[|\xi_L^\varepsilon|^p] \le C_L^{\varepsilon,p}$ for all $p \ge 1$.

Lemma 5.1. *Assume that* **(HL)** *and* **(D1)** *hold. Then, for each* $\varepsilon \in (0,1)$,

$$\begin{aligned}\mathbb{E}[|\tau_+^\ell - \bar\tau^\ell||\mathcal{F}_{\tau_+^\ell \wedge \bar\tau^\ell}] \le \xi_L^\varepsilon\big\{&h^{1/2} + (T-\bar\tau^\ell)^{1/2}\mathbb{P}[(A_\ell \cap B_\ell)^c|\mathcal{F}_{\bar\tau^\ell}]^{1-\varepsilon}\mathbf{1}_{\{\tau_+^\ell > \bar\tau^\ell\}} \\ &+ (T-\tau_+^\ell)^{1/2}\mathbb{P}[(\bar A_\ell \cap \bar B_\ell)^c|\mathcal{F}_{\tau_+^\ell}]^{1-\varepsilon}\mathbf{1}_{\{\tau_+^\ell < \bar\tau^\ell\}}\big\}\end{aligned}$$

for each $1 \le \ell \le m$.

Proof. 1. We first work on the event $\{\tau_+^\ell > \bar\tau^\ell\}$. It follows from (5.5) and Itô's lemma that

$$\begin{aligned}\mathbb{E}[\tau_+^\ell - \bar\tau^\ell|\mathcal{F}_{\bar\tau^\ell}] &\le \mathbb{E}\bigg[\mathbf{1}_{A_\ell\cap B_\ell}\int_{\bar\tau^\ell}^{\tau_+^\ell}\mathcal{L}F_\ell(X_s)\,\mathrm{d}s\bigg|\mathcal{F}_{\bar\tau^\ell}\bigg] + (T-\bar\tau^\ell)\mathbb{P}[(A_\ell\cap B_\ell)^c|\mathcal{F}_{\bar\tau^\ell}] \\ &\le \mathbb{E}\bigg[\mathbf{1}_{A_\ell\cap B_\ell}\bigg(\int_{\bar\tau^\ell}^{\tau_+^\ell}\mathcal{L}F_\ell(X_s)\,\mathrm{d}s + \int_{\bar\tau^\ell}^{\tau_+^\ell}DF_\ell(X_s)\sigma(X_s)\,\mathrm{d}W_s\bigg)\bigg|\mathcal{F}_{\bar\tau^\ell}\bigg] \\ &\quad - \mathbb{E}\bigg[\mathbf{1}_{A_\ell\cap B_\ell}\int_{\bar\tau^\ell}^{\tau_+^\ell}DF_\ell(X_s)\sigma(X_s)\,\mathrm{d}W_s\bigg|\mathcal{F}_{\bar\tau^\ell}\bigg]\end{aligned}$$

$$
\begin{aligned}
&+ (T-\bar\tau^\ell)\mathbb{P}[(A_\ell\cap B_\ell)^c|\mathcal{F}_{\bar\tau^\ell}]\\
&\le \gamma^{-1}\mathbb{E}\big[\big(d_\ell^2(X_{\tau_+^\ell})-d_\ell^2(X_{\bar\tau^\ell})\big)\mathbf{1}_{A_\ell\cap B_\ell}|\mathcal{F}_{\bar\tau^\ell}\big]\\
&\quad+\mathbb{E}\bigg[\mathbf{1}_{(A_\ell\cap B_\ell)^c}\int_{\bar\tau^\ell}^{\tau_+^\ell} DF_\ell(X_s)\sigma(X_s)\,\mathrm{d}W_s\bigg|\mathcal{F}_{\bar\tau^\ell}\bigg]\\
&\quad+(T-\bar\tau^\ell)\mathbb{P}[(A_\ell\cap B_\ell)^c|\mathcal{F}_{\bar\tau^\ell}],
\end{aligned}
$$

where, by the Hölder and Burkholder–Davis–Gundy inequalities, the Lipschitz continuity of σ and DF_ℓ (see **(HL)** and (5.3)) and Proposition 4.1,

$$
\mathbb{E}\bigg[\mathbf{1}_{(A_\ell\cap B_\ell)^c}\int_{\bar\tau^\ell}^{\tau_+^\ell} DF_\ell(X_s)\sigma(X_s)\,\mathrm{d}W_s\bigg|\mathcal{F}_{\bar\tau^\ell}\bigg]\le \xi_L^\varepsilon(T-\bar\tau^\ell)^{1/2}\mathbb{P}[(A_\ell\cap B_\ell)^c|\mathcal{F}_{\bar\tau^\ell}]^{1-\varepsilon}
$$

for all $\varepsilon\in(0,1)$. We now recall that $|d_\ell(X_{\tau_+^\ell})|\le h^{1/2-\eta}$ on B_ℓ, which implies that

$$
\mathbb{E}\big[\big(d_\ell^2(X_{\tau_+^\ell})-d_\ell^2(X_{\bar\tau^\ell})\big)\mathbf{1}_{A_\ell\cap B_\ell}|\mathcal{F}_{\bar\tau^\ell}\big]\le \mathbb{E}[d_\ell^2(X_{\tau_+^\ell})\mathbf{1}_{A_\ell\cap B_\ell}|\mathcal{F}_{\bar\tau^\ell}]\le h^{1-2\eta}.
$$

In view of the above inequalities, this provides the required estimate on the event set $\{\tau_+^\ell>\bar\tau^\ell\}$ since $\eta<1/4$.

2. We now work on the event $\{\tau_+^\ell<\bar\tau^\ell\}$. By Proposition 4.1,

$$
\mathbb{E}\bigg[\mathbf{1}_{\bar A_\ell\cap\bar B_\ell}\int_{\tau_+^\ell}^{\bar\tau^\ell}\big|\mathcal{L}^{\bar X_{\phi(s)}}F_\ell(\bar X_s)-\mathcal{L}^{\bar X_s}F_\ell(\bar X_s)\big|\,\mathrm{d}s\bigg|\mathcal{F}_{\tau_+^\ell}\bigg]\le\xi_L h^{1/2}
$$

with the notation $\mathcal{L}^yF_\ell:=\partial_tF_\ell+\langle b(y),DF_\ell\rangle+\frac12\operatorname{Tr}[a(y)D^2F_\ell]$, so that $\mathcal{L}^{\bar X_s}F_\ell(\bar X_s)=\mathcal{L}F_\ell(\bar X_s)$. Arguing as above, it follows that, on $\{\bar\tau^\ell>\tau_+^\ell\}$,

$$
\begin{aligned}
\mathbb{E}[\bar\tau^\ell-\tau_+^\ell|\mathcal{F}_{\tau_+^\ell}]&\le\xi_Lh^{1/2}+\gamma^{-1}\mathbb{E}\big[\big(d_\ell^2(\bar X_{\bar\tau^\ell})-d_\ell^2(\bar X_{\tau_+^\ell})\big)\mathbf{1}_{\bar A_\ell\cap\bar B_\ell}|\mathcal{F}_{\tau_+^\ell}\big]\\
&\quad+\mathbb{E}\bigg[\mathbf{1}_{(\bar A_\ell\cap\bar B_\ell)^c}\int_{\tau_+^\ell}^{\bar\tau^\ell}DF_\ell(\bar X_s)\sigma\big(\bar X_{\phi(s)}\big)\,\mathrm{d}W_s\bigg|\mathcal{F}_{\tau_+^\ell}\bigg]\\
&\quad+(T-\tau_+^\ell)\mathbb{P}[(\bar A_\ell\cap\bar B_\ell)^c|\mathcal{F}_{\tau_+^\ell}]\\
&\le\xi_Lh^{1/2}+\gamma^{-1}h^{1/2}+\xi_L^\varepsilon(T-\tau_+^\ell)^{1/2}\mathbb{P}[(\bar A_\ell\cap\bar B_\ell)^c|\mathcal{F}_{\tau_+^\ell}]^{1-\varepsilon}.
\end{aligned}
$$

□

It remains to control the different terms that appear in the upper bound of Lemma 5.1.

For notational convenience, we now introduce the sets (recall that $0<\eta<1/4$)

$$
E_\ell:=\{d_\ell(X_{\bar\tau^\ell})\le h^{1/2-\eta}\}\quad\text{and}\quad\bar E_\ell:=\{d_\ell(\bar X_{\tau_+^\ell})\le h^{1/2-\eta}\},\qquad 1\le\ell\le m.
$$

Remark 5.1. Observe that

$$
\mathbb{P}[E_\ell^c\cap\{\bar\tau^\ell<\tau_+^\ell\}]\le\mathbb{P}[E_\ell^c\cap\{\bar\tau^\ell<T\}]\le\mathbb{P}[\{d_\ell(X_{\bar\tau^\ell})-d_\ell(\bar X_{\bar\tau^\ell})\ge h^{1/2-\eta}\}\cap\{\bar\tau^\ell<T\}]
$$

since $d_\ell(\bar{X}_{\bar{\tau}^\ell}) \le 0$ on $\{\bar{\tau}^\ell < T\}$. Using (5.3), Chebyshev's inequality and Proposition 4.1, we then deduce that, for each $\varepsilon \in (0,1)$, there exists $C_L^\varepsilon > 0$ such that

$$\mathbb{P}[E_\ell^c \cap \{\bar{\tau}^\ell < \tau_+^\ell\}] \le C_L^\varepsilon h^{1-\varepsilon}.$$

Similarly, if τ^ℓ denotes the first exit time of $(t, X_t)_{t\ge 0}$ from $[0,T) \times \mathcal{O}^\ell$, then we have

$$\begin{aligned}
\mathbb{P}[\bar{E}_\ell^c \cap \{\bar{\tau}^\ell > \tau_+^\ell\}] \le{}& \mathbb{P}\Big[\big\{d_\ell(\bar{X}_{\tau_+^\ell}) - d_\ell(X_{\tau_+^\ell}) \ge \tfrac{1}{2}h^{1/2-\eta}\big\} \cap \big\{d_\ell(X_{\tau_+^\ell}) \le \tfrac{1}{2}h^{1/2-\eta}\big\} \cap \{\tau_+^\ell < T\}\Big] \\
&+ \mathbb{P}\Big[\big\{d_\ell(X_{\tau_+^\ell}) - d_\ell(X_{\tau^\ell}) > \tfrac{1}{2}h^{1/2-\eta}\big\} \cap \{\tau_+^\ell < T\}\Big] \\
\le{}& C_L^\varepsilon h^{1-\varepsilon},
\end{aligned}$$

where the last inequality follows from Chebyshev's inequality, Proposition 4.1 and the fact that $\tau_+^\ell - \tau^\ell \le h$. Note that the term $d_\ell(X_{\tau_+^\ell}) - d_\ell(X_{\tau^\ell})$ could be controlled by Bernstein-type inequalities in order to avoid the explosion of the constant with ε. However, to the best of our knowledge, such inequalities are not available in the existing literature for the term $d_\ell(\bar{X}_{\tau_+^\ell}) - d_\ell(X_{\tau_+^\ell})$ and Chebyshev's inequality remains the most natural tool to apply here.

Combining the above remark with the next two technical lemmas allows us to control the right-hand side terms in the upper bound of Lemma 5.1. Thus, the statement of Proposition 5.1 is a direct consequence of Lemma 5.1 combined with Remark 5.1, Lemmas 5.2 and 5.3 below, applied for η sufficiently small.

Lemma 5.2. *Assume that* **(HL)**, **(D1)** *and* **(C)** *hold. Then, for each* $\varepsilon \in (0,1)$,

$$\mathbb{P}[A_\ell^c|\mathcal{F}_{\bar{\tau}^\ell}]\mathbf{1}_{E_\ell \cap \{\tau_+^\ell > \bar{\tau}^\ell\}} + \mathbb{P}[\bar{A}_\ell^c|\mathcal{F}_{\tau_+^\ell}]\mathbf{1}_{\bar{E}_\ell \cap \{\tau_+^\ell < \bar{\tau}^\ell\}} \le \xi_L^\varepsilon h^{(1/2-\eta)(1-\varepsilon)} \qquad \forall \ell \le m. \tag{5.7}$$

Lemma 5.3. *Assume that* **(HL)**, **(D1)** *and* **(C)** *hold. Then, for each* $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\mathbb{P}[A_\ell \cap B_\ell^c|\mathcal{F}_{\bar{\tau}^\ell}]\mathbf{1}_{E_\ell \cap \{\tau_+^\ell > \bar{\tau}^\ell\}} + \mathbb{P}[\bar{A}_\ell \cap \bar{B}_\ell^c|\mathcal{F}_{\tau_+^\ell}]\mathbf{1}_{\bar{E}_\ell \cap \{\tau_+^\ell < \bar{\tau}^\ell\}} \\
&\quad \le \xi_L^\varepsilon \frac{h^{(1/2-\eta)(1-\varepsilon)}}{\sqrt{T - \bar{\tau}^\ell \wedge \tau_+^\ell}} \qquad \forall \ell \le m.
\end{aligned} \tag{5.8}$$

Proof of Lemma 5.2. 1. We first prove the bound for the first term. Let V be defined by $V_t := d_\ell(X_{\bar{\tau}^\ell + t})$ for $t \ge 0$ and let ϑ^y be the first time when V reaches $y \in \mathbb{R}$. Using $A_\ell^c = A_\ell^c \cap (\{\vartheta^0 \ge \vartheta^r\} \cup \{\vartheta^0 < \vartheta^r\})$, we deduce that on $\{\tau_+^\ell > \bar{\tau}^\ell\} \cap E_\ell$,

$$\mathbb{P}[A_\ell^c|\mathcal{F}_{\bar{\tau}^\ell}] \le \mathbb{P}[\vartheta^0 \ge \vartheta^r|\mathcal{F}_{\bar{\tau}^\ell}] + \mathbb{P}\Big[\Big\{\sup_{s\in[\tau^\ell,\tau_+^\ell]} |d_\ell(X_s)| \ge r\Big\} \cap \{\tau^\ell < T\}\Big|\mathcal{F}_{\bar{\tau}^\ell}\Big],$$

where, by (5.3), Chebyshev's inequality and Proposition 4.1, on $\{\tau_+^\ell > \bar\tau^\ell\} \subset \{\tau^\ell > \bar\tau^\ell\}$,

$$\mathbb{P}\Big[\Big\{\sup_{s\in[\tau^\ell,\tau_+^\ell]} |d_\ell(X_s)| \ge r\Big\} \cap \{\tau^\ell < T\}\big|\mathcal{F}_{\bar\tau^\ell}\Big] \le r^{-2}\mathbb{E}\Big[\sup_{s\in[\tau^\ell,\tau_+^\ell]} |d_\ell(X_s) - d_\ell(X_{\tau^\ell})|^2\big|\mathcal{F}_{\bar\tau^\ell}\Big]$$
$$\le \xi_L h;$$

recall that $\tau_+^\ell - \tau^\ell \le h$. It remains to provide a suitable bound for $\mathbb{P}[\vartheta^0 \ge \vartheta^r|\mathcal{F}_{\bar\tau^\ell}]$. From now on, we assume, without loss of generality, that

$$2h^{1/2-\eta} \le r. \tag{5.9}$$

Set $\vartheta := \vartheta^0 \wedge \vartheta^r$. Thanks to **(C)** and **(HL)**, we can define $\mathbb{Q} \sim \mathbb{P}$ by the density

$$H = \mathcal{E}_{\bar\tau^\ell+\vartheta}\bigg(-\int_0^\cdot \mathbf{1}_{E_\ell}\mathbf{1}_{s\ge\bar\tau^\ell}(n_\ell\sigma)(X_s)((n_\ell a n_\ell^*)(X_s))^{-1}\mathcal{L}d_\ell(X_s)\,\mathrm{d}W_s\bigg).$$

Let

$$W^{\mathbb{Q}} := W + \mathbf{1}_{[\bar\tau^\ell,\infty)}\mathbf{1}_{E_\ell}\int_{\bar\tau^\ell}^{(\bar\tau^\ell+\vartheta)\wedge\cdot} (n_\ell\sigma)^*(X_s)((n_\ell a n_\ell^*)(X_s))^{-1}\mathcal{L}d_\ell(X_s)\,\mathrm{d}s$$

be the Brownian motion associated to $\mathbb{Q}$ by Girsanov's theorem. We have

$$V_{t\wedge\vartheta} = V_0 + \int_{\bar\tau^\ell}^{\bar\tau^\ell+t\wedge\vartheta} n_\ell(X_s)\sigma(X_s)\,\mathrm{d}W_s^{\mathbb{Q}} \qquad \text{on } E_\ell.$$

Set

$$\Lambda_t := \int_{\bar\tau^\ell}^{\bar\tau^\ell+t} \big\|n_\ell\big(X_{s\wedge(\bar\tau^\ell+\vartheta)}\big)\sigma\big(X_{s\wedge(\bar\tau^\ell+\vartheta)}\big)\big\|^2\,\mathrm{d}s.$$

By the Dambis–Dubins–Schwarz theorem (see Theorem 4.6, Chapter 3 in [19]), there exists a one-dimensional $\mathbb{Q}$-Brownian motion Z such that

$$V_{t\wedge\vartheta} = V_0 + Z_{\Lambda_{t\wedge\vartheta}} \qquad \text{on } E_\ell \cap \{\tau_+^\ell > \bar\tau^\ell\} = \{V_0 \le h^{1/2-\eta}, \tau_+^\ell > \bar\tau^\ell\}.$$

This implies that

$$\mathbb{Q}[\vartheta^0 \ge \vartheta^r|\mathcal{F}_{\bar\tau^\ell}] \le h^{1/2-\eta}/r \qquad \text{on } E_\ell \cap \{\tau_+^\ell > \bar\tau^\ell\}$$

(see, e.g., Exercise 8.13, Chapter 2.8 in [19]). We conclude by using Hölder's inequality and (5.3).

2. The bound for the second term in (5.7) is derived similarly. We now write

$$V_t := d_\ell(\bar X_{\tau_+^\ell+t}), \qquad t \ge 0.$$

As above, we denote by ϑ^y the first time when V reaches $y \in \mathbb{R}$ and observe that, by (5.9),

$$\mathbb{P}[\bar{A}_\ell^c|\mathcal{F}_{\tau_+^\ell}] \leq \mathbb{P}[\vartheta^{-h^{1/2-\eta}} > \vartheta^r|\mathcal{F}_{\tau_+^\ell}] + \mathbb{P}\Big[\sup_{s\in[\tilde{\tau}^\ell,\tilde{\tau}^\ell+h]} |d_\ell(\bar{X}_s) - d_\ell(\bar{X}_{\tilde{\tau}^\ell})| > h^{1/2-\eta}\big|\mathcal{F}_{\tau_+^\ell}\Big],$$

where $\tilde{\tau}^\ell := \tau_+^\ell + \vartheta^{-h^{1/2-\eta}}$ and, by (5.3), Chebyshev's inequality and Proposition 4.1,

$$\mathbb{P}\Big[\sup_{s\in[\tilde{\tau}^\ell,\tilde{\tau}^\ell+h]} |d_\ell(\bar{X}_s) - d_\ell(\bar{X}_{\tilde{\tau}^\ell})| > h^{1/2-\eta}\big|\mathcal{F}_{\tau_+^\ell}\Big] \leq \xi_L^\eta h.$$

In order to bound the term $\mathbb{P}[\vartheta^{-h^{1/2-\eta}} > \vartheta^r|\mathcal{F}_{\tau_+^\ell}]$, we observe that (5.6) implies that, for h sufficiently small,

$$\|n_\ell(\bar{X}_s)\sigma(\bar{X}_{\phi(s)})\| \geq L^{-1/2}/\sqrt{2} \qquad \text{on } \bar{E}_\ell \cap \{s \in [\tau_+^\ell, \theta^\ell]\} \cap \{\|\bar{X}_s - \bar{X}_{\phi(s)}\| \leq r\},$$

where $\theta^\ell := \inf\{t \geq \tau_+^\ell : \bar{X}_t \notin B(\partial\mathcal{O}^\ell, r)\} \wedge T$. Moreover, it follows from Proposition 4.1 that

$$\mathbb{P}\Big[\sup_{s\leq T}\|\bar{X}_s - \bar{X}_{\phi(s)}\| > r\Big] \leq C_L r^{-4} h.$$

Up to obvious modifications, this allows us to reproduce the arguments of step 1 on the event set $\bar{E}_\ell$. □

Proof of Lemma 5.3. We only prove the bound for the first term. The second one can be derived from similar arguments (see step 2 in the proof of Lemma 5.2). We use the notation of the proof of Lemma 5.2. We first observe that, on $E_l \cap \{\tau^\ell > \bar{\tau}^\ell\}$,

$$\begin{aligned}\mathbb{P}[A_\ell \cap B_\ell^c|\mathcal{F}_{\bar{\tau}^\ell}] &\leq \mathbb{P}[A_\ell \cap \{\vartheta^0 > (T - \bar{\tau}^\ell)\}|\mathcal{F}_{\bar{\tau}^\ell}] \\ &\quad + \mathbb{P}\Big[\{\tau^\ell < T\} \cap \sup_{s\in[\tau^\ell,\tau_+^\ell]} |d_\ell(X_s) - d_\ell(X_{\tau^\ell})| \geq h^{1/2-\eta}\big|\mathcal{F}_{\bar{\tau}^\ell}\Big] \\ &\leq \mathbb{P}\Big[A_\ell \cap \Big\{\min_{t\in[0,T-\bar{\tau}^\ell]} Z_{\Lambda_t} > -h^{1/2-\eta}\Big\}\big|\mathcal{F}_{\bar{\tau}^\ell}\Big] + \xi_L^\eta h,\end{aligned}$$

where the second inequality follows from Chebyshev's inequality, **(HL)** and Proposition 4.1; recall that $\tau_+^\ell - \tau^\ell \leq h$. Using Hölder's inequality, we then observe that

$$\begin{aligned}&\mathbb{P}\Big[A_\ell \cap \Big\{\min_{t\in[0,T-\bar{\tau}^\ell]} Z_{\Lambda_t} > -h^{1/2-\eta}\Big\}\big|\mathcal{F}_{\bar{\tau}^\ell}\Big] \\ &\quad \leq \xi_L^\varepsilon \mathbb{Q}\Big[A_\ell \cap \Big\{\min_{t\in[0,T-\bar{\tau}^\ell]} Z_{\Lambda_t} > -h^{1/2-\eta}\Big\}\big|\mathcal{F}_{\bar{\tau}^\ell}\Big]^{1-\varepsilon}.\end{aligned}$$

Since, by (5.6),

$$\Lambda_{T-\bar{\tau}^\ell} \geq (T - \bar{\tau}^\ell)(2L)^{-1} \qquad \text{on } A_\ell \cap \{\vartheta^0 > (T - \bar{\tau}^\ell)\} \cap \{\bar{\tau}^\ell < \tau_+^\ell\} \subset A_\ell \cap \{\bar{\tau}^\ell < \tau_+^\ell = T\},$$

we deduce from Chapter 2 of [19] that, on $E_\ell \cap \{\bar\tau^\ell < \tau_+^\ell\}$,

$$\mathbb{Q}\Big[A_\ell \cap \Big\{\min_{t\in[0,T-\bar\tau^\ell]} Z_{\Lambda_t} > -h^{1/2-\eta}\Big\}\Big|\mathcal{F}_{\bar\tau^\ell}\Big] \le \mathbb{Q}\Big[\min_{t\in[0,(T-\bar\tau^\ell)(2L)^{-1}]} Z_t > -h^{1/2-\eta}\Big|\mathcal{F}_{\bar\tau^\ell}\Big]$$
$$\le C_L(T-\bar\tau^\ell)^{-1/2}h^{1/2-\eta}.$$

We conclude by combining the above estimates. □

6. Regularity of the BSDE and the related PDE

6.1. Interpretation in terms of parabolic semilinear PDEs with Dirichlet boundary conditions

In this section, we denote by $X^{t,x}$ the solution of (1.1) with initial condition $x \in \bar{\mathcal{O}}$ at time $t \le T$. We also denote by $\tau^{t,x}$ the first exit time of $(s, X_s^{t,x})_{s\ge t}$ from $\mathcal{O}\times[0,T)$ and write $(Y^{t,x}, Z^{t,x})$ for the solution of (1.2) with $(X^{t,x}, \tau^{t,x})$ in place of (X,τ).

As usual, the deterministic function $(t,x)\in\bar D \mapsto u(t,x) := Y_t^{t,x}$ can be related to the semilinear parabolic equation

$$\begin{cases} 0 = -\mathcal{L}u(t,x) - f(x,u(t,x),Du(t,x)\sigma(x)), & (t,x)\in\mathcal{O}\times[0,T), \\ u|_{\partial_p D} = g, \end{cases} \tag{6.1}$$

where we recall that $\mathcal{L}$ denotes the Dynkin operator associated to the diffusion X, $\mathcal{L}\psi := \partial_t\psi + \langle b, D\psi\rangle + \frac{1}{2}\operatorname{Tr}[aD^2\psi]$ with $a := \sigma\sigma^*$ and $\partial_p D := ([0,T)\times\partial\mathcal{O}) \cup (\{T\}\times\bar{\mathcal{O}})$ is the parabolic boundary of D.

Proposition 6.1. *Let* **(HL)**, **(D1)**, **(D2)**, **(C)** *and* **(Hg)** *hold. The function u then has linear growth and is the unique continuous viscosity solution of* (6.1) *in the class of continuous solutions with polynomial growth.*

A similar result is proved in [7], but in the elliptic case. For the sake of completeness, we provide a slightly different complete proof of the viscosity property in the Appendix, where the standard associated comparison result leading to uniqueness is also stated.

6.2. Boundary modulus of continuity

Adapting some barrier techniques for PDEs, we first prove the following bound for the modulus of continuity on the boundary.

Proposition 6.2. *Let* **(HL)**, **(D1)**, **(D2)**, **(C)** *and* **(Hg)** *hold. There then exists $C_L > 0$ such that for all $(t_0,x_0)\in[0,T)\times\partial\mathcal{O}$,*

$$\lim_{y\in\mathcal{O}, y\to x_0} \frac{|u(t_0,y)-u(t_0,x_0)|}{\|y-x_0\|} \le C_L. \tag{6.2}$$

In particular, if the gradient of u exists at (t_0, x_0), it is uniformly bounded.

Proof. Let $(t_0, x_0) \in [0, T) \times \partial\mathcal{O}$ and $\mathcal{A} := [t_0, T) \times \mathcal{N}$, where $\mathcal{N} \subset \mathcal{O}$ is an open set and $x_0 \in \partial\mathcal{N}$. We will only show that, for all $y \in \mathcal{N}$,

$$\frac{u(t_0, y) - u(t_0, x_0)}{\|y - x_0\|} \leq C_L. \tag{6.3}$$

The lower bound is obtained similarly. By **(D2)**, there exists $\varepsilon > 0$ and a family $(e_i)_{i \in [\![1,d]\!]}$ such that $x_0 + \varepsilon e_i \in \mathcal{N}$ for all $i \in [\![1, d]\!]$ and $\operatorname{span}(e_i, i \in [\![1, d]\!]) = \mathbb{R}^d$. Thus, (6.2) implies the statement concerning the gradient, whenever it is well defined. We now prove (6.3).

1. Assume that there exists a smooth function $\psi : \bar{\mathcal{A}} \to \mathbb{R}$ with first derivative bounded by C_L such that:

(a) $\psi \geq u$ on $\partial_p \mathcal{A} := ([t_0, T) \times \partial\mathcal{N}) \cup (\{T\} \times \bar{\mathcal{N}})$;
(b) $\mathcal{L}\psi(t, x) + f(x, \psi(t, x), D\psi(t, x)\sigma(x)) \leq 0$ for $(t, x) \in \mathcal{A}$;
(c) $\psi(t_0, x_0) = u(t_0, x_0) = g(t_0, x_0)$.

Using Proposition 6.1 and a standard maximum principle (see Lemma A.2 in the Appendix), we then derive that $u \leq \psi$ on $\bar{\mathcal{A}}$. In view of (c), this yields

$$\frac{u(t_0, y) - u(t_0, x_0)}{\|y - x_0\|} \leq \frac{\psi(t_0, y) - \psi(t_0, x_0)}{\|y - x_0\|} \leq C_L \qquad \forall y \in \bar{\mathcal{N}} \setminus \{x_0\}.$$

2. It remains to construct a smooth function satisfying (a), (b) and (c). Recall that the spatial boundary $\partial\mathcal{O}$ is compact. Since u is continuous on $\bar{D}$ (see Proposition 6.1), the compactness assumption **(D1)** ensures the uniform boundedness of u in a neighborhood of $[0, T] \times \partial\mathcal{O}$.

We specify the construction of the barrier function only for $x_0 \in \partial\mathcal{O} \setminus B(\mathcal{C}, L^{-1})$. Indeed, for $x_0 \in B(\mathcal{C}, L^{-1})$, assumption **(C)** ensures that the diffusion coefficient is uniformly elliptic in a neighborhood of x_0. The expression of the barriers below can then be simplified. Namely, we do not need the additional localization with the cone, that is, we can take $\kappa = 0$ in (6.6) below.

Let $y := y(x_0)$ be the point of $\bar{\mathcal{O}}^c$ associated to x_0 by the exterior sphere property; see **(D2)**. Set $r := r(x_0) = \|y(x_0) - x_0\|$. Recall that, by assumption, $B := B(y, r)$ satisfies $\bar{B} \cap \bar{\mathcal{O}} = \{x_0\}$.

It follows from **(HL)** and **(C)** that

$$\langle a(x) n(x_0), n(x_0)\rangle \geq L^{-1}/2 \qquad \text{on the set } \mathcal{D}_1 := \{x \in \mathcal{O} : \|x - x_0\| \leq \eta_L\} \tag{6.4}$$

for some $\eta_L > 0$ small enough, but depending only on L.

For $x \in \mathcal{O}$, we now set

$$d_B(x) := d(x, \partial B) = \|x - y\| - r$$

so that $d_B \in C^2(\bar{\mathcal{O}})$ with

$$Dd_B(x) = \frac{x - y}{\|x - y\|}, \qquad D^2 d_B(x) = \frac{I_d}{\|x - y\|} - \frac{(x - y)^*(x - y)}{\|x - y\|^3}, \tag{6.5}$$

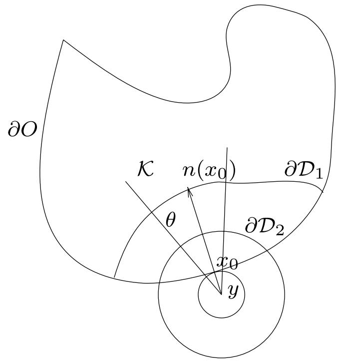

Figure 1. Domain for the barrier.

where I_d denotes the identity matrix of $\mathbb{M}^d$. We now introduce a cone

$$\mathcal{K} := \{x \in \mathbb{R}^d : \langle x - y, n(x_0)\rangle \geq \cos(\theta)\|x - y\|\}, \qquad \theta \in [0, \pi/2]$$

and

$$\mathcal{D}_2 := \{x \in \mathcal{O} : d_B(x) \leq \delta\}, \qquad \delta > 0,$$

where $\delta \leq \delta_L$ is sufficiently small to ensure $\mathcal{D}_2 \subset \mathcal{D}_1$. Finally, we set $\mathcal{N} := \mathcal{O} \cap \mathcal{K} \cap \mathcal{D}_2$, see Figure 1 above, and define the barrier function by

$$\psi(t,x) := g(t,x) + 4\alpha\big(\varphi(x)^{1/2} - \delta^{1/2}\big) + \kappa\langle x - y, n(x_0)\rangle\left(1 - \frac{\langle x - y, n(x_0)\rangle}{\|x - y\|}\right) \tag{6.6}$$

for $(t,x) \in [t_0, T] \times \bar{\mathcal{N}}$, where $\varphi(x) := \delta + d_B(x)$. For a suitable choice of the parameters $\alpha, \kappa, \delta > 0$ and $\theta \in (0, \pi/2)$, one can check that the above function ψ satisfies points (a), (b) and (c) of step 1; see [4] for details. $\square$

6.3. Representation and weak regularity of the gradient in the regular uniformly elliptic case

In this section, we strengthen the initial assumptions and work under:

(D′) $\mathcal{O}$ is a C^2 bounded domain satisfying **(D1)** and **(D2)** for the constant L;
(C′) a is uniformly elliptic with ellipticity constant L^{-1};
(H′) the coefficients b, σ, f and g satisfy **(Hg)**–**(HL)** and are uniformly $C^2(\bar{D})$.

From now on, given a matrix M, we denote by $M^{\cdot j}$ its jth column, viewed as a column vector.

Proposition 6.3 (Representation of the gradient). *Let the conditions* **(D′)**, **(C′)** *and* **(H′)** *hold. Then,* $u \in C^0(\bar{D}) \cap C^{1,2}(D)$, $Du \in C^0(\bar{D})$ *and, for all* $(t,x) \in \bar{D}$,

$$Du(t,x) = \mathbb{E}\left[Du(\tau^{t,x}, X^{t,x}_{\tau^{t,x}})\nabla X^{t,x}_{\tau^{t,x}} V^{t,x}_{\tau^{t,x}} + \int_t^{\tau^{t,x}} \partial_x f(\Theta^{t,x}_s)\nabla X^{t,x}_s V^{t,x}_s \, \mathrm{d}s\right], \tag{6.7}$$

where $\nabla X^{t,x}$ *is the first variation process of* $X^{t,x}$:

$$\nabla X^{t,x}_s = I_d + \sum_{j=1}^d \int_t^s D\sigma^{\cdot j}(X^{t,x}_v)\nabla X^{t,x}_v \, \mathrm{d}W^j_v + \int_t^s Db(X^{t,x}_v)\nabla X^{t,x}_v \, \mathrm{d}v, \qquad s \ge t,$$

and $V^{t,x}$ *is defined by*

$$V^{t,x}_s := \exp\left(\int_t^s \partial_y f(\Theta^{t,x}_v)\,\mathrm{d}v + \int_t^s \partial_z f(\Theta^{t,x}_v)\,\mathrm{d}W_v - \frac{1}{2}\int_t^s \|\partial_z f(\Theta^{t,x}_v)\|^2 \,\mathrm{d}v\right), \qquad s \ge t,$$

with $\Theta^{t,x} = (X^{t,x}, Y^{t,x}, Z^{t,x})$.

Proof. The result is obvious for $(t,x) \in \partial D$. We thus assume from now on that $(t,x) \in D$. We derive from Theorems 12.16 and 12.10 in [21] and the definition of Hölder spaces on page 46 of this reference that $Du \in C^0(\bar{D})$. Let us consider the systems of differential equations obtained by formally differentiating the PDE (6.1) w.r.t. $(x^i)_{i\in[\![1,d]\!]}$. For $i = 1, \dots, d$, we have

$$\begin{aligned} 0 &= \partial_t v^i + \left\langle b + \sigma^* D_z f(\Theta) + \frac{1}{2} D_{x^i} a^{\cdot i}, Dv^i \right\rangle + \frac{1}{2}\operatorname{Tr}[aD^2 v^i] \\ &\quad + \big(D_{x^i} b^i + D_y f(\Theta) + \langle D_z f(\Theta), D_{x^i}\sigma^{\cdot i}\rangle\big) v^i + D_{x^i} f(\Theta) + \sum_{k\neq i} h^{i,k}, \end{aligned} \tag{6.8}$$

$$\text{where } h^{i,k} = \big(D_{x^i} b^k + \langle D_z f(\Theta), D_{x^i}\sigma^{\cdot k}\rangle\big) D_{x^k} u + \sum_{l=1}^d D_{x^i} a^{kl} D_{x^k x^l} u$$

and $\Theta(t,x) = (x, u(t,x), Du\sigma(t,x))$.

Given n sufficiently large, set $\mathcal{O}_n := \{x + \bar{B}(0,n^{-1}), x \in \mathcal{O}^c\}^c \subset \mathcal{O}$, $T_n := T - n^{-1} > 0$ and $D_n := [0,T_n) \times \mathcal{O}_n$. Note that, by construction, $\mathcal{O}_n$ satisfies a uniform exterior sphere property (with radius $1/2n$). The PDE (6.8) on D_n with the boundary condition $D_{x^i}u$ on $\partial_p D_n = ([0,T_n) \times \partial\mathcal{O}_n) \cup (\{T_n\} \times \bar{\mathcal{O}}_n)$ then admits a unique $C^0(\bar{D}_n) \cap C^{1,2}(D_n)$ solution v^i_n; see Theorem 12.22 in [21]. Using the maximum principle, we can then identify $D_{x^i}u$ and v^i_n on $\bar{D}_n$ by considering the PDE satisfied by $\varepsilon^{-1}(u(\cdot, x + \varepsilon e_i) - u(\cdot, x)) - v^i_n(\cdot, x)$ on $\bar{D}_n$. Here, e_i is the ith canonical basis vector of $\mathbb{R}^d$ (see, e.g., Theorem 10, Chapter 3 in [11]). In particular, $Du \in C^0(\bar{D}_n) \cap C^{1,2}(D_n)$. By a usual localization argument, we then deduce from Itô's lemma applied to $Du(\cdot, X^{t,x})\nabla X^{t,x} V^{t,x}$, with $(t,x) \in D_n$, that

$$Du(t,x) = \mathbb{E}\left[Du(\tau_n, X^{t,x}_{\tau_n})\nabla X^{t,x}_{\tau_n} V^{t,x}_{\tau_n} + \int_t^{\tau_n} \partial_x f(\Theta^{t,x}_s)\nabla X^{t,x}_s V^{t,x}_s \, \mathrm{d}s\right],$$

where $\tau_n := \inf\{s \in [t, T_n] : (s, X_s^{t,x}) \notin D_n\}$. Observe that $\lim_n \tau_n = \tau$ $\mathbb{P}$-a.s. by continuity of X. We then derive the statement of the Proposition by sending $n \to \infty$, using the a priori smoothness of u, $Du \in C^0(\bar{D})$ and the dominated convergence theorem. □

Remark 6.1. Note that the various localizations in the previous proof are needed because we do not assume any compatibility condition on the parabolic boundary, that is, $\mathcal{L}g + f(\cdot, g, \sigma Dg) = 0$ on $\partial_p D$. Otherwise, Theorem 12.14 in [21] would give $u \in C^{1,2}(\bar{D})$, which would allow us to avoid the introduction of the subdomains $\mathcal{O}_n$.

Observe that by Proposition 6.2 and the continuity of Du stated in Proposition 6.3, we have $\|Du(\tau^{t,x}, X^{t,x}_{\tau^{t,x}})\| \le C_L$. The representation (6.7) and standard estimates then give $\|Du\|_{\infty,\bar{D}} \le C_L$.

Corollary 6.1. *Let* **(D′)**, **(C′)** *and* **(H′)** *hold. Then,* $\|Du\|_{\infty,\bar{D}} \le C_L$.

We can now prove Theorem 3.2 under the conditions **(D′)**, **(C′)** and **(H′)**.

Corollary 6.2. *Theorem* 3.2 *holds under the conditions* **(D′)**, **(C′)** *and* **(H′)**.

Proof. 1. **Proof of (3.4) and (3.5).** Recalling that $u \in C^{1,2}(D) \cap C^1(\bar{D})$ (see Proposition 6.3), we deduce from a standard verification argument that $Z = Du(\cdot, X)\sigma(X)$. Set $(\nabla X, V) := (\nabla X^{0,X_0}, V^{0,X_0})$ and observe that $(\nabla X_s^{t,X_t}, V_s^{t,X_t}) = (\nabla X_s \nabla X_t^{-1}, V_s V_t^{-1})$ for $s \ge t$, by the flow property. Thus, by Proposition 6.3,

$$Z_t = \mathbb{E}\left[Du(\tau, X_\tau)\nabla X_\tau V_\tau + \int_t^\tau \partial_x f(\Theta_s)\nabla X_s V_s \,\mathrm{d}s \Big| \mathcal{F}_t\right]\sigma(X_t)(\nabla X_t V_t)^{-1}, \qquad t \le \tau. \tag{6.9}$$

It then follows from Proposition 6.2 (boundedness of the gradient of u), **(HL)** and standard estimates that $\sup_{t\le\tau} \|Z_t\| \le \xi_L$. This readily implies (3.5), that is, $\mathbb{E}[\int_\theta^\vartheta \|Z_s\|^p \,\mathrm{d}s|\mathcal{F}_\theta] \le \mathbb{E}[\xi_L^p|\vartheta - \theta||\mathcal{F}_\theta]$, $p = 1, 2$. By the Burkholder–Davis–Gundy inequality, **(HL)** and Proposition 4.1, this also yields $\mathbb{E}[\sup_{t\in[\theta,\vartheta]} |Y_t - Y_\theta|^{2p}] \le \mathbb{E}[\xi_L^p|\vartheta - \theta|^p]$, $p \ge 1$.

2. **Proof of (3.6).** By the same arguments as above, we first obtain that $|u(t,x) - u(t,x')| \le C_L|x - x'|$. Moreover, for $t \le t' \le T$,

$$u(t,x) - u(t',x) = Y_t^{t,x} - u(t',x) = Y_t^{t,x} - Y_{t'}^{t,x} + u(t', X_{t'}^{t,x}) - u(t',x).$$

The Lipschitz continuity of u in space (Corollary 6.1) and standard estimates on SDEs imply that $|\mathbb{E}[u(t', X_{t'}^{t,x}) - u(t',x)]| \le C_L|t - t'|^{1/2}$. On the other hand, $\mathbb{E}[|Y_t^{t,x} - Y_{t'}^{t,x}|^2] \le C_L(t' - t)$, by the above estimate.

3. **Proof of (3.3).** The bound on $\mathcal{R}(Y)^\pi_{\mathcal{S}^2}$ follows from (3.4). Using (6.9) and exactly the same arguments as in the proof of Proposition 4.5 in [3] (see also [23]), we deduce that

$$\sum_{i=0}^{n-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} \|Z_t - Z_{t_i}\|^2 \,\mathrm{d}t\right] \le C_L h,$$

which implies that $\sum_{i=0}^{n-1}\mathbb{E}[\int_{t_i}^{t_{i+1}}\|Z_t-\hat{Z}_{t_i}\|^2\,\mathrm{d}t]\le C_Lh$ since $\hat{Z}$ is the best approximation of Z in $L^2(\Omega\times[0,T])$ by an element of $\mathcal{H}^2$ which is constant on each time interval $[t_i,t_{i+1})$. $\square$

6.4. Regularization procedure: proof of Theorem 3.2 in the general case

Step 1. Truncation of the domain: We first prove that Theorem 3.2 holds under the conditions **(D1)**, **(D2)**, **(C′)** and **(H′)**.

Let ϕ be a C^∞ density function with compact support on $\mathbb{R}^d$. Given $\varepsilon>0$, we define $\Delta_\varepsilon:=\varepsilon^{-d}\phi(\varepsilon^{-1}\cdot)\star(d\wedge d_{\varepsilon^{-1}})^+$, where $d_{\varepsilon^{-1}}$ denotes the algebraic distance to $\partial B(X_0,\varepsilon^{-1})$ and $\star$ denotes the convolution. Set $\mathcal{O}_\varepsilon:=\{x\in\mathbb{R}^d:\Delta_\varepsilon(x)>0\}$ and $D_\varepsilon:=[0,T)\times\mathcal{O}_\varepsilon$. It follows from the compact boundary assumption that $\partial\mathcal{O}\subset\bar{\mathcal{O}}_\varepsilon$, for ε sufficiently small. Note that $\mathcal{O}_\varepsilon$ is bounded, even if $\mathcal{O}$ is not. Let $(Y^\varepsilon,Z^\varepsilon)$ be defined as in (1.2) with $\mathcal{O}_\varepsilon$ in place of $\mathcal{O}$ and let τ^ε be the first exit time of $(\cdot,X)$ from D_ε. Observe that, by continuity of X, $\tau^\varepsilon\to\tau$ $\mathbb{P}$-a.s. Since, by **(Hg)**, **(HL)** and Theorem 1.5 in [26],

$$\begin{aligned}\|Y-Y^\varepsilon\|^2_{\mathcal{S}^2}+\|Z-Z^\varepsilon\|^2_{\mathcal{H}^2}&\le C_L\mathbb{E}\biggl[|g(\tau,X_\tau)-g(\tau^\varepsilon,X_{\tau^\varepsilon})|^2+\int_{\tau\wedge\tau^\varepsilon}^{\tau\vee\tau^\varepsilon}f(X_s,Y_s,Z_s)^2\,\mathrm{d}s\biggr]\\&\le C_L\mathbb{E}\biggl[\int_{\tau\wedge\tau^\varepsilon}^{\tau\vee\tau^\varepsilon}(1+\|X_s\|^2+|Y_s|^2+\|Z_s\|^2)\,\mathrm{d}s\biggr],\end{aligned}$$

we deduce from Proposition 4.1 and a dominated convergence argument that $\|Y-Y^\varepsilon\|^2_{\mathcal{S}^2}+\|Z-Z^\varepsilon\|^2_{\mathcal{H}^2}\to0$. Since the domain $\mathcal{O}_\varepsilon$ satisfies **(D′)**, we can apply Corollary 6.2 to $(Y^\varepsilon,Z^\varepsilon)$. Recalling that the associated constants depend only on L and are uniform in ε, we thus obtain the required controls on (Y,Z). Let u^ε be the solution of (6.1) associated to D_ε. The above stability result, applied to general initial conditions, implies that $u^\varepsilon\to u$ pointwise on $\bar{D}$. Corollary 6.2 thus implies that u satisfies (3.6).

Step 2. Regularization of the coefficients: We now prove that Theorem 3.2 holds under the conditions **(D1)**, **(D2)**, **(C)**, **(HL)** and **(Hg)**.

For $\varepsilon>0$, define b_ε, σ_ε and f_ε by

$$(b_\varepsilon,\sigma_\varepsilon,f_\varepsilon)(x,y,z):=(b,\sigma,f)\star\varepsilon^{-2d+1}\phi(\varepsilon^{-1}(x,y,z)),$$

where ϕ is a C^∞ density function with compact support on $\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$. Let us consider the FBSDE

$$\begin{cases}X_t^\varepsilon=x+\displaystyle\int_0^t b_\varepsilon(X_s^\varepsilon)\,\mathrm{d}s+\int_0^t\sigma_\varepsilon(X_s^\varepsilon)\,\mathrm{d}W_s+\sqrt{\varepsilon}\tilde{W}_t,\\ Y_t^\varepsilon=g(\tau^\varepsilon,X^\varepsilon_{\tau^\varepsilon})+\displaystyle\int_{t\wedge\tau^\varepsilon}^{\tau^\varepsilon}f_\varepsilon(X_s^\varepsilon,Y_s^\varepsilon,Z_s^\varepsilon)\,\mathrm{d}s-\int_{t\wedge\tau^\varepsilon}^{\tau^\varepsilon}Z_s^\varepsilon\,\mathrm{d}W_s-\int_{t\wedge\tau^\varepsilon}^{\tau^\varepsilon}\tilde{Z}_s^\varepsilon\,\mathrm{d}\tilde{W}_s,\end{cases}\tag{6.10}$$

where $(\tilde{W}_t)_{t\ge0}$ is an additional d-dimensional Brownian motion independent of W and

$$\tau^\varepsilon:=\inf\{s\ge0:(s,X_s^\varepsilon)\notin D\}.$$

This system satisfies the conditions of Step 1. Therefore, the estimates of Theorem 3.2 can be applied to $(Y^\varepsilon, Z^\varepsilon)$. Note that the associated constant depends only on L and is uniform in ε. Moreover, it follows from **(HL)** and Theorem 1.5 in [26] that

$$\|Y - Y^\varepsilon\|_{\mathcal{S}^2}^2 + \|Z - Z^\varepsilon\|_{\mathcal{H}^2}^2 \le C_L \mathbb{E}\Big[|g(\tau, X_\tau) - g(\tau^\varepsilon, X^\varepsilon_{\tau^\varepsilon})|^2 + \int_0^T \|X_s - X_s^\varepsilon\|^2 \, \mathrm{d}s\Big] \\ + \mathbb{E}\Big[\int_{\tau\wedge\tau^\varepsilon}^{\tau\vee\tau^\varepsilon} \big(|f(X_s, Y_s, Z_s)| + |f_\varepsilon(X_s^\varepsilon, Y_s, Z_s)|\big)^2 \, \mathrm{d}s\Big] + L\varepsilon.$$

Clearly, $X^\varepsilon \to X$ in $\mathcal{S}^2$. Since f and g are Lipschitz continuous, f and f_ε have linear growth and (X, X^ε, Y, Z) is bounded in $\mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{S}^2 \times \mathcal{H}^2$, it suffices to check that $\tau^\varepsilon \to \tau$ in probability to obtain the required controls on (Y, Z). This is implied by the non-characteristic boundary condition of **(C)** (see, e.g., the proof of Proposition 3 in [18]). The control (3.6) is obtained by arguing as above.

Appendix: Proof of Proposition 6.1

In the following, we use the notation

$$u^*(t, x) = \limsup_{(s,y)\in D\to(t,x)} u(s, y), \qquad u_*(t, x) = \liminf_{(s,y)\in D\to(t,x)} u(s, y), \qquad (t, x) \in \bar{D}.$$

The statement of Proposition 6.1 is a direct consequence of Lemmas A.1 and A.2 below.

Lemma A.1. *Let the conditions of Proposition 6.1 hold. The function u then has linear growth and u^* (resp. u_*) is a viscosity subsolution (resp. supersolution) of (6.1) with the terminal conditions $u^* \le g$ (resp. $u_* \ge g$) on $\partial_p D$.*

Proof. 1. The linear growth property property is an immediate consequence of Proposition 4.1.

2. It remains to prove that u^* and u_* are, respectively, a sub- and supersolution of (6.1) with the boundary conditions $u^* \le g$ and $u_* \ge g$ on $\partial_p D$. We concentrate on the supersolution property, the subsolution property being derived similarly. The proof is standard – as usual we argue by contradiction. Let $(t_0, x_0) \in [0, T] \times \bar{\mathcal{O}}$ and $\varphi \in C_b^2$ be such that $0 = \min_{(t,x)\in\bar{D}}(u_* - \varphi)(t, x) = (u_* - \varphi)(t_0, x_0)$, where the minimum is assumed, w.l.o.g., to be strict on $\bar{D}$. Assume that

$$\big(-\mathcal{L}\varphi(t_0, x_0) - f(x_0, \varphi(t_0, x_0), D\varphi\sigma(t_0, x_0))\big)\mathbf{1}_{(t_0,x_0)\in D} + (\varphi - g)(t_0, x_0)\mathbf{1}_{(t_0,x_0)\in\partial_p D} \\ =: -2\zeta < 0.$$

Recall from **(D2)** that if $x_0 \in \partial\mathcal{O}$, then we can find an open ball $B_0 \subset \mathcal{O}^c$ such that $\bar{B}_0 \cap \bar{\mathcal{O}} = \{x_0\}$.

If $x_0 \in \partial\mathcal{O}$, we denote by d_{B_0} the algebraic distance to B_0. On $\bar{D}$, we set

$$\tilde{\varphi}(t,x) = \varphi(t,x) - \left(\sqrt{T-t}\right)\mathbf{1}_{t_0=T} - d(x)\left(1 - \frac{d(x)}{\eta}\right)\mathbf{1}_{x_0\in\partial\mathcal{O}\setminus B(\mathcal{C},L^{-1})}$$
$$- d_{B_0}(x)\left(1 - \frac{d_{B_0}(x)}{\eta}\right)\mathbf{1}_{x_0\in\partial\mathcal{O}\cap B(\mathcal{C},L^{-1})}$$

for some $\eta > 0$. Observe that (t_0, x_0) is still a strict minimum of $(u_* - \tilde{\varphi})$ on $V_\eta \cap \bar{D}$ for some open neighborhood V_η of (t_0, x_0) on which $(d_{B_0} \vee d) \le \eta/2$ if $x_0 \in \partial\mathcal{O}$. Without loss of generality, we can then assume that

$$u \ge u_* \ge \tilde{\varphi} + \zeta \qquad \text{on } \partial V_\eta \setminus \bar{D}^c, \tag{A.11}$$

while

$$\tilde{\varphi} \le \varphi \le g - \zeta \qquad \text{on } \bar{V}_\eta \cap \partial_p D, \text{ if } (t_0, x_0) \in \partial_p D. \tag{A.12}$$

Moreover, observe that for F equal to d or d_{B_0}, $D(F(1 - F/\eta)) = DF(1 - 2\eta^{-1}F)$ and $D^2(F(1 - F/\eta)) = (1 - 2\eta^{-1}F)D^2F - 2\eta^{-1}DF^*DF$, where $\|DF\| = 1$. Thus, **(C)** implies that, for η and V_η small enough,

$$-\mathcal{L}\tilde{\varphi} - f(\cdot, \tilde{\varphi}, D\tilde{\varphi}\sigma) \le -\zeta < 0 \qquad \text{on } V_\eta \cap \bar{D}. \tag{A.13}$$

Let $(t_n, x_n)_n$ be a sequence in $D \cap V_\eta$ such that $(t_n, x_n, u(t_n, x_n)) \to (t_0, x_0, u_*(t_0, x_0))$. Let (X^n, Y^n, Z^n) be the solution of (1.1)–(1.2) associated to the initial conditions (t_n, x_n) and define θ_n as the first exit time of $D \cap V_\eta$ by $(\cdot, X^n)$. By applying Itô's lemma on $\tilde{\varphi}$ and using (A.12), (A.13), (A.11) and the identity $u = g$ on $\partial_p D$, we get

$$\tilde{\varphi}(t_n, x_n) = -\chi + u(\theta_n, X^n_{\theta_n}) + \int_{t_n}^{\theta_n} \left(f(X^n_s, \tilde{\varphi}(s, X^n_s), D\tilde{\varphi}\sigma(s, X^n_s)) - \eta_s\right) \mathrm{d}s$$
$$- \int_{t_n}^{\theta_n} D\tilde{\varphi}\sigma(s, X^n_s)\, \mathrm{d}W_s,$$

where χ is a bounded random variable satisfying $\chi \ge \zeta$ $\mathbb{P}$-a.s. and η is an adapted process in L^2 such that $\eta \ge \zeta$ $\mathrm{d}t \times \mathrm{d}\mathbb{P}$-a.e. Following the standard argument of the proof of Theorem 1.6 in [26], we deduce that $\tilde{\varphi}(t_n, x_n) \le Y^{t_n,x_n}_{t_n} - \zeta \mathrm{e}^{-LT} = u(t_n, x_n) - \zeta \mathrm{e}^{-LT}$. Since $\tilde{\varphi}(t_n, x_n) - u(t_n, x_n) \to 0$, this leads to a contradiction. □

We now state a comparison theorem for the PDE (6.1). The proof is standard (see, e.g., [6] or [4]).

Lemma A.2. *Let the conditions of Proposition 6.1 hold. Fix $t_0 \in [0, T)$ and $\mathcal{N} \subset \mathcal{O}$ an open set. Let U (resp. V) be an upper-semicontinuous subsolution (resp. lower-semicontinuous supersolution) with polynomial growth of (6.1) on $\mathcal{A} := [t_0, T) \times \mathcal{N}$ such that $V \ge U$ on $\partial_p\mathcal{A} := ([t_0, T) \times \partial\mathcal{N}) \cup (\{T\} \times \bar{\mathcal{N}})$. Then, $V \ge U$ on $\bar{\mathcal{A}}$.*

References

[1] Bally, V. and Pages, G. (2002). A quantization algorithm for solving discrete time multidimensional optimal stopping problems. *Bernoulli* **9** 1003–1049. MR2046816

[2] Bouchard, B. and Chassagneux, J.-F. (2008). Discrete-time approximation for continuously and discretely reflected BSDE's. *Stochastic Process. Appl.* **118** 2269–2293. MR2474351

[3] Bouchard, B. and Elie, R. (2005). Discrete time approximation of decoupled forward–backward SDE with jumps. *Stochastic Process. Appl.* **118** 53–75. MR2376252

[4] Bouchard, B. and Menozzi, S. (2008). Strong approximation of BSDEs in a domain. Available at http://hal.archives-ouvertes.fr/hal-00177481/fr/.

[5] Bouchard, B. and Touzi, N. (2004). Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* **111** 175–206. MR2056536

[6] Crandall, M.G., Ishii, H. and Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Amer. Math. Soc.* **27** 1–67. MR1118699

[7] Darling, R.W.R. and Pardoux, E. (1997). BSDE with random terminal time. *Ann. Probab.* **25** 1135–1159. MR1457614

[8] Delarue, F. and Menozzi, S. (2006). A forward–backward algorithm for quasi-linear PDEs. *Ann. Appl. Probab.* **16** 140–184. MR2209339

[9] Delarue, F. and Menozzi, S. (2008). An interpolated stochastic algorithm for quasi-linear PDEs. *Math. Comp.* **77** 125–158. MR2353946

[10] Freidlin, M. (1985). *Functional Integration and Partial Differential Equations*. *Annals of Mathematics Studies* **109**. Princeton, NJ: Princeton Univ. Press. MR0833742

[11] Friedman, A. (1964). *Partial Differential Equations of Parabolic Type.* Englewood Cliffs, NJ: Prentice Hall. MR0181836

[12] Gilbarg, D. and Trudinger, N.S. (1998). *Elliptic Partial Differential Equations of Second Order*. Reprint of 1998 edition. Classics in Mathematics. Berlin: Springer, 2001. xiv+517. MR1814364

[13] Gobet, E. (1998). Schéma d'Euler pour diffusions tuées. Application aux options barriére. Ph.D. thesis, Univ. Paris VII.

[14] Gobet, E. (2000). Weak approximation of killed diffusion using Euler schemes. *Stochastic Process. Appl.* **87** 167–197. MR1757112

[15] Gobet, E. and Labart, C. (2007). Error expansion for the discretization of backward stochastic differential equations. *Stochastic Process. Appl.* **117** 803–829. MR2330720

[16] Gobet, E., Lemor, J.P. and Warin, X. (2006). Rate of convergence of empirical regression method for solving generalized BSDE. *Bernoulli* **12** 889–916. MR2265667

[17] Gobet, E. and Menozzi, S. (2007). Stopped diffusion processes: Overshoots and boundary correction. Preprint PMA, Univ. Paris 7. MR2409016

[18] Gobet, E. and Menozzi, S. (2004). Exact approximation rate of killed hypoelliptic diffusions using the discrete Euler scheme. *Stochastic Process. Appl.* **114** 201–223. MR2073411

[19] Karatzas, I. and Shreve, S.E. (1990). *Brownian Motion and Stochastic Calculus*. New York: Springer. MR0917065

[20] Kobylanski, M. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* **28** 558–602. MR1782267

[21] Lieberman, G.M. (2005). *Second Order Parabolic Differential Equations.* River Edge, NJ: World Scientific. MR1465184

[22] Longstaff, F.A. and Schwartz, R.S. (2001). Valuing American options by simulation: A simple least-square approach. *Rev. Financial Stud.* **14** 113–147.

[23] Ma, J. and Zhang, J. (2002). Path regularity of solutions to backward stochastic differential equations. *Probab. Theory Related Fields* **122** 163–190. MR1894066

[24] Ma, J. and Zhang, J. (2005). Representations and regularities for solutions to BSDEs with reflections. *Stochastic Process. Appl.* **115** 539–569. MR2128629

[25] Milstein, G.N. and Tretyakov, M.V. (2001). Numerical solution of Dirichlet problems for nonlinear parabolic equations by a probabilistic approach. *IMA J. Num. Anal.* **21** 887–917. MR1867423

[26] Pardoux, E. (1998). Backward stochastic differential equations and viscosity solutions of semilinear parabolic and elliptic PDE's of second order. In *Stochastic Analysis and Related Topics: The Geilo Workshop 1996* (L. Decreusefond, J. Gjerd, B. Oksendal and A.S. Ustünel, eds.) 79–127. Boston: Birkhäuser. MR1652339

[27] Pardoux, E. and Peng, S. (1992). Backward stochastic differential equations and quasilinear parabolic partial differential equations. *Lecture Notes in Control and Inform. Sci.* **176** 200–217. MR1176785

[28] Peng, S. (1991). Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.* **37** 61–74. MR1149116

[29] Zhang, J. (2004). A numerical scheme for BSDEs. *Ann. Appl. Probab.* **14** 459–488. MR2023027

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