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Extreme lengths in Brownian and Bessel excursions

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We establish some strong limit theorems for the longest excursion lengths of a Bessel process of dimension $d \in (0, 2)$. In the special case $d = 1$, we recover and improve some well-known results for Wiener processes, and solve an open problem raised. The proof relies on exact distributions evaluated by Pitman and Yor and on a careful analysis of the Bessel sample paths.

Keywords: Bessel process; Brownian motion; excursion length; Lévy's class

1. Introduction

Let $\{R(t);\ t \geqslant 0\}$ be a Bessel process of dimension $0 < d < 2$, starting from 0 (in the particular case $d = 1$, R becomes a reflecting Wiener process). For notational convenience, we write throughout the paper $\nu \equiv \nu(d) = (d-2)/2$, which in Bessel language stands for the 'index' of R (cf. Revuz and Yor 1994, Chapter XI). For any $t > 0$, define $\Lambda^-(t) = \sup\{s \leqslant t\colon R(s) = 0\}$ and $\Lambda^+(t) = \inf\{s \geqslant t\colon R(s) = 0\}$, which represent respectively the left and right extremities of the excursion interval straddling t. We are interested in

$$V_1(t) \geqslant V_2(t) \geqslant \cdots \geqslant V_n(t) \geqslant \cdots, \tag{1.1}$$

the ordered excursion lengths of R over $(0, t)$, the last zero-free interval $(\Lambda^-(t), t)$ being considered as an (incomplete) excursion interval. Discussions on excursion intervals excluding $(\Lambda^-(t), t)$ are postponed to Section 5.

A remarkable development (Pitman and Yor 1992; 1997) in the study of ordered excursion lengths is that for any fixed positive numbers r, s and t and any integer $k \geqslant 1$,

$$\left(\frac{V_1(t)}{t}, \frac{V_2(t)}{t}, \ldots\right) \stackrel{(d)}{=} \left(\frac{V_1(\tau(s))}{\tau(s)}, \frac{V_2(\tau(s))}{\tau(s)}, \ldots\right), \tag{1.2}$$

$$\left(\frac{V_1(t)}{t}, \frac{V_2(t)}{t}, \ldots\right) \stackrel{(d)}{=} \left(\frac{V_1(H_k(r))}{H_k(r)}, \frac{V_2(H_k(r))}{H_k(r)}, \ldots\right), \tag{1.3}$$

where '$\stackrel{(d)}{=}$' stands for identity in distribution, τ is the right-continuous inverse process of the local time (in the sense of diffusion) of R at 0, and $H_k(r) = \inf\{t > 0\colon V_k(t) = r\}$ is the first

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hitting time of r by V_k. These identities in law, which bear similarities to Paul Lévy's celebrated arcsine laws, confirm random scaling properties for 'nice' clocks – a feature we shall exploit later. On the other hand, since τ is a stable subordinator of index $|\nu|$ (i.e. a non-decreasing $|\nu|$-stable process with independent increments; cf. Molchanov and Ostrovski 1969), the study in distribution of the V_k is closely related to that of ranked jumps of stable subordinators. The latter has been the subject of considerable interest in the literature, with various motivations; see the references in Perman *et al.* (1992). Let us also mention recent applications of ordered Brownian excursion lengths in financial mathematics (Chesney *et al.* 1997).

Following Csáki *et al.* (1985), this paper is concerned with *sample path* properties of the longest lengths of the excursions.

Let us first consider $t \mapsto \sum_{j=1}^{k} V_j(t)$, the sum of the k longest excursion lengths. Theorems 1.1 and 1.2 below characterize respectively the upper and lower functions of this process. Throughout the paper, 'i.o.' stands for 'infinitely often' as the appropriate index tends to ∞.

Theorem 1.1. *Let $k \geqslant 1$ and let $\phi > 0$ be a non-decreasing function. Then*

$$\mathbb{P}\left[\sum_{j=1}^{k} V_j(t) > t\left(1 - \frac{1}{\phi(t)}\right);\ \text{i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{\mathrm{d}t}{t(\phi(t))^{|\nu|}} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Theorem 1.2. *For any fixed $k \geqslant 1$ and non-decreasing function $\psi > 0$,*

$$\mathbb{P}\left[\sum_{j=1}^{k} V_j(t) < \frac{t}{\psi(t)};\ \text{i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{\mathrm{d}t}{t}\psi(t)\exp\left(-k\gamma\psi(t)\right) \begin{cases} < \infty \\ = \infty \end{cases},$$

where $\gamma \equiv \gamma(d)$ is the unique positive number such that

$$\sum_{n=1}^{\infty} \frac{\gamma^n}{n!(n - |\nu|)} = \frac{1}{|\nu|}.$$

Remarks. (i) Taking $d = k = 1$ in Theorems 1.1 and 1.2, we recover respectively the Chung–Erdös (1952) and Csáki–Erdös–Révész (1985) tests for the longest excursion of a Wiener process. For $d = 1$ and arbitrary k, Theorem 1.2 improves the following iterated logarithm law established in Csáki *et al.* (1985):

$$\liminf_{t \to \infty} \frac{\log\log t}{t} \sum_{j=1}^{k} V_j(t) = k\gamma \quad \text{a.s.}$$

The somewhat intricate constant γ appears naturally in Section 3.

(ii) Clearly Theorems 1.1 and 1.2 (as well as the forthcoming Theorems 1.3, 1.4 and 5.1) hold for all processes having the same zeros of a d-dimensional Bessel process.

In order to obtain a more complete description of the excursion lengths, Csáki *et al.* (1985) raise the problem of characterizing the set of those non-decreasing functions ϕ for which

$$\mathbb{P}\left[V_2(t) \geqslant \frac{t}{2}\left(1 - \frac{1}{\phi(t)}\right); \text{ i.o.}\right] = 1.$$

We solve this problem for all the V_k.

Theorem 1.3. *If $k \geqslant 2$ and if $\phi > 0$ is non-decreasing, we have*

$$\mathbb{P}\left[V_k(t) \geqslant \frac{t}{k}\left(1 - \frac{1}{\phi(t)}\right); \text{ i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{\mathrm{d}t}{t(\phi(t))^{(1+|\nu|)k-2}} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Finally, we complete the image of the limiting behaviours of the V_k with an integral test which characterizes the lower functions.

Theorem 1.4. *Assume $k \geqslant 2$ and $\psi > 0$ non-decreasing. Then*

$$\mathbb{P}\left[V_k(t) < \frac{t}{\psi(t)}; \text{ i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{\mathrm{d}t}{t(\psi(t))^{|\nu|}} \begin{cases} < \infty \\ = \infty \end{cases}.$$

Remark. Despite the resemblance, Theorem 1.3 is of a different nature than the other theorems presented above, and its proof needs more care.

The rest of the paper is organized as follows. Theorem 1.3 is proved in Section 2. Section 3 is devoted to the study of the upper and lower tails of $\sum_{j=1}^{k} V_j(t)$ and $V_k(t)$. The proofs of Theorems 1.1, 1.2 and 1.4 are provided in Section 4. The situation of completed excursion lengths is discussed in Section 5.

2. Proof of Theorem 1.3

Unless stated otherwise, $k \geqslant 2$ denotes a fixed integer in this section. The key ingredient in the proof of Theorem 1.3 is the following 'minimal inequality'. Recall that H_k is the inverse process of V_k.

Lemma 2.1. *There exists a constant $C \geqslant 1$ depending only on k and d such that for any $0 < x < 1/k$,*

$$C^{-1} x^{(|\nu|+1)k-2} \leqslant \mathbb{P}\left[\inf_{H_k(1) \leqslant t \leqslant H_k(2)} \left(\frac{1}{k} - \frac{V_k(t)}{t}\right) < x\right] \leqslant C x^{(|\nu|+1)k-2}. \tag{2.1}$$

Before we continue, some comments on notation are necessary. First, in the rest of the paper, $C \geqslant 1$ denotes a finite constant. Its value, which may change from line to line, depends only on k and on the underlying dimension d. Second, $a(x) \sim b(x)$ $(x \to x_0)$ means

$\lim_{x\to x_0} a(x)/b(x) = 1$. Third, $a(x) \asymp b(x)$ $(x \to x_0)$ means $0 < \liminf_{x\to x_0} a(x)/b(x) \leqslant \limsup_{x\to x_0} a(x)/b(x) < \infty$.

The proof of Lemma 2.1 relies on some exact distributions evaluated by Pitman and Yor (1997, Propositions 7 and 10), stated as follows.

Fact 2.2 (Pitman and Yor 1997). *Let $\{V_n(1)\}_{n\geqslant 1}$ be as in (1.1). Then*

$$\left\{\frac{V_{n+1}(1)}{V_n(1)}\right\}_{n\geqslant 1}$$

is a sequence of independent variables it such that $V_{n+1}(1)/V_n(1)$ has the Beta$(n|\nu|, 1)$ distribution. (2.2)

Furthermore, defining for $n \geqslant 1$ $(\sum_1^0 \equiv 0)$

$$\Xi_n = \sum_{j=n+1}^{\infty} \frac{V_j(1)}{V_n(1)} \text{ and } \Upsilon_n = \sum_{j=1}^{n-1}\left(\frac{V_j(1)}{V_n(1)} - 1\right),$$

we have

$$\mathbb{E}\exp(-\theta\Xi_n) = \left(1 + |\nu|\int_0^1 z^{-|\nu|-1}(1-\mathrm{e}^{-\theta z})\mathrm{d}z\right)^{-n}, \tag{2.3a}$$

$$\mathbb{E}\exp(-\theta\Upsilon_n) = \left(|\nu|\int_0^\infty (1+z)^{-|\nu|-1}\mathrm{e}^{-\theta z}\mathrm{d}z\right)^{n-1}, \qquad \theta \geqslant 0. \tag{2.4a}$$

Remarks. (i) Elementary computations using (2.3a) and (2.4a) yield, for each fixed $n \geqslant 1$,

$$\mathbb{E}\exp(-\theta\Xi_n) \sim (\Gamma(1-|\nu|))^{-n}\theta^{-n|\nu|}, \tag{2.3b}$$

$$\mathbb{E}\exp(-\theta\Upsilon_n) \sim |\nu|^{n-1}\theta^{-(n-1)}, \qquad \theta \to \infty. \tag{2.4b}$$

(ii) By definition and (2.2), it immediately follows that

$$\frac{1}{V_n(1)} = n + \Upsilon_n + \Xi_n, \qquad n \geqslant 1, \tag{2.5}$$

$$\frac{1}{V_n(1)} - 1 = \sum_{j=1}^{n-1}\frac{V_j(1)}{V_n(1)} + \Xi_n, \qquad n \geqslant 1, \tag{2.6}$$

$$\frac{1}{V_n(1)} - 1 = \frac{V_{n-1}(1)}{V_n(1)}(n-1+\Upsilon_{n-1}) + \Xi_n, \qquad n \geqslant 2. \tag{2.7}$$

Proof of Lemma 2.1. Of course we only have to treat the case when x is in the (positive) neighbourhood of 0. For notational simplicity (i.e. by an abuse of notation!), we write $H_k = H_k(1)$ and $V_k = V_k(1)$ within the proof. Thus, with probability one,

$$H_k - \Lambda^-(H_k) = 1. \tag{2.8}$$

Define

$$X_k = \inf_{H_k(1)\leqslant t\leqslant H_k(2)}\left(\frac{1}{k}-\frac{V_k(t)}{t}\right),$$
$$E = \{1 < V_{k-1}(H_k) < 2\},$$
$$F = \{\Lambda^+(H_k) - \Lambda^-(H_k) \leqslant V_{k-1}(H_k)\}.$$

Accordingly, the probability term in (2.1) can be written as

$$\mathbb{P}(X_k < x) = \mathbb{P}(X_k < x;\ E) + \mathbb{P}(X_k < x;\ E^{\mathrm{c}})$$
$$\equiv \mathrm{I} + \mathrm{II},$$

with obvious notation. We now evaluate I and II. Observe that if $\omega \in E$, then $H_k(2)$ is strictly larger than $\Lambda^+(H_k)$, and several lines of elementary calculation show that $\sup_{\Lambda^+(H_k)\leqslant t\leqslant H_k(2)} V_k(t)/t$ is either realized at $t = \Lambda^+(H_k)$ or smaller than $1/(k+\frac{1}{2})$. Thus in this situation,

$$\{X_k < x\} = \left\{\inf_{H_k\leqslant t\leqslant \Lambda^+(H_k)}\left(\frac{1}{k}-\frac{V_k(t)}{t}\right) < x\right\}.$$

We distinguish two subcases. First, let $\omega \in E\cap F$. We have $V_k(t) = t - \Lambda^-(H_k)$ for $t \in [H_k, \Lambda^+(H_k)]$, which means that the supremum of $V_k(t)/t$ over $[H_k, \Lambda^+(H_k)]$ is reached at $t = \Lambda^+(H_k)$, or equivalently that $X_k = \Lambda^-(H_k)/\Lambda^+(H_k) - (k-1)/k$. Accordingly,

$$(\{X_k < x\}\cap E\cap F) \subset \left(E\cap F\cap\left\{\Lambda^-(H_k) \leqslant \frac{k-1+kx}{1-kx}V_{k-1}(H_k)\right\}\right). \tag{2.9}$$

In the second subcase, $\omega \in E\cap F^{\mathrm{c}}$, it is easily seen that the supremum of $V_k(t)/t$ over $[H_k, \Lambda^+(H_k)]$ is realized at $t = \Lambda^-(H_k) + V_{k-1}(H_k)$. Thus $X_k = 1/k - V_{k-1}(H_k)/(\Lambda^-(H_k) + V_{k-1}(H_k))$, which yields

$$(\{X_k < x\}\cap E\cap F^{\mathrm{c}}) = \left(E\cap F^{\mathrm{c}}\cap\left\{\Lambda^-(H_k) \leqslant \frac{k-1+kx}{1-kx}V_{k-1}(H_k)\right\}\right). \tag{2.10}$$

By (2.9) and (2.10), we obtain

$$\begin{aligned}\mathrm{I} &= \mathbb{P}(X_k < x;\ E;\ F) + \mathbb{P}(X_k < x;\ E;\ F^{\mathrm{c}})\\ &\leqslant \mathbb{P}\left(E;\ \Lambda^-(H_k) \leqslant \frac{k-1+kx}{1-kx}V_{k-1}(H_k)\right)\\ &\leqslant \mathbb{P}(1 < V_{k-1}(H_k) < 2;\ H_k - 1 \leqslant (k-1+2k^2x)V_{k-1}(H_k)),\end{aligned} \tag{2.11}$$

the last inequality being due to (2.8) and the trivial estimate $(k-1+kx)/(1-kx) \leqslant k-1+2k^2x$ (for small x). Since $V_k(H_k)=1$, by means of (1.3) and (2.7), we have, for any $z>0$,

$$\mathbb{P}(1<V_{k-1}(H_k)<2;\ H_k-1 \leqslant (k-1+z)V_{k-1}(H_k))$$

$$= \mathbb{P}\left(1<\frac{V_{k-1}}{V_k}<2;\ \frac{1}{V_k}-1<(k-1+z)\frac{V_{k-1}}{V_k}\right)$$

$$= \mathbb{P}\left(1<\frac{V_{k-1}}{V_k}<2;\ \mathrm{Y}_{k-1}+\frac{V_k}{V_{k-1}}\Xi_k<z\right) \tag{2.12}$$

$$\leqslant \mathbb{P}\left(\mathrm{Y}_{k-1}+\frac{1}{2}\Xi_k<z\right). \tag{2.13}$$

Since Y_{k-1} and Ξ_k are independent (cf. (2.2)), using (2.3b) and (2.4b), we have, for some unimportant (but computable) constant $C>0$,

$$\mathbb{E}\exp\left(-\theta\mathrm{Y}_{k-1}-\frac{\theta}{2}\Xi_k\right) \sim C\theta^{-(1+|\nu|)k+2}, \qquad \theta\to\infty.$$

It follows from a Tauberian theorem (cf. Feller 1971, p. 445) that

$$\mathbb{P}\left(\mathrm{Y}_{k-1}+\frac{1}{2}\Xi_k<y\right) \asymp y^{(1+|\nu|)k-2}, \qquad y\to 0. \tag{2.14}$$

Going back to (2.11) and (2.13), we obtain

$$\mathrm{I} \leqslant Cx^{(1+|\nu|)k-2}. \tag{2.15}$$

To estimate II, consider the event $E^{\mathrm{c}}=\{V_{k-1}(H_k)\geqslant 2\}$. A key observation is that in order to realize $\{X_k<x\}$ (x being sufficiently small), we must have $H_k(2)\leqslant \Lambda^+(H_k)$. Moreover, the supremum of $t\mapsto V_k(t)/t$ over $[H_k(1), H_k(2)]$ is realized at $H_k(2)=H_k(1)+1$, and it is easily seen that $V_1(H_k)\leqslant 2+2k^2x/(1-kx)$ (which is smaller than $2+3k^2x$). Consequently,

$$\mathrm{II} = \mathbb{P}(X_k<x;\ E^{\mathrm{c}})$$

$$\leqslant \mathbb{P}\left(\frac{1}{k}-\frac{2}{H_k+1}<x;\ 2\leqslant V_j(H_k)\leqslant 2+3k^2x,\ j=1,\ldots,k-1\right)$$

$$= \mathbb{P}\left(\frac{1}{k}-\frac{2}{1+1/V_k}<x;\ 2\leqslant \frac{V_j}{V_k}\leqslant 2+3k^2x,\ j=1,\ldots,k-1\right),$$

the last identity being due to (1.3). Using (2.6) and writing $V_j/V_k=\prod_{i=j}^{k-1}(V_i/V_{i+1})$, we obtain

$$\mathrm{II} \leqslant \mathbb{P}\left(\Xi_k<\frac{2}{1/k-x}-2k;\ 2\leqslant \frac{V_{k-1}}{V_k}\leqslant 2+3k^2x;\ 1\leqslant \frac{V_j}{V_{j+1}}\leqslant 1+2k^2x,\ j=1,\ldots,k-2\right).$$

Since Ξ_k and $(V_{j+1}/V_j)_{1\leqslant j\leqslant k-1}$ are independent such that V_{j+1}/V_j has Beta $(j|\nu|, 1)$, law (cf. (2.2)), this implies

$$\mathrm{II} \leqslant Cx^{k-1}\mathbb{P}(\Xi_k < 3k^2x).$$

The lower tail of Ξ_k is easily computed. Indeed, by (2.3b) and the Tauberian theorem mentioned above,

$$\mathbb{P}(\Xi_k < y) \asymp y^{k|\nu|}, \qquad y \to 0. \tag{2.16}$$

Consequently,

$$\mathrm{II} \leqslant Cx^{(1+|\nu|)k-1}. \tag{2.17}$$

Since $\mathbb{P}(X_k < x) = \mathrm{I} + \mathrm{II}$, combining (2.15) and (2.17) yields the second part of Lemma 2.1. It remains to verify its first part. Observe from (2.10) and (2.8) that

$$(\{X_k < x\} \cap E \cap B) = (E \cap A \cap B), \tag{2.18}$$

with

$$A = \left\{\Lambda^-(H_k) \leqslant \frac{k-1+kx}{1-kx} V_{k-1}(H_k)\right\},$$

$$B = \left\{1 < \frac{\Lambda^+(H_k) - H_k + 1}{V_{k-1}(H_k)} < 13\right\}.$$

Let us now estimate $\mathbb{P}(E \cap A \cap B)$. First, the random variable $(\Lambda^+(H_k) - H_k)/(R(H_k))^2$ is independent of $\mathscr{F}_{H_k}$ ($\mathscr{F}$ being the completed natural filtration of R), and has the same law as the first hitting time at 0 of a d-dimensional Bessel process starting from 1. Thus

$$\mathbb{P}(E \cap A \cap B) \geqslant \mathbb{P}\left(E;\ A;\ B;\ 2 < \frac{\Lambda^+(H_k) - H_k}{(\mathrm{R}(H_k))^2} < 3\right)$$

$$\geqslant C^{-1}\mathbb{P}\left(E \cap A;\ \frac{1}{4} < \frac{V_{k-1}(H_k)}{(R(H_k))^2} < 2\right),$$

with $C^{-1} = \mathbb{P}(2 < (\Lambda^+(H_k) - H_k)/(R(H_k))^2 < 3)$. Since $R(H_k)$ is independent of $\mathscr{F}\Lambda^-(H_k)$ – this is pointed out in Lecture 6 of Yor (1995) for the Brownian case; the independence holds, however, for general Markov processes (cf. Jeulin 1980, Theorem 6.3) – we have

$$\mathbb{P}(E \cap A \cap B) \geqslant C^{-1}\mathbb{P}(1 < R(H_k) < 2)\mathbb{P}(E \cap A) = \hat{C}^{-1}\mathbb{P}(E \cap A).$$

By means of (2.12) and the independence of Υ_{k-1}, Ξ_k and V_{k-1}/V_k (cf. (2.2)), we obtain

$$\mathbb{P}(E \cap A) \geqslant \mathbb{P}(\Upsilon_{k-1} + \Xi_k < kx)\mathbb{P}\left(1 < \frac{V_{k-1}}{V_k} < 2\right)$$

$$\geqslant C^{-1}\mathbb{P}\left(\Upsilon_{k-1} + \frac{1}{2}\Xi_k < \frac{1}{2}kx\right),$$

which, according to (2.18) and (2.14), implies

$$\mathbb{P}(X_k < x;\ E;\ B) \geqslant C^{-1} x^{(1+|\nu|)k-2}. \tag{2.19}$$

This is the desired first part of Lemma 2.1. □

From now on, we shall properly write $V_k(1)$ for the variable, V_k denoting exclusively the process $t \mapsto V_k(t)$.

Proof of Theorem 1.3. Let $\phi > 0$ be non-decreasing such that $\int^{\infty}(\mathrm{d}t/t)(\phi(t))^{-(1+|\nu|)k+2} < \infty$. Define $t_n = 2^n$. By scaling and Lemma 2.1,

$$\mathbb{P}\left[\inf_{H_k(t_n)\leqslant t\leqslant H_k(t_{n+1})}\left(\frac{1}{k}-\frac{V_k(t)}{t}\right) < \frac{1}{\phi(t_n)}\right] = \mathbb{P}\left[\inf_{H_k(1)\leqslant t\leqslant H_k(2)}\left(\frac{1}{k}-\frac{V_k(t)}{t}\right) < \frac{1}{\phi(t_n)}\right]$$

$$\leqslant C(\phi(t_n))^{-(1+|\nu|)k+2},$$

which is summable for n. According to the Borel–Cantelli lemma, when n is sufficiently large and $t \in [H_k(t_n),\ H_k(t_{n+1})]$, we have

$$\frac{1}{k} - \frac{V_k(t)}{t} \geqslant \frac{1}{\phi(t_n)} \geqslant \frac{1}{\phi(V_k(t))} \geqslant \frac{1}{\phi(t)},$$

which yields the convergent half of Theorem 1.3. For its divergent half, we assume that $\int^{\infty}(\mathrm{d}t/t)(\phi(t))^{-(1+|\nu|)k+2} = \infty$ and let $t_n = 2^n$. Define $\hat{\phi}(t) = \phi(t^2)$ and consider

$$D_n = \left\{\inf_{H_k(t_n)\leqslant t\leqslant H_k(t_{n+1})}\left(\frac{1}{k}-\frac{V_k(t)}{t}\right) < \frac{1}{\hat{\phi}(t_n)}\right\},$$

$$E_n = \left\{1 < \frac{V_{k-1}(H_k(t_n))}{t_n} < 2\right\},$$

$$B_n = \left\{1 < \frac{\Lambda^+(H_k(t_n)) - \Lambda^-(H_k(t_n))}{V_{k-1}(H_k(t_n))} < 13\right\},$$

$$G_n = D_n \cap E_n \cap B_n,$$

for $n \geqslant n_0$. Recall that $t_{n+1} = 2t_n$ and $H_k(t_n) = \Lambda^-(H_k(t_n)) + t_n$; by scaling and (2.19), we obtain

$$\mathbb{P}(G_n) \geqslant C^{-1}(\hat{\phi}(t_n))^{-(1+|\nu|)k+2}, \tag{2.20}$$

which implies $\sum_n \mathbb{P}(G_n) = \infty$. Of course, we intend to apply the Borel–Cantelli lemma. To this end, observe that on $D_n \cap E_n$ (with sufficiently large n), the infimum expression in D_n must be realized at some (random) time $t \in [H_k(t_n),\ \Lambda^+(H_k(t_n))]$; furthermore, in this case the length of any excursion before $H_k(t_n)$ must be less than $3t_n$. Accordingly, if

$\omega \in D_n \cap E_n \cap B_n$, then any excursion before $\Lambda^+(H_k(t_n))$ has length no longer than $26t_n$, i.e.

$$G_n = E_n \cap B_n \cap \left\{ \inf_{H_k(t_n) \leqslant t \leqslant \Lambda^+(H_k(t_n))} \left(\frac{1}{k} - \frac{V_k(t)}{t} \right) < \frac{1}{\hat{\phi}(t_n)} \right\}$$

$$\cap \{ V_1(\Lambda^+(H_k(t_n))) < 26t_n \}. \tag{2.21}$$

Consider now $j - 6 \geqslant i \geqslant n_0$ (with a sufficiently large initial value n_0). Let $\{\hat{R}(t) \equiv R(t + \Lambda^+(H_k(t_i))); \ t \geqslant 0\}$, which is a d-dimensional Bessel process starting from 0, independent of $\mathscr{F}_{\Lambda^+(H_k(t_i))}$ ($\mathscr{F}$ being the filtration of R). Define the corresponding $(\hat{V}_k(t))$ and $(\hat{H}_k(t))$ in an obvious way. Since $t_j > 26t_{i+1}$, we deduce from (2.21) that on the event G_i, $H_k(s) = \hat{H}_k(s) + \Lambda^+(H_k(t_i))$ for $t_j \leqslant s \leqslant t_{j+1}$ and $V_k(t) = \hat{V}_k(t - \Lambda^+(H_k(t_i)))$ for $H_k(t_j) \leqslant t \leqslant H_k(t_{j+1})$. The strong Markov property yields

$$\begin{aligned}
\mathbb{P}(G_i \cap G_j) &\leqslant \mathbb{P}\left(G_i; \inf_{\hat{H}_k(t_j) \leqslant u \leqslant \hat{H}_k(t_{j+1})} \left(\frac{1}{k} - \frac{\hat{V}_k(u)}{u + \Lambda^+(H_k(t_i))} \right) < \frac{1}{\hat{\phi}(t_i)} \right) \\
&\leqslant \mathbb{P}\left(G_i; \inf_{\hat{H}_k(t_j) \leqslant u \leqslant \hat{H}_k(t_{j+1})} \left(\frac{1}{k} - \frac{\hat{V}_k(u)}{u} \right) < \frac{1}{\hat{\phi}(t_i)} \right) \\
&= \mathbb{P}(G_i)\mathbb{P}(D_j) \\
&\leqslant C\mathbb{P}(G_i)\mathbb{P}(G_j),
\end{aligned}$$

where in the last inequality we have used the scaling property, Lemma 2.1, as well as (2.20). Consequently,

$$\liminf_{n \to \infty} \sum_{i=n_0}^{n} \sum_{j=n_0}^{n} \mathbb{P}(G_i \cap G_j) \Bigg/ \left(\sum_{i=n_0}^{n} \mathbb{P}(G_i) \right)^2 \leqslant C.$$

According to Kochen and Stone's (1964) Borel–Cantelli lemma, We have $\mathbb{P}(G_n; \text{ i.o.}) \geqslant 1/C$. Now observe that with probability one, we have $V_k(t) \geqslant 2t^{1/2}$ for sufficiently large t (this can easily be verified by means of the convergent part of the Borel–Cantelli lemma; actually more is true for this kind of estimate, cf. Theorem 1.4). Hence

$$\mathbb{P}\left[V_k(t) \geqslant \frac{t}{k} \left(1 - \frac{1}{\phi(t)} \right); \text{ i.o.} \right] > 0.$$

This yields the divergent half once we prove the following lemma. □

Lemma 2.3. *Let $k \geqslant 1$, and let $f > 0$ be a measurable function such that $f(t) \to \infty$ (as $t \to \infty$). The event*

$$\{ V_k(t) \geqslant f(t); \text{ i.o.} \}$$

has probability zero or one.

Proof. Fix $s > 0$. Consider the new Bessel process $\{R^{(s)}(t) \equiv R(t + \Lambda^+(s));\ t \geqslant 0\}$. Denote by $V_1^{(s)} \geqslant V_2^{(s)} \geqslant \cdots$ the associated ranked excursion lengths processes. Clearly,

$$\{V_k(t) \geqslant f(t);\ \text{i.o.}\} = \{V_k^{(s)}(t) \geqslant f(t + \Lambda^+(s));\ \text{i.o.}\}. \tag{2.22}$$

Since $\Lambda^+(s)$ and the process $R^{(s)}$ (hence $V_k^{(s)}$) are measurable with respect to $\mathscr{G}_s \equiv \sigma\{R(u);\ u \geqslant s\}$, so is the event on the right-hand-side of (2.22). Consequently, $\{V_k(t) \geqslant f(t)$: i.o.$\}$ is $\cap_{r \geqslant s} \mathscr{G}_r$-measurable, for all $s > 0$, which means that it is a trivial event (this is easily seen, for example, by virtue of Bessel time inversion and Blumenthal's 0–1 law). □

3. Tails

The following results give useful estimates of the tails of $V_k(1)$ and of $\sum_{j=1}^k V_j(1)$.

Theorem 3.1. *As* $x \to 0$,

$$\mathbb{P}\left(\sum_{j=1}^k V_j(1) < x\right) \asymp \exp\left(-\frac{k\gamma}{x}\right), \qquad k \geqslant 1, \tag{3.1}$$

$$\mathbb{P}\left(\sum_{j=1}^k V_j(1) > 1 - x\right) \asymp x^{|\nu|}, \qquad k \geqslant 1, \tag{3.2}$$

$$\mathbb{P}(V_k(1) < x) \asymp x^{|\nu|}, \qquad k \geqslant 2, \tag{3.3}$$

where γ *is defined in Theorem 1.2.*

Before proving Theorem 3.1, we state a Tauberian theorem which may be of independent interest.

Theorem 3.2. *Let a, b and c be positive constants, and A a non-increasing function with* $\lim_{s \to \infty} A(s)$ *such that*

$$F(z) \equiv -\int_0^\infty e^{zs}\, dA(s)$$

converges for $\operatorname{Re}(z) < a$. *Define* $G(s) \equiv F(a - s)/(a - s) - c/s^b$ *for* $0 < \operatorname{Re}(s) < a$. *If for any fixed* $T > 0$,

$$\eta(x, T) \equiv x^{b-1} \int_{-T}^{T} |G(2x + iy) - G(x + iy)|\, dy = o(1), \qquad x \to 0^+,$$

then

$$A(t) = \left(\frac{c}{\Gamma(b)} + O(\rho(t))\right) e^{-at} t^{b-1}, \qquad t \to \infty,$$

with $\rho(t) \equiv \inf_{T \geqslant 1}(T^{-1} + \eta(t^{-1}, T) + (tT)^{-b})$. *The implicit constants in* $O(\rho(t))$ *above depend only on a, b and c.*

Remark. In most probabilistic applications, $A(t)$ is the tail distribution of some random variable, say X, and $F(z)$ the corresponding exponential moment which explodes at $\mathrm{Re}\,(z) = a$. If we know the rate of explosion of $F(z)$ at $\mathrm{Re}\,(z) = a$ (and under the regularity condition on $\eta(x, T)$, of course), Theorem 3.2 provides useful information about the upper tail behaviour of X. Note that no additional information is required about $G(s)$ when $\mathrm{Re}\,(s) \to a$ (which corresponds to $\mathrm{Re}\,(z) \to 0$ for the exponential moment $F(z)$). The 'critical line' for $G(s)$ is $\mathrm{Re}\,(s) = 0$ (the value of b contains the main information about the explosion rate), not $\mathrm{Re}\,(s) = a$.

Proof of Theorem 3.2. Suppose without loss of generality that $A(0^+) = 1$. Define $A(t) \equiv 1$ for $t \leqslant 0$ and $g_\sigma(t) \equiv (1 - \mathrm{e}^{-\sigma t})\,\mathrm{e}^{(a-\sigma)t} A(t)$, $0 \leqslant \sigma < a/2$. The proof is along the lines presented in Tenenbaum (1995, p. 234, Theorem 11) who treats the case when A is non-decreasing. The details are omitted. We only mention the key estimate: there exists a constant $K > 0$ depending only on a, b and c, such that for all $T \geqslant 1$, $0 \leqslant y \leqslant 1/T$, $0 \leqslant \sigma < a/2$,

$$\sup_{-\infty < x < \infty} (g_\sigma(x + y) - g_\sigma(x)) \leqslant K \|g_\sigma\|_\infty y,$$

('$\|\cdot\|_\infty$' denoting the L_∞-norm). This is verified via several lines of careful though elementary calculation. □

Proof of Theorem 3.1. By (2.3a) and analytic continuation, we have, for $\theta < \gamma$,

$$\mathbb{E} \exp(\theta \Xi_k) = \left(1 - |\nu| \int_0^1 z^{-|\nu|-1}(\mathrm{e}^{\theta z} - 1)\,\mathrm{d}z\right)^{-k}$$

$$= \left(1 - |\nu| \sum_{n=1}^{\infty} \frac{\theta^n}{n!(n - |\nu|)}\right)^{-k}.$$

Applying Theorem 3.2 to $A(t) = \mathbb{P}(\Xi_k > t)$, $a = \gamma$, $b = k$ and $c = \gamma^{-1}(|\nu| \int_0^1 x^{-|\nu|} \mathrm{e}^{-\gamma x}\,\mathrm{d}x)^{-k}$, this yields

$$\mathbb{P}(\Xi_k > y) \sim \frac{c}{\Gamma(k)} y^{k-1} \exp(-\gamma y), \qquad y \to \infty. \tag{3.4}$$

On the other hand, using (2.4b) gives

$$\mathbb{E}[(k + \Upsilon_k)^{k-1} \exp(-\lambda \Upsilon_k)] \asymp \lambda^{-(k-1)}, \qquad \lambda \to \infty. \tag{3.5}$$

By (2.5) and (2.6),

$$\frac{1}{\sum_{j=1}^k V_j(1)} = 1 + \frac{\Xi_k}{k + \Upsilon_k}. \tag{3.6}$$

Since Ξ_k and Υ_k are independent, (3.4)–(3.6) together imply

$$\mathbb{P}\left(\frac{1}{\sum_1^k V_j(1)} > z+1\right) \sim \frac{c}{\Gamma(k)} z^{k-1}\mathbb{E}[(k+\Upsilon_k)^{k-1}\exp(-\gamma z(k+\Upsilon_k))]$$

$$\asymp \exp(-k\gamma z), \qquad z \to \infty,$$

which yields (3.1). Similarly, combining (2.4b) with (2.16) and (3.6) implies (3.2), and (3.3) is a straightforward consequence of (2.5), (2.3b) and (2.4b). □

4. Proofs of Theorems 1.1, 1.2 and 1.4

The convergent part of these theorems follows from the tail estimations in Theorem 3.1, the scaling property, and a standard argument combining the easy half of the Borel–Cantelli lemma and monotonicity.

Proof of the convergent part of Theorems 1.1, 1.2 and 1.4. Let $k \geqslant 1$ and let ϕ be non-decreasing such that $\int^\infty (\mathrm{d}t/t)\phi(t)^{-|\nu|} < \infty$. Define $t_n = 2^n$ for $n \geqslant n_0$. Since $\sum_{j=1}^\infty V_j(t_n) = 1$ and $t_{n+1} = 2t_n$, by scaling and (3.2), we have

$$\mathbb{P}\left(\sum_{j=k+1}^\infty V_j(t_n) < \frac{t_{n+1}}{\phi(t_n)}\right) = \mathbb{P}\left(\sum_{j=1}^k V_j(1) > 1 - \frac{2}{\phi(t_n)}\right) \leqslant C(\phi(t_n))^{|\nu|},$$

and the sum over n of the latter quantity is finite. Thus by the Borel–Cantelli lemma, almost surely for sufficiently large n, we have $\sum_{j=k+1}^\infty V_j(t_n) \geqslant t_{n+1}/\phi(t_n)$. Let $t \in [t_n, t_{n+1}]$. Then

$$\sum_{j=1}^k V_j(t) = t - \sum_{j=k+1}^\infty V_j(t) \leqslant t - \sum_{j=k+1}^\infty V_j(t_n) \leqslant t - \frac{t_{n+1}}{\phi(t_n)} \leqslant t\left(1 - \frac{1}{\phi(t)}\right),$$

which implies the convergent half of Theorem 1.1. That of Theorems 1.2 and 1.4 can be proved along the same lines. □

Proof of the divergent half of Theorem 1.2. In view of the obvious relation $\sum_{j=1}^k V_j(t) \leqslant kV_1(t)$, it suffices to treat the case $k = 1$. Let $\int^\infty (\mathrm{d}t/t)\psi(t)\exp(-\gamma\psi(t)) = \infty$. As usual for this kind of exponential integral test, we can limit ourselves to the study of a 'critical case':

$$\frac{1}{2\gamma}\log\log t \leqslant \psi(t) \leqslant \frac{2}{\gamma}\log\log t \tag{4.1}$$

(cf. Erdös 1942). Define $t_i = \exp(i/\log i)$ and $A_i = \{V_1(t_i) < t_i/\psi(t_i)\}$ for $i \geqslant n_0$ (n_0 being sufficiently large). By scaling and taking $k = 1$ in (3.1),

$$\mathbb{P}(A_i) \geqslant C^{-1}\exp(-\gamma\psi(t_i)), \tag{4.2}$$

which implies $\sum_i \mathbb{P}(A_i) = \infty$. Let $j > i \geqslant n_0$. By the Markov property, we have

$$\mathbb{P}(A_i \cap A_j) \leqslant \mathbb{P}\left(A_i; \text{ the longest length of excursions over } (t_i, t_j) < \frac{t_j}{\psi(t_j)}\right)$$

$$= \mathbb{E}\left[\mathbb{1}_{A_i} \mathbb{P}_{R(t_i)}\left(V_1(t_j - t_i) < \frac{t_j}{\psi(t_j)}\right)\right], \tag{4.3}$$

where $\mathbb{P}_a$ denotes the probability under which the Bessel process starts from a (thus $\mathbb{P} = \mathbb{P}_0$). By (3.1), $\mathbb{P}(V_1(u) < y) \leqslant C \exp(-\gamma u/y)$ (for *any* u and y). Therefore, if $T(0) = \inf\{t > 0: R(t) = 0\}$ denotes the first hitting time at 0 by R, then for any positive numbers a and $t > y$,

$$\begin{aligned}\mathbb{P}_a(V_1(t) < y) &= \int_0^y \mathbb{P}_a(T(0) \in \mathrm{d}s)\mathbb{P}(V_1(t-s) < y) \\ &\leqslant C\mathrm{e}^{\gamma}\mathrm{e}^{-\gamma t/y} \int_0^y \mathbb{P}_a(T(0) \in \mathrm{d}s) \\ &\leqslant C\mathrm{e}^{\gamma}\mathrm{e}^{-\gamma t/y}.\end{aligned}$$

Going back to (4.3), we obtain

$$\mathbb{P}(A_i \cap A_j) \leqslant C\mathbb{P}(A_i) \exp\left(-\gamma\left(1 - \frac{t_i}{t_j}\right)\psi(t_j)\right). \tag{4.4}$$

From here, we use an argument due to Erdös (1942). Let us distinguish two possible situations. First, assume $j < i + (\log i)^2$. In this case, using (4.1) and the definition of the sequence (t_n), we easily arrive at the estimate $(1 - t_i/t_j)\psi(t_j) \geqslant \min(j - i, \log i)/C$, which according to (4.4) yields

$$\begin{aligned}\sum\sum_{n_0 \leqslant i \leqslant j \leqslant n; j < i + (\log i)^2} \mathbb{P}(A_i \cap A_j) &\leqslant C \sum_{i=n_0}^{n} \mathbb{P}(A_i) \sum_{j=i}^{\infty} \mathrm{e}^{-(j-i)/C} + C \sum_{i=n_0}^{n} \mathbb{P}(A_i) \sum_{j=i}^{i+(\log i)^2} i^{-1/C} \\ &\leqslant \hat{C} \sum_{i=n_0}^{n} \mathbb{P}(A_i),\end{aligned} \tag{4.5}$$

for some constant $\hat{C}$ depending only on d. If on the other hand $j \geqslant i + (\log i)^2$, then by (4.1), $(t_i/t_j)\psi(t_j)$ is bounded (above). Therefore we have, by (4.4) and (4.2), $\mathbb{P}(A_i \cap A_j) \leqslant C\mathbb{P}(A_i)\mathbb{P}(A_j)$, which implies

$$\sum\sum_{n_0 \leqslant i \leqslant j \leqslant n; j \geqslant i + (\log i)^2} \mathbb{P}(A_i \cap A_j) \leqslant C\left(\sum_{i=n_0}^{n} \mathbb{P}(A_i)\right)^2. \tag{4.6}$$

Since $\sum \mathbb{P}(A_i) = \infty$, combining (4.5) and (4.6) gives

$$\liminf_{n \to \infty} \sum_{i=n_0}^{n} \sum_{j=n_0}^{n} \mathbb{P}(A_i \cap A_j) \Big/ \left(\sum_{i=n_0}^{n} \mathbb{P}(A_i)\right)^2 \leqslant C,$$

which, by means of Kochen and Stone's (1964) Borel–Cantelli lemma, tells us that $\mathbb{P}(A_n;\ \text{i.o.}) \geqslant 1/C$. The proof of the divergent part of Theorem 1.2 is completed by a 0–1 law similar to Lemma 2.3. □

The divergent half of Theorem 1.1 can be proved using the Borel–Cantelli lemma, by considering the events $A_n = \{V_1(\tau(t_n))/\tau(t_n) > 1 - 1/\hat{\phi}(t_n)\}$, where τ is the inverse of the local time at 0 introduced in Section 1, and $\hat{\phi}(t) = \phi(t^3)$. Moreover, it proves the first result in Theorem 5.1 (cf. Section 5). However, it is more convenient to apply a known integral test for the last passage process.

Fact 4.1 (Chung and Erdös 1952; Hobson 1992; 1994; Bertoin 1995). *Let* $\Lambda^-(t)$, *be, as before, the last zero of R before t. For any non-decreasing function* $f > 0$,

$$\mathbb{P}[f(\Lambda^-(t)) < t;\ \text{i.o.}] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{\mathrm{d}t}{t^{1-|\nu|}(f(t))^{|\nu|}} \begin{cases} < \infty \\ = \infty \end{cases}.$$

In the case $d = 1$, the above test was established by Chung and Erdös (1952) and Hobson (1994). It was extended to Bessel processes by Hobson (1992) and to general Markov processes by Bertoin (1995).

Proof of the divergent half of Theorem 1.1. Let ϕ be a non-decreasing function such that $\int^{\infty}(\mathrm{d}t/t)\phi(t)^{-|\nu|} = \infty$. Let $f(t) = t\phi(t^2)$, which is non-decreasing. It is easily seen that $\int^{\infty} t^{-(1-|\nu|)}(f(t))^{-|\nu|}\,\mathrm{d}t = \infty$. Therefore, by the test in Fact 4.1, $\mathbb{P}(f(\Lambda^-(t)) < t;\ \text{i.o.}) = 1$. Since by the same test, $\Lambda^-(t) \geqslant t^{1/2}$ for large t, this yields $\mathbb{P}(\Lambda^-(t)\phi(t) < t;\ \text{i.o.}) = 1$. The divergent half of Theorem 1.1 follows using the obvious relation $\sum_1^k V_j(t) \geqslant t - \Lambda^-(t)$. □

Proof of the divergent half of Theorem 1.4. If $\int^{\infty}(\mathrm{d}t/t)\psi(t)^{-|\nu|} = \infty$, then by Theorem 1.1, $\mathbb{P}(V_1(t) > t(1 - 1/\psi(t));\ \text{i.o.}) = 1$. Since for any $k \geqslant 2$, $V_k(t) \leqslant t - V_1(t)$, this yields the desired conclusion. □

Remark. J. Bertoin kindly points out that Theorem 1.1 can also be obtained by means of subordinator techniques. The present proof is retained because it gives a unified approach for this kind of problem.

5. Excursion lengths *à la* Knight

In the previous sections, we have studied the ordered lengths of Bessel excursions before t, by considering $(\Lambda^-(t), t)$ as an excursion interval. It looks natural to investigate also the *completed* excursion intervals, i.e. those before time $\Lambda^-(t)$. Knight (1986) established some interesting exact results for the longest completed excursion length of a Wiener process. Extensions to Bessel processes can be found in Perman (1993). Let

$$\tilde{V}_1(t) \geqslant \tilde{V}_2(t) \geqslant \cdots \geqslant \tilde{V}_n(t) \geqslant \cdots$$

be the ordered lengths of completed excursions of R over $(0, t)$. One may ask whether the $\tilde{V}_k$ have the same asymptotic behaviours as the V_k.

Theorem 5.1. *Let $k \geqslant 1$ and let $\phi > 0$ be a non-decreasing function. Then*

$$\mathbb{P}\left[\sum_{j=1}^{k} \tilde{V}_j(t) > t\left(1 - \frac{1}{\phi(t)}\right);\ \text{i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{\mathrm{d}t}{t(\phi(t))^{|\nu|}} \begin{cases} < \infty \\ = \infty \end{cases},$$

$$\mathbb{P}\left[\sum_{j=1}^{k} \tilde{V}_j(t) < \frac{t}{\phi(t)};\ \text{i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{\mathrm{d}t}{t(\phi(t))^{|\nu|}} \begin{cases} < \infty \\ = \infty \end{cases},$$

$$\mathbb{P}\left[\tilde{V}_k(t) < \frac{t}{\phi(t)};\ \text{i.o.}\right] = \begin{cases} 0 \\ 1 \end{cases} \Leftrightarrow \int^{\infty} \frac{\mathrm{d}t}{t(\phi(t))^{|\nu|}} \begin{cases} < \infty \\ = \infty \end{cases}.$$

We feel free to omit the proof, which is more or less in the same vein as for the V_k. Comparing Theorem 5.1 with the corresponding results for the V_k, it is immediately seen that $\sum_{j=1}^{k} \tilde{V}_j(t)$ ($\tilde{V}_l(t)$, $l \geqslant 2$) has the same upper (lower) functions as $\sum_{j=1}^{k} V_j(t)$ ($V_l(t)$, $l \geqslant 2$). On the other hand, $\sum_{j=1}^{k} \tilde{V}_j(t)$ has completely different lower functions from $\sum_{j=1}^{k} V_j(t)$. This is due to the fact that the last zero-free interval $(\Lambda^-(t), t)$ has an overwhelming contribution in the lim inf behaviour of $\sum_{j=1}^{k} V_j(t)$ (or simply in that of $V_1(t)$).

The problem of characterizing Lévy's upper class for $\tilde{V}_k(t)$ ($k \geqslant 2$) remains, to the best of our knowledge, open.

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