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Hausdorff–Besicovitch dimension of graphs and *p*-variation of some Lévy processes

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The connection between Hausdorff–Besicovitch dimension of graphs of trajectories and various Blumenthal–Getoor indices is well known for α-stable Lévy processes as well as for some stationary Gaussian processes possessing Orey index. We show that the same relationship holds for several classes of Lévy processes that are popular in financial mathematics models – in particular, the Carr–Geman–Madan–Yor, normal inverse Gaussian, generalized hyperbolic, generalized z and Meixner processes.

Keywords: Blumenthal–Getoor indices; Carr–Geman–Madan–Yor process; generalized hyperbolic process; generalized z-process; graph; Hausdorff–Besicovitch dimension; Lévy process; Meixner process; normal inverse Gaussian process; p-variation

1. Introduction

Recent advances in the theory and applications of Lévy processes have strengthened their appeal as replacements for Brownian motion, advocated in the famous papers by Black and Scholes (1973) and Merton (1973), as well as for the pure jump processes of Cox and Ross (1976) and the jump diffusions of Merton (1976), to model returns of financial assets. Among the most popular examples are the variance-gamma (VG) process developed by Madan and Seneta (1990) and later treated by Madan *et al.* (1998), the Carr–Geman–Madan–Yor (CGMY) process (a generalization of VG) introduced by Carr *et al.* (2002), the hyperbolic model (HM) considered by Eberlein *et al.* (1998), the normal inverse Gaussian (NIG) model suggested by Barndorff-Nielsen (1997, 1998), and the generalized hyperbolic (GH) Lévy motion introduced to model wind-blown sand grain size by Barndorff-Nielsen and Halgreen (1977) and applied to finance by Eberlein (2001). An alternative direction is taken by Grigelionis (2001) who introduced the generalized z-process (GZ) that includes the Meixner process studied by Schoutens and the z-process of Prentice (1975). Yet another fruitful direction is to use non-Gaussian Ornstein–Uhlenbeck type models for stochastic volatility of financial markets, as proposed by Barndorff-Nielsen and Shephard (2001).

The CGMY process – or, even more broadly, the generalized tempered stable process (GTSP) considered by Cont and Tankov (2004) – generalizes the α-stable Lévy motion and is defined by specifying its Lévy measure, and not the closed form of the characteristic function as is the case with the NIG, VG, HM, GZ and Meixner models. The closed form of the CGMY characteristic function is known – see Carr *et al.* (2002) for the case $Y < 0$, Miyahara (2002) for all cases and Zhou *et al.* (2005) for the GTSP – but should be understood in the limiting sense if $Y = 0, 1$ (see (3.1)). So numerical stability, rate of convergence and error issues should be addressed if Y is near 0 or 1. The financial data analysis provided by (Carr *et al.* 2002), seems to ignore this altogether. An alternative would be to use the exact form of the characteristic function whenever ar $Y = 0, 1$, but this requires prior knowledge of Y. So we propose estimating Y from the Hausdorff–Besicovitch dimension of the graph of the CGMY process.

In this paper we show how the parameter Y can be computed for the CGMY process whenever $Y \in [1, 2)$. If $Y \in [\frac{1}{2}, 1)$, one can recover Y by using subordination to Brownian motion (Manstavičius 2005).

More generally, we show how the Hausdorff–Besicovitch dimension of the graph of a real-valued Lévy process X can be quickly computed, provided the upper index β as well as the least lower Blumenthal and Getoor (1961) index β'' are equal. This result is well known for many processes, among them Brownian motion (Taylor 1955), α-stable processes (Blumenthal and Getoor 1962; Jain and Pruitt 1968; Pruitt and Taylor 1969) and stationary Gaussian processes with Orey index (Orey 1970). In many cases, such as α-stable processes, even the correct Hausdorff measure is known (Jain and Pruitt 1968; Pruitt and Taylor 1969). But for general Lévy processes on $\mathbb{R}^d$ there are still some unanswered questions (for details, see Pruitt and Taylor 1996).

If $\beta \geqslant 1$, then β is also equal to the p-variation index $v(X)$ of the process X (for details, see Norvaiša and Salopek 2002) and can be used for statistical inference as described by Dudley and Norvaiša (1999). For example, in the case of the CGMY process with $Y \in [1, 2)$, the reciprocal $1/Y$ becomes the oscillation η-summing index and can be estimated statistically using oscillation η-summing estimators, as described in Norvaiša and Salopek (2002).

Following our discussion of the CGMY process, we also include a short exposition (see Sections 3.2–3.4) on the NIG, GH, GZ and Meixner processes. Their upper Blumenthal–Getoor index as well as their p-variation index is equal to 1 almost surely, which distinguishes them from the CGMY model with $Y \in (1, 2)$. So variational properties of data can also be used to narrow down the choice of a particular model. In fact, variational properties of Lévy processes have recently become very popular; see, for example, Barndorff-Nielsen and Shephard (2003).

Section 4 contains the proof of Theorem 2.1 followed by a discussion of some open problems in the case $\beta''_X < \beta_X$.

2. Preliminaries and results

Throughout we will assume that $X = \{X_t, t \in [0, 1]\}$ is a Lévy process, that is, a process with stationary and independent increments, which starts at the origin at $t = 0$ and has almost all trajectories cadlag. By now standard references for the theory of Lévy processes are Bertoin (1996) and Sato (2000). For more on Lévy processes in finance, see also the recent books by Applebaum (2004), Cont and Tankov (2004), Schoutens (2003) and Bingham and Kiesel (2004).

We will denote by $u \cdot v$ the standard scalar product of vectors u and v in $\mathbb{R}^d$ and by $\|u\|$ the Euclidean norm of u. The notation $f(x) \sim g(x)$ as $x \to a$ will mean $\lim_{x \to a} f(x)/g(x) = 1$.

Furthermore, let $\phi(\xi)$ be the characteristic exponent of X, that is, the function satisfying $\mathrm{E} \exp(\mathrm{i}\xi \cdot X_t) = \exp(-t\phi(\xi))$ for $\xi \in \mathbb{R}^d$ and $t \geqslant 0$. The famous Lévy–Khinchine formula (Sato 2000: Theorem 8.1) provides the exact expression for the function $\phi(\xi)$, which is characterized by a triplet $(\mathbf{b}, A, L)$, where $\mathbf{b} \in \mathbb{R}^d$, A is a symmetric non-negative definite $d \times d$ matrix, and L is the Lévy measure, a σ-finite Borel measure such that

$$\int_{\mathbb{R}^d \setminus \{0\}} (\|\mathbf{x}\|^2 \wedge 1) L(\mathrm{d}\mathbf{x}) < \infty.$$

Then the Lévy–Khinchine formula states that

$$\phi(\xi) = -\mathrm{i}\mathbf{b} \cdot \xi + \frac{1}{2}(A\xi) \cdot \xi - \int_{\mathbb{R}^d \setminus \{0\}} \left(\mathrm{e}^{\mathrm{i}\mathbf{y} \cdot \xi} - 1 - \mathrm{i}\mathbf{y} \cdot \xi 1_{\{\|\mathbf{y}\| \leqslant 1\}}(\mathbf{y})\right) L(\mathrm{d}\mathbf{y}). \tag{2.1}$$

We also let $\mathrm{Gr}X = \mathrm{Gr}X(\omega) = \{(t, X_t(\omega)), t \in [0, 1]\} \subset \mathbb{R}^{d+1}$ be the graph of a random trajectory of the process X.

Now recall the definitions of Hausdorff measures and Hausdorff–Besicovitch dimension (Mattila 1995; Rogers, 1998). Given $s \geqslant 0$, $\varepsilon > 0$ and a subset $A \subset \mathbb{R}^{d+1}$, we set

$$\mathcal{H}^s_\varepsilon(A) = \inf\left\{ \sum_j \left(\mathrm{diam}(B_j)\right)^s : A \subset \bigcup_j B_j, \, \mathrm{diam}(B_j) \leqslant \varepsilon, \, B_j \subset \mathbb{R}^{d+1} \right\}.$$

It is well known (Mattila 1995: 54) that the limit $\lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(A)$ exists. This limit is called the s-Hausdorff measure of A and denoted $\mathcal{H}^s(A)$. Furthermore, by Mattila (1995: Theorem 4.4) it is enough to consider only open (or closed, or even convex) covers of A in the definition of $\mathcal{H}^s_\varepsilon(A)$ above. The common value

$$\inf\{s > 0 : \mathcal{H}^s(A) = 0\} = \sup\{s \geqslant 0 : \mathcal{H}^s(A) = +\infty\}$$

is called the Hausdorff–Besicovitch dimension of A and denoted by $\dim_{\mathrm{HB}} A$.

We will also use several indices related to the Lévy process X. The first is the upper index β of Blumenthal and Getoor (1961) defined as

$$\beta = \inf\left\{ r > 0 : \int_{0 < \|y\| \leqslant 1} \|y\|^r L(\mathrm{d}y) < +\infty \right\}, \tag{2.2}$$

which can also be computed as (Blumenthal and Getoor 1961: Theorem 2.1)

$$\beta = \inf\{r > 0 : t^r L(\{y : \|y\| > t\}) \to 0, \text{ as } t \to 0\}. \tag{2.3}$$

If the process X is assumed to have no Gaussian component and no drift then, by Blumenthal and Getoor (1961: Theorem 3.2),

$$\begin{aligned} \beta &= \inf\{r \geqslant 0 : \|\xi\|^{-r} |\phi(\xi)| \to 0, \text{ as } \|\xi\| \to \infty\} \\ &= \inf\{r \geqslant 0 : \|\xi\|^{-r} \Re\phi(\xi) \to 0, \text{ as } \|\xi\| \to \infty\}. \end{aligned} \tag{2.4}$$

We will also require a pair of lower indices of Blumenthal and Getoor (1961). The smaller of these is

$$\beta'' = \sup\{r \geqslant 0 : \|\xi\|^{-r}\Re\phi(\xi) \to \infty, \text{ as } \|\xi\| \to \infty\}, \tag{2.5}$$

and will be used to control the rate of growth of $\Re\phi(\xi)$, as $\|\xi\| \to \infty$. The larger is

$$\beta' = \sup\left\{ r \geqslant 0 : \int_{\mathbb{R}^d} \|\xi\|^{r-d} \frac{1 - e^{-\Re\phi(\xi)}}{\Re\phi(\xi)} \, d\xi < \infty \right\}. \tag{2.6}$$

By Blumenthal and Getoor (1961: Theorem 5.1), $0 \leqslant \beta'' \leqslant \beta' \leqslant \beta \leqslant 2$.

If X has no Gaussian component, but has a linear drift term (this is often the case in financial models; see Sections 3.2–3.4), then we will use the first equality in (2.4) as the definition for β. This is essentially the same as taking $\beta = 1 \vee \beta_{X'}$, where $\beta_{X'}$ is the upper index of the process X' obtained from X by subtracting its linear drift. If a Gaussian component is present, then we use (2.4) to define β and obtain $\beta = \beta' = \beta'' = 2$. Also it is well known (Blumenthal and Getoor 1961) that for an α-stable Lévy process (with drift removed if $\alpha < 1$) we have $\beta = \beta' = \beta'' = \alpha$.

Pruitt (1969) introduced another index γ given by

$$\gamma = \sup\left\{ r \geqslant 0 : \limsup_{a \to 0} a^{-r} \int_0^1 P(\|X(t)\| \leqslant a) dt < +\infty \right\}$$

and showed that the Hausdorff–Besicovitch dimension of the range $X[0, 1]$ is equal to γ almost surely. If, in addition, $\Re\phi(\xi) \geqslant 2\log\|\xi\|$ for all large $\|\xi\|$, then from Pruitt (1969: Theorem 5) we have

$$\gamma = \sup\left\{ r < d : \int_{\mathbb{R}^d} \|\xi\|^{r-r} \Re\left(\frac{1 - e^{-\phi(\xi)}}{\phi(\xi)} \right) d\xi < +\infty \right\}.$$

Recently Khoshnevisan *et al.* (2003) established the following equality which holds under no restrictions on the growth of $\Re\phi(\xi)$

$$\gamma = \sup\left\{ r < d : \int_{\|\xi\|>1} \|\xi\|^{r-d} \Re\left(\frac{1}{1 + \phi(\xi)} \right) d\xi < +\infty \right\}. \tag{2.7}$$

We will use (2.7) to investigate the Hausdorff–Besicovitch dimension of the graph GrX; I am very grateful to a referee for pointing out that this can be viewed as the range of a new Lévy process $G_t = (t, X_t)$, $t \in [0, 1]$. The characteristic exponent of G is easily seen to be $\phi_G(\xi_1, \xi_2) = -i\xi_1 + \phi_X(\xi_2)$, where $\xi_1 \in \mathbb{R}$ and $\xi_2 \in \mathbb{R}^d$. Also the Lévy measure of G is given by $L_G(dx, dy) = \delta_0(dx) L_X(dy)$, where δ_0 is the point mass at $x = 0$. To discriminate between various indices of X and G we will use subscripts, for example β_X, γ_G.

Our main result is the following.

Theorem 2.1. *Let X be a real-valued Lévy process with Blumenthal–Getoor indices* β_X *and* β''_X *defined in* (2.2) *and* (2.5)*, respectively. If* $G_t = (t, X_t)$, $t \in [0, 1]$*, then the following statements hold:*

(i) *If* $1 < \beta''_X = \beta_X \leqslant 2$, *then* $\gamma_G = 2 - \beta_X^{-1}$.
(ii) *If* $\beta_X \leqslant 1$, *then* $\gamma_G = 1$.

Remark 2.1. It is worth mentioning that if X is a Lévy process in $\mathbb{R}^d$ with $d \geqslant 2$, then the condition $\beta''_X = \beta_X$ is very restrictive and automatically implies $\gamma_G = \beta_X$. This follows easily from Pruitt (1969: Theorem 5), Blumenthal and Getoor (1961: Theorem 3.1) and (2.6), that is,

$$\beta''_X = \beta'_X = d \wedge \beta'_X = \beta'_G \leqslant \gamma_G \leqslant \beta_G = 1 \vee \beta_X = \beta_X;$$

here β_G is defined using the first equality in (2.4). On the other hand, if $d = 1$, then the above inequalities only yield $1 \leqslant \gamma_G \leqslant 1 \vee \beta_X$.

Remark 2.2. Only the inequality $\gamma_G \geqslant 2 - \beta_X^{-1}$ needs to be established in part (i) of the above theorem. By Norvaiša and Salopek (2002: Proposition 1), $\gamma_G \leqslant 2 - \beta_X^{-1} \leqslant \beta_X$, whenever $\beta_X \geqslant 1$.

Remark 2.3. Part (ii) of the theorem is not new, and is only included for completeness. It follows most easily from the fact that $\gamma_G \geqslant 1$ almost surely (projection of G onto the t-axis does not increase the Hausdorff–Besicovitch dimension) and the inequality $\gamma_G \leqslant \beta_G = 1 \vee \beta_X = 1$ mentioned in Remark 2.1.

For many Lévy processes used in financial mathematics computation of the indices β and β'' is relatively easy, so the above theorem provides a quick way of computing the Hausdorff–Besicovitch dimension of the graphs. We illustrate this in Section 3 by first considering the CMGY process, as defined in Carr *et al.* (2002), and obtain the following result.

Corollary 2.2. *Let* X *be the CGMY process with parameter* $Y \in [1, 2)$. *Then* $\beta_X = \beta''_X = Y$ *and* $\gamma_X = 2 - 1/Y$.

For many other interesting processes X mentioned at the beginning of this paper — NIG, GH, GZ and Meixner processes – their index $\beta_X = 1$, so their graphs have Hausdorff–Besicovitch dimension equal to 1 almost surely. For more details, see Sections 3.2–3.4.

3. Applications

3.1. The CGMY process

To model financial asset returns Carr *et al.* (2002) introduced the CGMY process which generalized the variance-gamma (VG) model developed by Madan and Seneta (1990) and later extended by Madan and Milne (1991) and Madan *et al.* (1998). The CGMY process is defined as a real-valued Lévy process with the characteristic triplet $(b, 0, L)$, where the Lévy measure L has density $k_{\mathrm{CGMY}}(x)$ with respect to Lebesgue measure on $\mathbb{R}$ given by

$$k_{\mathrm{CGMY}}(x) = \frac{C}{|x|^{1+Y}} \exp\left\{-\frac{M+G}{2}|x| + \frac{G-M}{2}x\right\},$$

with $C > 0$, $G \geqslant 0$, $M \geqslant 0$ and $Y < 2$. If $Y \leqslant 0$, then $G \wedge M > 0$ is assumed. As a special case, when $Y = 0$, the function k_{CGMY} is the Lévy density of the VG process.

By Carr *et al.*, (2002: Theorem 1), Miyahara (2002: Section 7.1) or Zhou *et al.* (2005: Proposition 2.1), setting $\psi(\xi) = (\phi(\xi) + \mathrm{i}b_1\xi)/C$, we have

$$\psi(\xi) = \begin{cases} \Gamma(-Y)[(M-\mathrm{i}\xi)^Y - M^Y + (G+\mathrm{i}\xi)^Y - G^Y], & \text{if } Y \neq 0, 1, \\ \log(1+\mathrm{i}\xi/G) + \log(1-\mathrm{i}\xi/M), & \text{if } Y = 0, \\ (G+\mathrm{i}\xi)\log(1+\mathrm{i}\xi/G) + (M-\mathrm{i}\xi)\log(1-\mathrm{i}\xi/M), & \text{if } Y = 1, \end{cases} \tag{3.1}$$

where $b_1 = b + \int_{|x|>1} xL(\mathrm{d}x)$ and tb_1 is the mean of the process.

It is easy to see that

$$\int_{0<|x|\leqslant 1} |x|^p k_{\mathrm{CGMY}}(x)\mathrm{d}x < \infty \text{ if and only if } p > Y.$$

So $\beta_X = Y \vee 0$. To show that $\beta''_X = \beta_X$, and hence prove Corollary 2.2, we only need to establish $\beta''_X \geqslant Y$. But this easily follows from the following lemma.

Lemma 3.1. *Let X be the CGMY process with characteristic exponent $\phi(\xi)$. There exists a finite constant $K = K(C, c, G, M, Y) > 0$ such that $\Re\phi(\xi) \geqslant K|\xi|^Y$ for all $|\xi| \geqslant c > 0$.*

Proof. If $\xi \in \mathbb{R}\backslash\{0\}$, then formula (2.1) yields

$$\begin{aligned} \Re\phi(\xi) &= \int_{\mathbb{R}\backslash\{0\}} (1-\cos(y\xi))k_{\mathrm{CGMY}}(y)\mathrm{d}y \\ &= C\int_0^\infty (1-\cos(y\xi))\frac{\mathrm{e}^{-My} + \mathrm{e}^{-Gy}}{y^{1+Y}}\,\mathrm{d}y \\ &= C|\xi|^Y\int_0^\infty (1-\cos(z))\frac{\mathrm{e}^{-Mz/|\xi|} + \mathrm{e}^{-Gz/|\xi|}}{z^{1+Y}}\,\mathrm{d}z. \end{aligned}$$

Now if $|\xi| \geqslant c > 0$, we obtain

$$\Re\phi(\xi) \geqslant C|\xi|^Y\int_0^\infty (1-\cos(z))\frac{\mathrm{e}^{-Mz/c} + \mathrm{e}^{-Gz/c}}{z^{1+Y}}\,\mathrm{d}z = K|\xi|^Y,$$

where $K = K(C, c, M, G, Y) > 0$. □

3.2. The NIG process

The normal inverse Gaussian distributions and the corresponding Lévy processes have also proved to be well suited to modelling financial asset returns, as seen in Barndorff-Nielsen

(1997, 1998) and references therein. There are several definitions of the NIG$(\alpha, \beta, \mu, \delta)$ distribution and the corresponding process. One can first define the density $g_{\mathrm{NIG}}(x; \alpha, \beta, \mu, \delta)$ of the NIG distribution with respect to the Lebesgue measure

$$g_{\mathrm{NIG}}(x; \alpha, \beta, \mu, \delta) = c(\alpha, \beta, \mu, \delta) q\Big(\frac{x-\mu}{\delta}\Big)^{-1} C_1\Big(\alpha\delta q\Big(\frac{x-\mu}{\delta}\Big)\Big) \mathrm{e}^{\beta x}, \qquad x \in \mathbb{R},$$

where

$$q(x) = \sqrt{1+x^2},$$

$$c(\alpha, \beta, \mu, \delta) = \frac{\alpha}{\pi} \exp\Big\{\delta\sqrt{\alpha^2-\beta^2} - \beta\mu\Big\}$$

and the function K_1 denotes the modified Bessel function of the third order of index 1. The parameters α, β, μ and δ satisfy $0 \leqslant |\beta| < \alpha$, $\mu \in \mathbb{R}$ and $\delta > 0$. It is then shown that the NIG distribution is infinitely divisible and yields a Lévy process (Barndorff-Nielsen 1997).

The second approach is to start with the inverse Gaussian IG(δ, γ) distribution by specifying its density (Barndorff-Nielsen 1997: equation (2.5))

$$d(z; \delta, \gamma) = \frac{\delta \mathrm{e}^{\delta\gamma}}{\sqrt{2\pi} z^{3/2}} \exp\Big\{-\frac{1}{2}\Big(\frac{\delta^2}{z} + \gamma^2 z\Big)\Big\}, \qquad z > 0.$$

Then NIG$(\alpha, \beta, \mu, \delta)$ is the variance–mean mixture of a normal with the IG as the mixing distribution, that is, it can be described as the marginal distribution of x for a pair of random variables (x, z), where z has IG$(\delta, \sqrt{\alpha^2-\beta^2})$ distribution while, conditionally on z, the distribution of x is $N(\mu+\beta z, z^2)$.

The third way is to define the NIG process by subordination as follows:

$$X_{\mathrm{NIG}}(t) = \mu t + W(Z_t),$$

where W is the Brownian motion with drift β and diffusion coefficient 1, and where Z_t is the IG$(\delta, \sqrt{\alpha^2-\beta^2})$ process, independent of W. The process Z_t can also be thought of as the first passage time to level δt of a Brownian motion with drift $\sqrt{\alpha^2-\beta^2}$ and diffusion coefficient 1. This is where the name NIG comes from.

But for us it is most convenient to define the NIG$(\alpha, \beta, \mu, \delta)$ process as in Barndorff-Nielsen (1998: 47) by specifying its Lévy triplet $(b, 0, L)$:

$$b = \mu + \frac{2\delta\alpha}{\pi}\int_0^1 \sinh(\beta x) K_1(\alpha x)\,\mathrm{d}x,$$

$$\frac{\mathrm{d}L}{\mathrm{d}y} = f(y; \alpha, \beta, \delta) = \frac{\delta\alpha}{\pi|y|}\mathrm{e}^{\beta y} K_1(\alpha|y|), \qquad y \in \mathbb{R}\backslash\{0\}.$$

It is well known (Olver 1970: equation (9.6.9)) that, as $y \downarrow 0$, the function $K_1(y) \sim 1/y$, and so

$$f(y; \alpha, \beta, \delta) \sim \frac{\delta}{\pi} y^{-2}, \qquad \text{as } y \downarrow 0.$$

Therefore,

$$\int_{0<|y|\leqslant 1} |y|^p f(y;\, \alpha,\, \beta,\, \delta)\,\mathrm{d}y < \infty \text{ if and only if } p > 1.$$

So $\beta_X = 1$ for the NIG process.

3.3. The GH process

The generalized hyperbolic (GH) distributions form an even larger class of infinitely divisible distributions popular in financial models (see Bingham and Kiesel 2001). They include the hyperbolic as well as the NIG distributions as subclasses, and even allow representation of the VG distributions as limiting cases. Just as with the NIG distributions, we first define the density $g_{\mathrm{GH}}(x;\, \lambda,\, \alpha,\, \beta,\, \mu,\, \delta)$ of the NIG distribution (Barndorff-Nielsen 1997: equation (4.3)) with respect to the Lebesgue measure on $\mathbb{R}$ as

$$g_{\mathrm{GH}}(x;\, \lambda,\, \alpha,\, \beta,\, \mu,\, \delta) = c(\lambda,\, \alpha,\, \beta,\, \mu,\, \delta) q\Big(\frac{x-\mu}{\delta}\Big)^{\lambda-1/2} K_{\lambda-1/2}\Big(\alpha\delta q\Big(\frac{x-\mu}{\delta}\Big)\Big)\mathrm{e}^{\beta x},$$

where

$$q(x) = \sqrt{1+x^2},$$

$$c(\lambda,\, \alpha,\, \beta,\, \mu,\, \delta) = \frac{\alpha^{1/2-\lambda}}{\sqrt{2\pi\delta}}(\alpha^2-\beta^2)^{\lambda/2}\Big(K_\lambda\Big(\delta\sqrt{\alpha^2-\beta^2}\Big)\Big)^{-1}\mathrm{e}^{-\beta\mu},$$

and the function K_ν denotes the modified Bessel function of the third order of index ν. The parameters α, β, λ, μ and δ satisfy $0 \leqslant |\beta| < \alpha$, $\lambda, \mu \in \mathbb{R}$ and $\delta > 0$.

The cases $\lambda = -\frac{1}{2}$ and $\lambda = 1$ correspond to the NIG$(\alpha, \beta, \mu, \delta)$ and hyperbolic distributions, respectively. The other two cases of interest are $\lambda = 0$ (the hyperboloid distribution) and $\lambda = \frac{1}{2}$. Moreover, letting $\delta \to 0$ yields the VG$(\lambda,\, \alpha,\, \beta,\, \mu)$ distribution. See Raible (2000) for details, as well as more properties of the GH distributions.

Yet only the expression for the Lévy density of the GH process and its behaviour near the origin are of interest to us. The results of Prause (1999: Theorem 1.43) (for the case $\lambda \geqslant 0$) and Wiesendorfer Zahn (1999: Anhang C) (for general λ) give the density of the Lévy measure L as

$$\frac{\mathrm{d}L}{\mathrm{d}x} = \frac{\mathrm{e}^{\beta x}}{|x|}\left\{\int_0^\infty \frac{\exp\Big\{-\sqrt{2y+\alpha^2}|x|\Big\}}{\pi^2 y\Big(J^2_{|\lambda|}(\delta\sqrt{2y}) + Y^2_{|\lambda|}(\delta\sqrt{2y})\Big)}\,\mathrm{d}y + \lambda \mathrm{e}^{-\alpha|x|}\mathbf{1}_{\{\lambda\geqslant 0\}}\right\},$$

where J_ν and Y_ν are, respectively, the Bessel functions of the first and second kind of order ν (for the definitions and properties, see Olver 1970). Furthermore, Raible (2000: Proposition 2.18) gives the asymptotics of this density:

$$\frac{\mathrm{d}L}{\mathrm{d}y} = \frac{1}{y^2}\Big(\frac{\delta}{\pi} + \frac{\lambda+1/2}{2}|y| + \frac{\delta\beta}{\pi}y + o(|y|)\Big), \qquad \text{as } y \to 0.$$

This implies that

$$\int_{0<|y|\leqslant 1} |y|^p \left(\frac{\mathrm{d}L}{\mathrm{d}y}\right) \mathrm{d}y < \infty, \text{ if and only if } p > 1,$$

and so, as for the NIG process, we have $\beta_X = 1$.

3.4. The GZ processes

The generalized z-processes defined by Grigelionis (2001) provide an alternative class of processes well suited to modelling financial data. They generalize z-processes considered by Prentice (1975) and include the Meixner process investigated by Schoutens (2002).

A generalized z-distribution $\mathrm{GZD}(\alpha, \beta_1, \beta_2, \delta, \mu)$ on $\mathbb{R}$ is specified by its characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\xi X_1} = \left(\frac{B(\beta_1 + \mathrm{i}\alpha/(2\pi), \beta_2 - \mathrm{i}\alpha/(2\pi))}{B(\beta_1, \beta_2)}\right)^{2\delta} \mathrm{e}^{\mathrm{i}\mu\xi}, \qquad \xi \in \mathbb{R}, \delta > 0,$$

where $\alpha > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\mu \in \mathbb{R}$, and $B(\beta_1, \beta_2)$ denotes the Euler beta function. In particular, if $\delta = \frac{1}{2}$, we have a z-distribution $\mathrm{ZD}(\alpha, \beta_1, \beta_2, \mu)$, and if $\beta_1 = 1/2 + \beta/(2\pi)$, $\beta_2 = 1/2 - \beta/(2\pi)$ with $\beta \in (-\pi, \pi)$, then such a GZD distribution is a Meixner distribution $\mathrm{MD}(\alpha, \beta, \mu, \delta)$ (Grigelionis 2001: Proposition 4). Furthermore, the characteristic triplet of a GZD distribution is $(b, 0, L)$, where, by Grigelionis (2001: Proposition 1),

$$b = \mu + \frac{\alpha\delta}{\pi} \int_0^{2\pi/\alpha} \frac{\mathrm{e}^{-\beta_2 x} - \mathrm{e}^{-\beta_1 x}}{1 - \mathrm{e}^{-x}} \mathrm{d}x,$$

and the Lévy measure L has a density with respect to the Lebesgue measure on $\mathbb{R}\backslash\{0\}$ given by

$$\frac{\mathrm{d}L}{\mathrm{d}x} = \frac{2\delta}{|x|(1 - \mathrm{e}^{-2\pi|x|/\alpha})} \exp\left\{-\frac{2\pi}{\alpha}\left(\frac{\beta_2 + \beta_1}{2}|x| + \frac{\beta_2 - \beta_1}{2}x\right)\right\}.$$

The latter formula immediately yields

$$\frac{\mathrm{d}L}{\mathrm{d}x} \sim \frac{\delta\alpha}{\pi}|x|^{-2}, \qquad \text{as } x \to 0,$$

and so, as with the GH and NIG processes, the upper index $\beta_X = 1$.

4. Proof of Theorem 2.1

Due to Remark 2.2, we only need to establish $\gamma_G \geqslant 2 - 1/\beta_X$ whenever $\beta''_X = \beta_X > 1$. Since $\phi_G(\xi_1, \xi_2) = -\mathrm{i}\xi_1 + \phi_X(\xi_2)$, we can write

$$H(\xi_1, \xi_2) := \Re\left(\frac{1}{1 + \phi_Y(\xi_1, \xi_2)}\right) = \frac{1 + \Re\phi_X(\xi_2)}{(1 + \Re\phi_X(\xi_2))^2 + (\xi_1 - \Im\phi_X(\xi_2))^2}.$$

Using (2.7), it is enough to show that

$$I(r) := \int_{\xi_1^2+\xi_2^2 \geqslant 1} \frac{H(\xi_1, \xi_2)}{(\xi_1^2 + \xi_2^2)^{1-r/2}} \, d\xi_1 \, d\xi_2 < +\infty, \tag{4.1}$$

for any $1 < r < 2 - 1/\beta_X$. For such r we can always find an $\varepsilon \in (0, \beta_X - 1)$ so that $r < 1 + (q-1)/p = (2\beta_X - 1)/(\beta_X + \varepsilon) < 2 - 1/\beta_X$, where $p = \beta_X + \varepsilon$ and $q = \beta_X - \varepsilon$.

From Blumenthal and Getoor (1961: Lemma 3.1) we obtain

$$\Re\phi_X(\xi_2) \leqslant C_1(1 \vee |\xi_2|^p) \quad \text{and} \quad |\Im\phi_X(\xi_2)| \leqslant C_1(1 \vee |\xi_2|^p), \tag{4.2}$$

for some constant $C_1 = C_1(p) > 0$. Without loss of generality, we can assume that $C_1 \geqslant 1$.

The definition of β'' in (2.5) yields the existence of some constant $C_2 = C_2(q) > 0$ such that

$$\Re\phi_X(\xi_2) \geqslant C_2|\xi_2|^q, \qquad \text{for all } |\xi_2| \geqslant 1.$$

Now split the compliment of the unit disc in $\mathbb{R}^2$ into four disjoint sets as follows:

$$\mathbb{R}^2 \backslash \{\|u\| \leqslant 1\} = A_1 \cup A_2 \cup A_3 \cup A_4,$$

where, using C_1 from (4.2) we set (see Figure 1)

$$A_1 = \{(u_1, u_2) : |u_1| \geqslant 2C_1(1 \vee |u_2|^p)\},$$

$$A_2 = \{(u_1, u_2) : |u_2| \geqslant 1, |u_1| < 2C_1|u_2|\},$$

$$A_3 = \{(u_1, u_2) : |u_2| > 1, 2C_1|u_2| \leqslant |u_1| < 2C_1|u_2|^p\},$$

$$A_4 = (\mathbb{R}^2 \backslash \{\|u\| \leqslant 1\}) \backslash (A_1 \cup A_2 \cup A_3).$$

On the set A_1 we have $|\xi_1| - |\Im\phi_X(\xi_2)| \geqslant |\xi_1|/2$, so

$$H(\xi_1, \xi_2) \leqslant 4\frac{1 + \Re\phi_X(\xi_2)}{\xi_1^2} \leqslant 8C_1\frac{1 \vee |\xi_2|^p}{\xi_1^2}, \quad \text{and} \quad (\xi_1^2 + \xi_2^2)^{r/2-1} \leqslant |\xi_1|^{r-2}.$$

On A_2 we will use

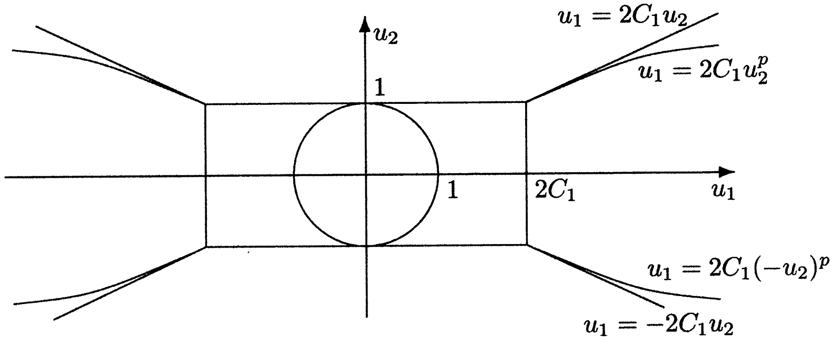

Figure 1. Subdivision of integration plane.

$$H(\xi_1, \xi_2) \leqslant \frac{1}{1 + \Re\phi_X(\xi_2)} \leqslant \frac{1}{C_2|\xi_2|^q}, \quad \text{and} \quad (\xi_1^2 + \xi_2^2)^{r/2-1} \leqslant |\xi_2|^{r-2}.$$

On A_3 we will use the same bound as on A_2 for $H(\xi_1, \xi_2)$ and the bound used on the set A_1 for $(\xi_1^2 + \xi_2^2)^{r/2-1}$. Finally, on A_4 the function $H(\xi_1, \xi_2)(\xi_1^2 + \xi_2^2)^{r/2-1}$ is bounded and continuous, so the integral of this function over A_4 will be a finite constant no matter what r is chosen.

Now we can bound the integral in (4.1) as follows:

$$I(r) = \left(\int_{A_1} + \int_{A_2} + \int_{A_3} + \int_{A_4}\right) \frac{H(\xi_1, \xi_2)}{(\xi_1^2 + \xi_2^2)^{r/2-1}} \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2 = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4,$$

where, using symmetry with respect to ξ_1,

$$\mathbf{I}_1 \leqslant 16C_1 \int_{2C_1}^{\infty} \frac{\mathrm{d}\xi_1}{\xi_1^{4-r}} \int_{|\xi_2|^p \leqslant \xi_1/(2C_1)} (|\xi_2|^p \vee 1) \mathrm{d}\xi_2$$

$$\leqslant \frac{32C_1}{p+1} \int_{2C_1}^{\infty} \left(p + \frac{\xi_1^{1+1/p}}{(2C_1)^{1+1/p}}\right) \frac{\mathrm{d}\xi_1}{\xi_1^{4-r}} < \infty,$$

provided $r < 2 - 1/p$.

Similarly, if $r < q$, we obtain

$$\mathbf{I}_2 \leqslant \frac{4}{C_2} \int_1^{\infty} \frac{1}{\xi_2^{q+2-r}} \left(\int_0^{2C_1\xi_2} \mathrm{d}\xi_1\right) \mathrm{d}\xi_2 \leqslant \frac{8C_1}{C_2} \int_1^{\infty} \xi_2^{r-q-1} \, \mathrm{d}\xi_2 < \infty.$$

The third integral is bounded as follows:

$$\mathbf{I}_3 \leqslant \frac{4}{C_2} \int_1^{\infty} \left(\int_{2C_1\xi_2}^{2C_1\xi_2^p} \frac{\mathrm{d}\xi_1}{\xi_1^{2-r}}\right) \frac{\mathrm{d}\xi_2}{\xi_2^q} \leqslant \frac{4(2C_1)^{r-1}}{C_2(r-1)} \int_1^{\infty} \xi_2^{-q+p(r-1)} \, \mathrm{d}\xi_2 < \infty,$$

if $1 < r < 1 + (q-1)/p$.

Combining the bounds for $\mathbf{I}_i$, $i = 1, 2, 3$, and recalling that $\mathbf{I}_4 < \infty$ for any r, we obtain $I(r) < +\infty$ as long as

$$r < \min\{q, 2 - 1/p, 1 + (q-1)/p\} = 1 + (q-1)/p.$$

This implies that $\gamma_G \geqslant 1 + (q-1)/p$. Increasing r to $2 - 1/\beta_X$ yields the desired inequality $\gamma_G \geqslant 2 - 1/\beta_X$ and completes the proof of Theorem 2.1.

5. Discussion

To complement Theorem 2.1 it is important to look at the case $\beta_X'' < \beta_X$. If $d \geqslant 2$, then γ_G can be anywhere in the interval $[2 - 1/\beta_X, \beta_X]$. Indeed, following Pruitt and Taylor (1969), consider a Lévy process X with stable components, $X = (X_{\alpha_1}, X_{\alpha_2})$, where X_{α_1} is a real-valued α_1-stable process and X_{α_2} is an α_2-stable Lévy process in $\mathbb{R}^{d-1}$, independent of X_{α_1}. If we take $\alpha_2 < \alpha_1$ and $\alpha_1 > 1$, then it is easy to see that $\alpha_1 = \beta_X$, $\alpha_2 = \beta_X''$. Moreover, by

Pruitt and Taylor (1969: Theorem 8.1), $\gamma_G = (1 + \alpha_2(1 - 1/\alpha_1)) \vee (2 - 1/\alpha_1)$. So, varying $\alpha_2 \in [1, \alpha_1]$, we can achieve any value of γ_G in $[2 - 1/\alpha_1, \alpha_1]$.

On the other hand, if $d = 1$, then processes with $1 \vee \beta_X'' < \beta_X$ are not hard to find. Following an idea due to Orey (1968), consider a real-valued Lévy process X with the triplet $(0, 0, L)$ where the Lévy measure L is defined by

$$L(\mathrm{d}x) = \sum_{n=1}^{\infty} a_n^{-\alpha_1}(\delta_{a_n}(\mathrm{d}x) + \delta_{-a_n}(\mathrm{d}x)) + |x|^{-1-\alpha_2}\,\mathrm{d}x,$$

where $0 < \alpha_2 < \alpha_1 < 2$, $\alpha_1 > 1$, and $a_n = 2^{-c^n}$ for some integer $c > 2/(2 - \alpha_1)$. Then it is easy to check that $\beta_X = \alpha_1$ and $\beta_X'' = \alpha_2$. The latter follows from the fact that if $\phi_X(z) = \phi_1(z) + C_{\alpha_2}|z|^{\alpha_2}$, where $C_{\alpha_2} > 0$ and

$$\phi_1(z) = 2\sum_{n=1}^{\infty}(1 - \cos(za_n))a_n^{-\alpha_1},$$

then $\phi_1(z_k) \to 0$ along the sequence $z_k = 2\pi a_k^{-1}$ which converges to infinity as $k \to \infty$. Unfortunately, we were unable to compute γ_G for such a process or show that the whole range $[1, 2 - 1/\beta_X]$ can be attained by γ_G.

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