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Asymptotic error rates in third-generation wireless systems

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The introduction of the so-called third-generation wireless communication system, also known as UMTS or IMT-2000, is a large-scale revolution in telecommunications. It uses a technique called code division multiple access (CDMA). An advanced algorithm to improve the performance of such a CDMA system is called hard-decision parallel interface cancellation and was studied by van der Hofstad and Klok for a rather basic model. We extend many of their results to a more realistic model, where different users transmit at different powers and where additive noise is present.

Keywords: code division multiple access; exponential rate; hard decision parallel interference cancellation; large-deviation theory

1. Introduction

The third-generation (3G) wireless communication system is currently being rolled out in Europe. In Japan the 3G system was launched in 2001, while in the USA the process has been delayed by the mobile providers until 2006 at the earliest. The 3G system is based on a digital broadband technique called code division multiple access (CDMA). In CDMA, k users transmit data across a channel simultaneously. In order to do so, each user multiplies his binary data signal by an individual coding sequence. At the receiver, the signal of the mth ($0 \leqslant m \leqslant k-1$) user can be retrieved by taking the inner product of the transformed total signal and the mth coding sequence.

When the coding sequences are orthogonal, all data that does not originate from the mth user will be annihilated. However, in practice, there will always be interference from the other users. Therefore, one tries to find techniques to get rid of this interference. The best-known technique is a maximum likelihood estimator, introduced in Verdú (1986), which obtains jointly optimal decisions for all users. Unfortunately, this technique is of such high complexity that it cannot be performed in real time. A more straightforward technique is called *interference cancellation* (see Prasad *et al.* 2000, Chapter 4) and the references therein. The idea is that we try to cancel the interference due to the other users. Interference cancellation, especially hard-decision parallel interference cancellation (HD-PIC), is the most promising and the most practical technique for base-station receivers. In this paper we will focus on the HD-PIC system. Another practical and promising technique is soft-decision PIC (SD-PIC), which is studied in van der Hofstad *et al.* (1999).

In van der Hofstad and Klok (2004) the HD-PIC system for a simple model is

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investigated. In this model it is assumed that all users are equivalent, in the sense that they have the same characteristics. Furthermore, it is assumed that the only noise in the system comes from the users themselves, i.e., the received signal is only disturbed by other users in the system, *not* by external sources. The probability that a bit is received incorrectly indicates the quality of the system. Van der Hofstad and Klok focus on the the decay of the bit error probability as the bandwidth tends to infinity. It turns out that this decay is exponential with a certain exponential rate. More precisely, as the bandwidth tends to infinity, we have that minus the logarithm of the bit error probability equals the bandwidth multiplied by a (strictly positive) exponential rate plus lower-order terms.

We will next describe the results obtained in van der Hofstad and Klok (2004). It has been shown that when we do *not* apply interference cancellation, the exponential rate, denoted by $H_k^{(1)}$, equals

$$H_k^{(1)} = \frac{k-2}{2}\log\left(\frac{k-2}{k-1}\right) + \frac{k}{2}\log\left(\frac{k}{k-1}\right),$$

$$H_k^{(1)} = \frac{1}{2k} + \mathcal{O}\left(\frac{1}{k^2}\right), \qquad \text{for } k \to \infty. \tag{1}$$

For a system where HD-PIC is applied, van der Hofstad and Klok (2004) proved that the rate, denoted by $H_k^{(2)}$, is given by

$$H_k^{(2)} = \frac{1}{2\sqrt{k}}\left(1 + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right)\right).$$

This increase from $1/(2k)$ to $1/(2\sqrt{k})$ is impressive and implies a significant improvement in performance and efficiency.

In this paper we will extend these results to more general cases. Rather than assuming that all users are equivalent, we assume that the signals of different users arrive with different powers. This is much more realistic, since it incorporates the effect that when a transmitter is far away from the base station, its signal will be received with a smaller power than the signal of a user close to the base station. We denote the powers by $P_0, P_1, \ldots, P_{k-1}$. We further abbreviate $P = \sum_{j=0}^{k-1} P_j$. When we investigate the quality of the system with respect to user 0, the equivalent of $k \to \infty$ turns out to be $P/P_0 \to \infty$, i.e., the relative total power tends to infinity. In this situation we are able to prove that

$$H_k^{(1)} = \frac{P_0}{2P} + \mathcal{O}\left(\frac{P_0^2}{P^2}\right) \quad \text{and} \quad H_k^{(2)} = \frac{1}{2}\sqrt{\frac{P_0}{P}} + \mathcal{O}\left(\frac{P_0}{P}\right). \tag{2}$$

This result is very similar to the results obtained above, where k is replaced by P/P_0. We should see P_0/P as a signal-to-noise ratio.

Another important extension addressed in this paper is the inclusion of noise from other (unknown) sources. In practice, it is inevitable that undesirable noise disturbs the system. For example, transmission of data from users to another base station often results in many weak signals, interfering with the signals of the base station of interest. This noise is modelled as a white noise process with intensity σ^2, and is often called additive white Gaussian noise (AWGN). When we take the AWGN into account, we obtain

$$H_k^{(1)} = \frac{P_0}{2(P+\sigma^2)} + \mathcal{O}\left(\frac{P_0^2}{(P+\sigma^2)^2}\right).$$

However, expecting that $H_k^{(2)}$ would asymptotically equal $\frac{1}{2}\sqrt{P_0/(P+\sigma^2)}$ would be too optimistic, since the AWGN is *not* cancelled. The result is split into two cases, depending on whether the noise is dominant or not. In the first scenario, the AWGN is dominant, i.e., σ^2 is large. Then, we show that

$$H_k^{(2)} = \frac{P_0}{2\sigma^2} + \mathcal{O}\left(\frac{P_0^2}{\sigma^4}\right).$$

When σ^2 is sufficiently small, different behaviour can be observed. When the powers obey a certain technical condition (see Section 3 below), we obtain

$$H_k^{(2)} = \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right).$$

The factor $P_0/(P+\sigma^2)$ is a signal-to-noise ratio, just as P_0/P was in (2). The term $\sigma^2/(8(P+\sigma^2))$ is a correction term, due to the fact that the AWGN is not cancelled. When the powers do not obey the technical condition, it is not possible to derive the asymptotic form of the rate. However, we *are* able to derive a lower bound on the rate:

$$H_k^{(2)} \geqslant \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right).$$

The results in this paper allow investigation of various realistic scenarios. For example, the case where one user walks away from the base station, so that the power decreases, can be investigated.

The remainder of this paper is organized as follows. In the next section we will describe the CDMA model. In Section 3 the results are described in more detail and some examples are given. Section 4 deals with the asymptotic behaviour of $H_k^{(1)}$. In Section 5 the necessary preparations are made to enable us to prove the results for $H_k^{(2)}$. Finally, in Section 6 the asymptotic behaviour for $H_k^{(2)}$ is proven.

2. Model description

We consider a system with k users, transmitting binary data. The mth user transmits data $b_m = (\ldots, b_{m,-1}, b_{m0}, b_{m1}, \ldots) \in \{-1, +1\}^{\mathbb{Z}}$. The data signal $b_m(t)$ of the mth user is now defined as $b_m(t) = b_{m\,,t/T}$, for $0 \leqslant m \leqslant k-1$, where for $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer larger than or equal to x. The variable T represents the bit duration, i.e., it is the time used to transmit one bit. For each m, $0 \leqslant m \leqslant k-1$, we have a coding sequence $a_m = (\ldots, a_{m,-1}, a_{m0}, a_{m1}, \ldots) \in \{-1, +1\}^{\mathbb{Z}}$ and we put $a_m(t) = a_{m\,,t/T_c}$, where $T_c = T/n$, for some integer n. The variable T_c is often called the chip duration. The variable n is the actual number of transmitted bits, and plays a key role throughout this paper. In practice, the value of n ranges from 30 to 180.

The transmitted coded signal of the mth user is then

$$s_m(t) = (2P_m)^{1/2}\, b_m(t) a_m(t) \cos(\omega_c t), \qquad 0 \leqslant m \leqslant k-1, \tag{3}$$

where P_m is the power of the mth user and ω_c the carrier frequency. We allow the powers to be different, but we assume that the powers change slowly, so that within one bit-period, the power is constant. This is known as coarse power control. The factor $\cos(\omega_c t)$ can be understood as follows. It is not desirable to transmit signals in a frequency band around zero. Multiplying with $\cos(\omega_c t)$ results in shifting the frequency to ω_c and $-\omega_c$. Thus, for the transmission of $s_m(t)$ a frequency band $(-\omega_c - \Delta, -\omega_c + \Delta) \cup (\omega_c - \Delta, \omega_c + \Delta)$ is reserved, where the width Δ depends mainly on n. This holds for every m, so that indeed all users transmit at the same frequency band.

The code $a_m(t)$ is known to the transmitter (e.g., the mobile phone of the transmitting person), the base stations and the receiver (e.g., the mobile phone of the receiving person). When a mobile phone user wants a connection, the base station keeps contact with both mobile phones. The actual conversation can start as soon as the base station has verified that both mobiles use the correct codes.

The total received signal is given by

$$r(t) = \sum_{j=0}^{k-1} s_j(t) + \eta\, n(t), \tag{4}$$

where $n(t)$ is a white noise process, i.e., it is the derivative of Brownian motion in the distribution sense, and $\eta \geqslant 0$. The white noise represents the noisy channels of the users and all interference of other sources that are not yet taken into account. Therefore, η may depend on k. In practice the signals do not need to be synchronized, i.e., it is not necessary that all users transmit using the same time grid. However, for technical reasons we assume that they do.

To retrieve the data bit b_{m1}, the signal $r(t)$ is multiplied by $a_m(t)\cos\omega_c t$ and then averaged over $[0, T]$. In practice $\omega_c T_c$ is large. For simplicity, we pick $\omega_c T_c = \pi f_c$, where $f_c \in \mathbb{N}$, to obtain

$$\frac{1}{T}\int_0^T r(t) a_m(t) \cos(\omega_c t)\,\mathrm{d}t = \left(\frac{P_m}{2}\right)^{1/2} b_{m1} + \sum_{\substack{j=0\\ j\neq m}}^{k-1} \left(\frac{P_j}{2}\right)^{1/2} b_{j1}\, \frac{1}{n}\sum_{i=1}^{n} a_{ji} a_{mi} + \frac{1}{n}\sum_{i=1}^{n} a_{mi} \frac{\eta}{\sqrt{2T_c}} N_i, \tag{5}$$

where the N_i are independent standard normal variables. As will be seen from (5), the decoded signal consists of the desired bit, interference due to the other users, and AWGN. In the ideal situation the vectors $(a_{m1}, \ldots, a_{m,n})$ and $(a_{j1}, \ldots, a_{j,n})$, $j \neq m$, would be orthogonal, so that $\sum_{i=1}^{n} a_{ji} a_{mi} = 0$. However, more efficiency can be achieved when non-orthogonal codes are allowed. In practice, the a-sequences are generated by a random number generator. To model the pseudo-random sequence a, let A_{mi}, $0 \leqslant m \leqslant k-1$, $i = 1, 2, \ldots, n$, be an array of independent and identically distributed (i.i.d.) random variables with distribution

$$\mathbb{P}(A_{01} = +1) = \mathbb{P}(A_{01} = -1) = 1/2. \tag{6}$$

Assuming the coding sequences to be random, we model the signal of (5) as

$$\left(\frac{P_m}{2}\right)^{1/2} b_{m1} + \sum_{\substack{j=0 \\ j\neq m}}^{k-1} \left(\frac{P_j}{2}\right)^{1/2} b_{j1}\, \frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi} + \frac{1}{n}\sum_{i=1}^{n} A_{mi}\frac{\eta}{\sqrt{2T_c}} N_i. \tag{7}$$

Note that for each m and j with $m \neq j$, the sequence $A_{ji}A_{mi}$, $i = 1, \ldots, n$, is an i.i.d. sequence with mean 0 and so by the strong law of large numbers $\frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi} \to 0$, almost surely, for $n \to \infty$. This demonstrates the annihilation of the interference of the other users for $n \to \infty$.

An estimator for b_{m1} is given by

$$\hat{b}_{m1}^{(1)} = \mathrm{sgnr}\left\{ \left(\frac{P_m}{2}\right)^{1/2} b_{m1} + \sum_{\substack{j=0 \\ j\neq m}}^{k-1} \left(\frac{P_j}{2}\right)^{1/2} b_{j1}\, \frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi} + \frac{1}{n}\sum_{i=1}^{n} A_{mi}\frac{\eta}{\sqrt{2T_c}} N_i \right\},$$

where, for $x \in \mathbb{R}$, the randomized sign function is defined as

$$\mathrm{sgnr}(x) = \begin{cases} +1, & x > 0, \\ U, & x = 0, \quad \text{with } \mathbb{P}(U=-1) = \mathbb{P}(U=+1) = 1/2. \\ -1, & x < 0. \end{cases} \tag{8}$$

The random variable U is independent of all other random variables in the system and every time we need the sgnr function another independent trial U is performed. The superscript $^{(1)}$ indicates that the estimates above are initial estimates. We will see later how we can obtain better estimates.

In the Introduction, we saw that advanced receivers have been proposed to increase performance. We focus on the hard-decision procedure, which is described below. In this procedure, it is assumed that the powers P_j are known. We estimate the data signal $s_j(t)$ for $t \in [0, T]$ by (recall (3))

$$\hat{s}_j^{(1)}(t) = (2P_j)^{1/2}\, \hat{b}_{j1}^{(1)}\, a_j(t)\cos(\omega_c t).$$

Then we estimate the total interference for the mth user in $r(t)$ due to the other users by (recall (4))

$$\hat{r}_m^{(1)}(t) = \sum_{\substack{j=0 \\ j\neq m}}^{k-1} \hat{s}_j^{(1)}(t).$$

We use the above to find a better estimate of the data bit b_{m1}:

$$\hat{b}^{(2)}_{m1} = \operatorname{sgnr}\left\{\frac{1}{\mathrm{T}}\int_0^T (r(t) - \hat{r}^{(1)}_m(t))a_m(t)\cos(\omega_c t)\mathrm{d}t\right\}$$

$$= \operatorname{sgnr}\left\{\left(\frac{P_m}{2}\right)^{1/2} b_{m1} + \sum_{\substack{j=0\\ j\neq m}}^{k-1}\left(\frac{P_j}{2}\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi}\right)\left(b_{j1} - \hat{b}^{(1)}_{j1}\right) + \frac{1}{n}\sum_{i=1}^{n} A_{mi}\frac{\eta}{\sqrt{2T_c}}N_i\right\}. \tag{9}$$

We are now interested in $\mathbb{P}(\hat{b}^{(2)}_{m1} \neq b_{m1})$, which is the probability of a bit error after one stage of interference cancellation. We will see that this probability is indeed smaller than $\mathbb{P}(b^{(1)}_{m1} \neq b_{m1})$, the probability of a bit error without interference cancellation.

2.1. Reformulation of the problem

It is important to observe that as n tends to ∞, the interference due to other users will diminish. However, since T is fixed and $\operatorname{var}(n^{-1}\sum A_{mi}(\eta/\sqrt{2T_c})N_i) = \eta^2/(2nT_c) = \eta^2/(2T)$, the AWGN does not vanish. More mathematically,

$$\operatorname{var}\left\{\left(\frac{P_m}{2}\right)^{1/2} + \sum_{\substack{j=0\\ j\neq m}}^{k-1}\left(\frac{P_j}{2}\right)^{1/2} b_{j1}\frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi}\right\} = \mathcal{O}(1/n),$$

$$\text{while } \operatorname{var}\left(\frac{1}{n}\sum A_{mi}\frac{\eta}{\sqrt{2T_c}}N_i\right) = \mathcal{O}(1). \tag{10}$$

In practice, the powers are always adjusted in such a way that the AWGN is not dominant. In our model, we replace P_m by nP_m, so that the variance in (10) is also $\mathcal{O}(1)$. We further introduce $\sigma^2 = \eta^2/T_c$, which is fixed. Together with $b^2_{m1} = 1$, the signal in (7) becomes

$$\left(\frac{n}{2}\right)^{1/2} b_{m1}\left(P_m^{1/2} + \sum_{\substack{j=0\\ j\neq m}}^{k-1} P_j^{1/2}\frac{1}{n}\sum_{i=1}^{n} b_{j1}A_{ji}b_{m1}A_{mi} + \frac{1}{n}\sum_{i=1}^{n} b_{m1}A_{mi}N_i\right),$$

where (with a slight abuse of notation) $(N_i)_{i=1}^n$ are i.i.d. with $N_i \sim \mathcal{N}(0, \sigma^2)$. We emphasize that the power adjustment condition above is realistic, especially in a model where interference cancellation is applied.

Since $b_{j1}A_{ji} \stackrel{d}{=} A_{ji}$ and $N_i \stackrel{d}{=} A_{0i}N_i$, we can therefore write the probability of a bit error more conveniently as

$$\mathbb{P}(\hat{b}^{(1)}_{m1} \neq b_{m1}) = \mathbb{P}\left(\frac{\hat{b}^{(1)}_{m1}}{b_{m1}} \neq 1\right) = \mathbb{P}(\operatorname{sgnr}_m(Z^{(1)}_m) < 0) = \mathbb{P}(Z^{(1)}_m < 0) + \frac{1}{2}\mathbb{P}(Z^{(1)}_m = 0),$$

where $Z^{(1)}_m$, for $0 \leqslant m \leqslant k-1$, is defined as

$$Z_m^{(1)} = P_m^{1/2} + \sum_{\substack{j=0 \\ j\neq m}}^{k-1} P_j^{1/2} \frac{1}{n}\sum_{i=1}^{n} A_{ji}A_{mi} + \frac{1}{n}\sum_{i=1}^{n} A_{mi}A_{0i}N_i.$$

We note that when $\sigma^2 = 0$, the event $\{Z_m^{(1)} = 0\}$ should be taken into account, while for $\sigma^2 > 0$, this events has probability zero, so that $\mathbb{P}(\hat{b}_{m1}^{(1)} \neq b_{m1}) = \mathbb{P}(Z_m^{(1)} \leqslant 0)$.

Without loss of generality, we focus on user 0. Therefore, we prefer to introduce the random variables

$$X_{mi} = A_{0i}A_{mi}, \qquad 1 \leqslant i \leqslant n,\ m = 0, \ldots, k-1.$$

It is straightforward to prove that the matrix $(X_{mi})_{m=1,\ldots,k-1,i=1,\ldots,n}$ has i.i.d. elements and $X_{0i} = A_{0i}^2 = 1$ for all i. We obtain

$$Z_m^{(1)} = P_m^{1/2} + \sum_{\substack{j=0 \\ j\neq m}}^{k-1} P_j^{1/2} \frac{1}{n}\sum_{i=1}^{n} X_{ji}X_{mi} + \frac{1}{n}\sum_{i=1}^{n} X_{mi}N_i. \tag{11}$$

Similarly,

$$Z_m^{(2)} = P_m^{1/2} + \sum_{\substack{j=0 \\ j\neq m}}^{k-1} P_j^{1/2} \frac{1}{n}\sum_{i=1}^{n} X_{ji}X_{mi}(1 - \operatorname{sgnr}(Z_j^{(1)})) + \frac{1}{n}\sum_{i=1}^{n} X_{mi}N_i. \tag{12}$$

We finally define the exponential rates $H_k^{(s)}$, $s = 1, 2$, as follows:

$$H_k^{(s)} = -\lim_{n\to\infty} \frac{1}{n} \log\biggl(\mathbb{P}(Z_m^{(s)} < 0) + \frac{1}{2}\mathbb{P}(Z_m^{(s)} = 0)\biggr). \tag{13}$$

3. Results and examples

3.1. Results

This paper extends the results for the model without interference cancellation and the one-stage HD-PIC model of van der Hofstad and Klok (2004) to the model where unequal powers and AWGN are incorporated. The limit $k \to \infty$ is replaced by $(P + \sigma^2)/P_0 \to \infty$, where $P = \sum_{j=0}^{k-1} P_j$. Note that this is quite general, since we no longer require $k \to \infty$.

We prove that the analogue of (1) is

$$\frac{P_0}{2(P+\sigma^2)} \leqslant H_k^{(1)} \leqslant \frac{P_0}{2(P+\sigma^2)} + \mathcal{O}\biggl(\frac{P_0^2}{(P+\sigma^2)^2}\biggr), \qquad \text{as} \quad \frac{P_0}{P+\sigma^2} \to 0. \tag{14}$$

We see that the result holds as long as the ratio between the mean $(P_0^{1/2})$ and standard deviation $((P+\sigma^2)^{1/2})$ of $Z_0^{(1)}$ tends to 0. The results are obtained from standard large-deviation theory together with a Taylor expansion of the moment generating function. For the HD-PIC model with equal powers and no AWGN, investigated in van der Hofstad and Klok (2004), a parameter r was introduced to denote the number of errors in the first stage. It was shown that typically $r = \frac{1}{2}\sqrt{k}$ errors are made in the first stage, where 'typically' means that bit errors caused by different r are a tiny fraction of the total bit error probability for large n. For the general model, the analogue of r is $R \subseteq \{1, \ldots, k-1\}$, the set of users with a wrongly estimated bit. We further define $P_R = \sum_{j \in R} P_j$ and

$$\rho = \frac{1}{2}\sqrt{\frac{P+\sigma^2}{P_0} - \frac{\sigma^2}{4P_0}}.$$

As seen below, the relevant parameter is not R itself – it turns out that only P_R is relevant. We will prove that when $P_R = \rho\, P_0$, the probability of a bit error in the second stage is dominant, i.e., $P_R = \rho\, P_0$ is typically observed.

The main result of the paper is the following theorem, which is proven in Section 6.

Theorem 3.1. *Assume that* $P_0/(P+\sigma^2) \to 0$.
(i) *If* $\rho \geqslant 0$, *then*

$$H_k^{(2)} \geqslant \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2} - \frac{\sigma^2}{8(P+\sigma^2)}} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right).$$

Moreover, when

$$\min_{R \subseteq \{1,\ldots,k-1\}} \left| \frac{P_R}{P_0} - \rho \right| = \mathcal{O}\left(\frac{P+\sigma^2}{P_0}\right)^{1/4}, \qquad (*)$$

equality is attained:

$$H_k^{(2)} = \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2} - \frac{\sigma^2}{8(P+\sigma^2)}} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right).$$

(ii) *If* $\rho \leqslant 0$, *then*

$$\frac{P_0}{\sigma^2} \to 0 \quad \textit{and} \quad H_k^{(2)} = \frac{P_0}{2\sigma^2} + \mathcal{O}\left(\frac{P_0^2}{\sigma^4}\right).$$

In the case of equal powers, it is straightforward that ρ can be approximated with an error of at most 1/2. For the general case, where we do not have control over the powers, $(*)$ describes the condition under which ρ can be approximated closely enough. Due to the increased variety in possible scenarios for $P_0, \ldots, P_{k-1}$ and σ^2, it is hard to find a non-technical condition that is equivalent to $(*)$. Therefore, the results are illustrated by examples of typical scenarios in Section 3.2 below.

When we compare Theorem 3.1 with the result for $H_k^{(1)}$, given in (14), we conclude that HD-PIC gives a significant increase in performance for both small and large σ^2 (case (i) and (ii), respectively). In case (i), we see that

$$H_k^{(2)} \geqslant \frac{1}{4}\sqrt{\frac{P_0}{P+\sigma^2}} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right),$$

which is a significant improvement over $H_k^{(1)} = P_0/(2(P+\sigma^2)) + \mathcal{O}(P_0^2/(P+\sigma^2)^2)$. In case (ii), we observe that $\rho \leqslant 0$ implies $R = \varnothing$. In other words, typically *all* interference due to other users is cancelled successfully.

3.2. Examples

In this subsection, we will give some practical scenarios. We first observe that an increase in σ^2 should result in a decrease in ρ. Indeed, since $P + \sigma^2 \geqslant P \geqslant P_0$,

$$\frac{\partial \rho}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2}\left(\frac{1}{2}\sqrt{\frac{P+\sigma^2}{P_0}} - \frac{\sigma^2}{4P_0}\right) = \frac{1}{4\sqrt{P_0(P+\sigma^2)}} - \frac{1}{4P_0} \leqslant 0.$$

Therefore, $\rho \leqslant \frac{1}{2}\sqrt{P/P_0}$.

Furthermore, it is straightforward to show that $(P - P_0)/P_0 \geqslant \frac{1}{2}\sqrt{P/P_0}$ when $P_0/(P+\sigma^2)$ is sufficiently small and $\rho \geqslant 0$. This means that there always exists an R such that $|P_R/P_0 - \rho| \leqslant \frac{1}{2}\max_m P_m/P_0$.

The first example is a typical scenario in 3G mobile networks. In order to avoid users transmitting data with a relatively high or low power level, the base station ensures that $M^{-1} \leqslant P_j \leqslant M$ for all $0 \leqslant m \leqslant k-1$, for some $M < \infty$.

Example 3.1. Assume that there exists an $M > 0$ such that uniformly in k, $M^{-1} \leqslant P_j \leqslant M$ for all $0 \leqslant j \leqslant k-1$. Then P_R/P_0 can approximate any value between 0 and $(P - P_0)/P_0$ with an error of at most $M/2$. Thus, once $P_0/(P+\sigma^2)$ is sufficiently small, $(P-P_0)/P_0 \geqslant \frac{1}{2}\sqrt{P/P_0} \geqslant \rho$, and thus ρ can be approximated with an error of at most $M/2$. This implies that $(*)$ is fulfilled.

In Example 3.1, the error $\min|P_R/P_0 - \rho|$ is bounded from above by the quantity $\frac{1}{2}\max_j P_j/P_0$, and it is shown that this upper bound is sufficiently small. In the next proposition, a wider class of power scenarios is given.

Proposition 3.2. *Denote by* $\beta : \{1, \ldots, k-1\} \mapsto \{1, \ldots, k-1\}$ *the function that orders* $P_1, \ldots, P_k$*, e.g.,* $P_{\beta(1)}$ *is the smallest interfering power,* $P_{\beta(k-1)}$ *the largest. Note that* P_0 *is discarded. Then* $(*)$ *is fulfilled if there exists a constant* $0 < C < \infty$ *such that, for all* $m \geqslant 1$*,*

$$\sum_{j=1}^{m-1} \frac{P_{\beta(j)}}{P_0} \geqslant \frac{P_{\beta(m)}}{P_0} - C\left(\frac{P}{P_0}\right)^{1/4}. \tag{$**$}$$

Proof. We prove the proposition by construction. We approximate ρ from below by adding one by one the largest relative powers one by one. Let R denote the obtained set. Take the unique smallest $m = m^*$ such that $\beta(m^*) \notin R$. By construction

$$\sum_{m\in\beta^{-1}(R)} \frac{P_{\beta(m)}}{P_0} \leqslant \rho < \sum_{\substack{m\in\beta^{-1}(R)\\ m>m^*}} \frac{P_{\beta(m)}}{P_0} + \frac{P_{\beta(m^*)}}{P_0} = \sum_{m\in\beta^{-1}(R)} \frac{P_{\beta(m)}}{P_0} + \left(\frac{P_{\beta(m^*)}}{P_0} - \sum_{m=1}^{m^*-1} \frac{P_{\beta(m)}}{P_0}\right).$$

Condition $(**)$ implies that the last term on the right-hand side is sufficiently small. Furthermore, $\sum_{m\in\beta^{-1}(R)} P_{\beta(\mathrm{m})} = \sum_{m\in R} P_m$, so that $(*)$ holds using this R. □

A (somewhat artificial) example is $P_m = 2^{m-1}$. In this case $(**)$ is fulfilled ($C = 2$ suffices), so that $(*)$ is fulfilled by the proposition above. Of course, since every integer can be written as a sum of powers of 2, we knew in advance that ρ could be approximated with an error of at most 1/2. We note that the scenario in Example 3.1 is included in the proposition.

Finally, we treat a practical example, where the user we are interested in walks away from the base station, without changing its transmitted power, i.e., the user of interest has a power tending to zero. This phenomenon is known as the near-far effect. The behaviour of the receiver in the case of a near-far scenario is considered to be extremely relevant, since it characterizes the robustness of the receiver against scenarios where extremely low or high powers cannot be avoided.

Example 3.2. We consider a system where only the desired user is moving. To be more precise, $P_1, \ldots, P_{k-1}$ and σ^2 are fixed and user 0 walks away from the base station, so that $P_0 \to 0$. When $P_0 \to 0$, clearly $P_0/(P+\sigma^2) \to 0$. If σ^2 is small but strictly positive, we are in case (i) of Theorem 3.1 for a while, so that when $(*)$ holds,

$$H_k^{(2)} = \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}}\left(1 + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right)\right).$$

However, it is inevitable that at a certain moment $\rho = 0$. It is easy to see that this happens when $P_0 = \sigma^4/(4(P+\sigma^2))$. When P_0 becomes smaller, we are in case (i) of Theorem 3.1 and thus

$$H_k^{(2)} = \frac{P_0}{2\sigma^2} + \mathcal{O}\left(\frac{P_0}{2\sigma^2}\right) \to 0.$$

When $\sigma^2 = 0$, a different situation occurs. It is possible to derive (but not shown here) that

$$H_k^{(2)} \geqslant \frac{P_{\beta(1)}}{2P} > 0.$$

This indicates that the inclusion of AWGN in the model is extremely relevant.

4. Asymptotic behaviour of $H_k^{(1)}$

In this section we are interested in the behaviour of $\mathbb{P}(Z_0^{(1)} \leqslant 0)$. Since this probability turns out to be exponentially small as $n \to \infty$, we will focus on the exponential rate $H_k^{(1)}$, defined in (13). It turns out that analytical formulae cannot be given. Therefore, we only focus on the asymptotic behaviour when the ratio between desired signal and noise tends to zero, i.e., $P_0/(P+\sigma^2) \to 0$, where $P = \sum_{j=0}^{k-1} P_j$.

For this system, where no interference cancellation is applied, the analysis is not difficult.

However, the proof is set up in such a way that it carries over to the technically more involved proof regarding $H_k^{(2)}$.

Proposition 4.1. *As* $P_0/(P+\sigma^2) \to 0$,

$$\frac{P_0}{2(P+\sigma^2)} \leqslant H_k^{(1)} \leqslant \frac{P_0}{2(P+\sigma^2)} + \mathcal{O}\left(\frac{P_0^2}{(P+\sigma^2)^2}\right).$$

Proof. Here and throughout this paper, C denotes a strictly positive constant that may not depend on P_j and σ^2. The constant C may change from line to line.

We investigate $Z_0^{(1)} = \frac{1}{n}\sum_{i=1}^n (P_0^{1/2} + \sum_{j=1}^{k-1} P_j^{1/2} X_{ji} + N_i)$. Clearly

$$\frac{1}{2}\mathbb{P}(Z_0^{(1)} \leqslant 0) \leqslant \mathbb{P}(Z_0^{(1)} < 0) + \frac{1}{2}\mathbb{P}(Z_0^{(1)} = 0) \leqslant \mathbb{P}(Z_0^{(1)} \leqslant 0),$$

and at an exponential scale the factor $1/2$ vanishes, so that it suffices to investigate $\mathbb{P}(Z_0^{(1)} \leqslant 0)$. According to Cramér's theorem (den Hollander 2000, Theorem I.4),

$$H_k^{(1)} = \sup_{t \leqslant 0}\{-\log h(t)\}, \qquad \text{where } h(t) = \mathbb{E}\exp\left(t\left(P_0^{1/2} + \sum_{j=1}^{k-1} P_j^{1/2} X_{j1} + N_1\right)\right).$$

Since $\mathbb{E}\,\mathrm{e}^{xN_1} = \mathrm{e}^{\sigma^2 x^2/2}$ and $\mathbb{E}\,\mathrm{e}^{yX_{11}} = \cosh y$,

$$-\log h(t) = -tP_0^{1/2} - \sigma^2 t^2/2 - \sum_{j=1}^{k-1} \log\cosh(P_j^{1/2} t). \tag{15}$$

Step 1: Lower bound. A simple comparison of the coefficients in the power series reveals that $\cosh x \leqslant \mathrm{e}^{x^2/2}$. Indeed, since $(2m)! \geqslant 2^m m!$ for all $m \geqslant 0$,

$$\cosh t = \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!} \leqslant \sum_{m=0}^{\infty} \frac{(t^2)^m}{2^m m!} = \mathrm{e}^{t^2/2}. \tag{16}$$

Substitution in (15) leads to a lower bound for the rate:

$$H_k^{(1)} \geqslant \sup_{t \leqslant 0}\left\{-P_0^{1/2} t - t^2\left(\sum_{j=1}^{k-1} P_j + \sigma^2\right)/2\right\} \tag{17}$$

$$\geqslant \left\{-P_0^{1/2} t - t^2\left(\sum_{j=1}^{k-1} P_j + \sigma^2\right)/2\right\}\Bigg|_{t=-P_0^{1/2}/(P+\sigma^2-P_0)} = \frac{P_0}{2(P+\sigma^2-P_0)} \geqslant \frac{P_0}{2(P+\sigma^2)}.$$

Step 2: Upper bound. To obtain an upper bound, we will define an ellipse $\mathcal{E}$ with $0 \in \mathcal{E}^0$, the interior of $\mathcal{E}$. In order to show that the supremum of $-\log h(t)$ (i.e., the infimum of $h(t)$) is attained in $\mathcal{E}^0$, it is sufficient to show that on $\partial\mathcal{E}$, the boundary of the ellipse, $h(t) > 1$. Since $h(0) = 1$ and h is convex, we can then conclude that $h(t) > 1$ outside the ellipse and thus the supremum is never attained there. Indeed, whenever $t \notin \mathcal{E}$, there exists a unique $0 < \alpha < 1$ such that $\alpha t \in \partial\mathcal{E}$. From convexity of h and $h(\alpha t) > 1$ it follows that

$$1 < h(\alpha t) = h(\alpha t + (1-\alpha)\cdot 0) \leqslant \alpha h(t) + (1-\alpha)h(0) = \alpha h(t) + (1-\alpha). \qquad (18)$$

It immediately follows that $h(t) > 1$. Once we are allowed to restrict $t \in \mathcal{E}^0$, we can prove the desired upper bound.

We observe that $e^x \geqslant 1 + x + x^2/2 + x^3/6$ and $e^y \geqslant 1 + y$ to obtain (recall (15))

$$h(t) \geqslant e^{tP_0^{1/2}+t^2\sigma^2/2}\mathbb{E}\left(1 + t\sum_{j=1}^{k-1} P_j^{1/2}X_{j1} + \frac{t^2}{2}\left(\sum_{j=1}^{k-1} P_jX_{j1}\right)^2 + \frac{t^3}{6}\left(\sum_{j=1}^{k-1} P_j^{1/2}X_{ji}\right)^3\right)$$

$$= e^{tP_0^{1/2}+t^2\sigma^2/2}\left(1 + t^2\sum_{j=1}^{k-1} P_j/2\right) \geqslant \left(1 + tP_0^{1/2} + t^2\sigma^2/2)(1 + t^2\sum_{j=1}^{k-1} P_j/2\right)$$

$$= 1 - \frac{P_0}{2(P+\sigma^2-P_0)} + \frac{P+\sigma^2-P_0}{2}(t-t^*)^2 + \frac{P_0^{1/2}(P-P_0)}{2}t^3 + \frac{(P-P_0)\sigma^2}{4}t^4,$$

where $t^* = -P_0^{1/2}/(P+\sigma^2-P_0)$. The terms t^3 and t^4 turn out to be of no importance. With the use of a well-chosen ellipse, we can show that they are indeed error terms. We define the ellipse as

$$\mathcal{E} = \left\{t\colon \frac{P+\sigma^2-P_0}{2}(t-t^*)^2 \leqslant \frac{P_0}{P+\sigma^2-P_0}\right\}.$$

For $t \in \mathcal{E}$, the triangular inequality yields $|t| \leqslant (1+\sqrt{2})P_0^{1/2}/(P+\sigma^2-P_0)$. Therefore, on $\partial\mathcal{E}$, we have

$$\left|\frac{P_0^{1/2}(P-P_0)}{2}t^3 + \frac{(P-P_0)\sigma^2}{4}t^4\right| \leqslant C\frac{P_0^2(P-P_0)}{(P+\sigma^2-P_0)^3} + C\frac{P_0^2(P-P_0)\sigma^2}{(P+\sigma^2-P_0)^4}$$

$$\leqslant C\frac{P_0^2}{(P+\sigma^2-P_0)^2}.$$

We can now conclude that on $\partial\mathcal{E}$ the minimum over $h(t)$ is never attained. Indeed, for $t \in \partial\mathcal{E}$,

$$h(t) \geqslant 1 - \frac{P_0}{2(P+\sigma^2-P_0)} + \frac{P_0}{P+\sigma^2-P_0} - C\frac{P_0^2}{(P+\sigma^2-P_0)^2} > 1,$$

when $P_0/(P+\sigma^2-P_0)$ is sufficiently small.

When we restrict to $t \in \mathcal{E}$, we have

$$h(t) \geqslant 1 - \frac{P_0}{2(P+\sigma^2-P_0)} + \frac{P+\sigma^2-P_0}{2}(t-t^*)^2 - C\frac{P_0^2}{(P+\sigma^2-P_0)^2},$$

and the minimum of the right-hand side is attained at $t = t^*$. This results in

$$h(t) \geqslant 1 - \frac{P_0}{2(P+\sigma^2-P_0)} - C\frac{P_0^2}{(P+\sigma^2-P_0)^2}.$$

The upper bound for the supremum of $-\log h(t)$ is now obtained by observing that

$$\frac{P_0}{P+\sigma^2-P_0}=\frac{P_0}{P+\sigma^2}\left(1+\mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right)\right)$$

and $-\log(1-x+\mathcal{O}(x^2))=x+\mathcal{O}(x^2),\ x\to 0.$ □

5. Preparations for $H_k^{(2)}$

For a system in which interference cancellation is applied, the proofs will become more difficult. In this section, we prepare the main ingredients for the proof of Theorem 3.1. We will follow the outline of the proof of Proposition 4.1 closely. Recall formulae (11) and (12). Clearly $1-\mathrm{sgnr}(\cdot)$ is either 0 or 2. Thus, user m contributes to $Z_0^{(2)}$ if and only if $\mathrm{sgnr}(Z_m^{(1)})<0$. We write

$$\mathbb{P}(Z_0^{(2)}\leqslant 0)=\sum_{R\subseteq\{1,\dots,k-1\}}\mathbb{P}(Z_0^{(2)}\leqslant 0,\ B_R),\qquad \text{where } B_R$$

$$=\left\{\max_{m\in R}\mathrm{sgnr}(Z_m^{(1)})<0,\ \min_{m\in\{1,\dots,k-1\}\setminus R}\mathrm{sgnr}(Z_m^{(1)})>0\right\}.$$

One may verify from (8) that

$$\frac{1}{2}\mathbb{P}(Z_m\leqslant 0,\cdot)\leqslant\mathbb{P}(\mathrm{sgnr}(Z_m)<0,\cdot)\leqslant\mathbb{P}(Z_m\leqslant 0,\cdot)$$

so that

$$2^{1-k}\sum_{R\subseteq\{1,\dots,k-1\}}\mathbb{P}\left(\max_{m\in R}Z_m^{(1)}\leqslant 0,\ \min_{m\in\{1,\dots,k-1\}\setminus R}Z_m^{(1)}\geqslant 0,\ P_0^{1/2}+2\sum_{j\in R}P_j^{1/2}\frac{1}{n}\sum_{i=1}^{n}X_{ji}\leqslant 0\right)$$

$$\leqslant\mathbb{P}(Z_0^{(2)}\leqslant 0)\tag{19}$$

$$\leqslant\sum_{R\subseteq\{1,\dots,k-1\}}\mathbb{P}\left(\max_{m\in R}Z_m^{(1)}\leqslant 0,\ \min_{m\in\{1,\dots,k-1\}\setminus R}Z_m^{(1)}\geqslant 0,\ P_0^{1/2}+2\sum_{j\in R}P_j^{1/2}\frac{1}{n}\sum_{i=1}^{n}X_{ji}\leqslant 0\right).$$

Subsequently, we will denote $\overline{Z}_0^{(2)}=P_0^{1/2}+2\sum_{j\in R}P_j^{1/2}n^{-1}\sum_{i=1}^{n}X_{ji}$. The bar denotes that we have knowledge of stage 1 and we have inserted the correct values of the sgnr functions.

We next apply the 'largest-exponent-wins' principle – the probability with the smallest exponential rate will dominate all others (den Hollander 2000, equation (I.2)) – to the bounds in (19) and find (the factor 2^{1-k} vanishes)

$$H_k^{(2)}=-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(Z_0^{(2)}\leqslant 0)=\min_{R\subseteq\{1,\dots,k-1\}}H_{k,R}^{(2)},\tag{20}$$

where

$$H_{k,R}^{(2)}=-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\max_{m\in R}Z_m^{(1)}\leqslant 0,\ \min_{m\in\{1,\dots,k-1\}\setminus R}Z_m^{(1)}\geqslant 0,\ \overline{Z}_0^{(2)}\leqslant 0\right).$$

Existence of $H^{(2)}_{k,R}$ follows from Cramér's theorem.

For $A \subseteq \{0, \ldots, k-1\}$ we abbreviate $P_A = \sum_{j \in A} P_j$ and $P = \sum j = 0^{k-1} P_j$. We define, for $R \subseteq \{1, \ldots, k-1\}$,

$$\mathcal{H} = \frac{P_0}{2(4P_R + \sigma^2)} + \frac{P_R}{2(P + \sigma^2)}.$$

Theorem 5.1. *For* $\mathcal{H} \to 0$,

$$H^{(2)}_{k,R} = \mathcal{H}(1 + \mathcal{O}(\mathcal{H})).$$

Proof. For $A \in \mathbb{N} \cup \{0\}$ and $(t_{10}, t_{11}, \ldots) \in \mathbb{R}^{\mathbb{N}}$, we introduce $S_A = \sum_{m \in A} P_m^{1/2} X_{m1}$ and $T_A = \sum_{m \in A} t_{1m} X_{m1}$. It is useful to observe that

$$\mathbb{E}\, T_A^2 = \sum_{m \in A} t_{1m}^2 \quad \text{and} \quad \mathbb{E}\, T_A^m \leqslant c_m \left(\sum_{m \in A} t_{1m}^2 \right)^{m/2}, \tag{21}$$

for some c_m not depending on A. Substituting $t_{1m} = P_m^{1/2}$ leads to bounds of moments of S_R.

Similar to the situation with equal powers, we write $R_0 = \{0, \ldots, k-1\}$ and $R^c = R_0 \backslash R$. Further, we define $R^* = \{1, \ldots, k-1\} \backslash R$, which is in fact an abbreviation for $R^c \backslash \{0\}$.

Step 1: Lower bound. Since for any events A, B, $\mathbb{P}(A \cap B) \leqslant \mathbb{P}(A)$, we can discard the event $\{\min_{m \in R^*} Z_m^{(1)} \geqslant 0\}$. This results in

$$H^{(2)}_{k,R} \geqslant -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\max_{m \in R} Z_m^{(1)} \leqslant 0,\ \overline{Z}_0^{(2)} \leqslant 0 \right) = -\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} \underline{Y}_i \leqslant \underline{0} \right), \tag{22}$$

where, for $i = 1, \ldots, n$, $\underline{Y}_i$ is a $(|R| + 1)$-dimensional vector with elements

$$Y_{1m,i} = P_m^{1/2} + \sum_{\substack{j=0 \\ j \neq m}}^{k-1} P_j^{1/2} X_{ji} X_{mi} + X_{mi} N_i, \qquad m \in R,$$

$$Y_{20,i} = P_0^{1/2} + 2 \sum_{j \in R} P_j^{1/2} X_{ji} + N_i,$$

and where, for a vector $\underline{x}$, the statement $\underline{x} \leqslant \underline{0}$ implies that each entry of $\underline{x}$ is less than or equal to zero. Since $(\underline{Y}_i)_{i=1}^n$ is i.i.d., we have, according to Cramér's theorem, that the rate on the right-hand side of (22) is given by

$$\sup_{\underline{t} \leqslant \underline{0}} \{-\log h(\underline{t})\}, \qquad \text{where } h(\underline{t}) = \mathbb{E}\, \mathrm{e}^{\langle \underline{t}, \underline{Y}_1 \rangle},$$

with

$$Y_{1m,1} = P_m^{1/2} X_{m1} S_{R_0} + X_{m1} N_1, \qquad m \in R,$$

$$Y_{20,1} = P_0^{1/2} + 2S_R + N_1.$$

We can rewrite the inner product as $\langle \underline{t}, \underline{Y}_1 \rangle = T_R S_{R_0} + t_{20} P_0^{1/2} + 2t_{20} S_R + N_1(T_R + t_{20})$, so that (using the fact that $\mathbb{E}\, \mathrm{e}^{xN_1} = \mathrm{e}^{\sigma^2 x^2/2}$)

$$h(\underline{t}) = \mathbb{E}\, \mathrm{e}^{T_R S_{R_0} + t_{20} P_0^{1/2} + 2t_{20} S_R + \sigma^2 (T_R + t_{20})^2/2}.$$

Similar to the proof of the lower bound of $H_k^{(1)}$, we will make use of the fact that we can substitute any value for t_{1m}. Therefore, we only consider $\underline{t}$ in the point $t_{1m}^* = -P_m^{1/2}/(P + \sigma^2)$ and $t_{20}^* = -P_0^{1/2}/(4P_R + \sigma^2)$. Note that for this particular choice of $\underline{t}$, $T_R = -S_R/(P + \sigma^2)$. We write $h(\underline{t}^*) = \mathbb{E}\, \mathrm{e}^{Y_q + Y_a}$, where

$$Y_q = -\frac{1}{P + \sigma^2} S_R^2 - \frac{P_0}{4P_R + \sigma^2} + \frac{\sigma^2}{2} \left(\frac{S_R}{P + \sigma^2} + \frac{P_0^{1/2}}{4P_R + \sigma^2} \right)^2, \tag{23}$$

$$Y_a = -\frac{1}{P + \sigma^2} S_R S_{R^c} - \frac{2P_0^{1/2}}{4P_R + \sigma^2} S_R. \tag{24}$$

This partition into a quadratic part and an asymmetric part will simplify the proof, because it turns out that the asymptotic behaviour depends only on the first moment of Y_q and the second moment of Y_a.

Using $\mathrm{e}^y = 1 + y + y^2 \mathrm{e}^{\zeta y}/2$ for some $\zeta = \zeta_y \in [0, 1]$ and $\mathrm{e}^x = 1 + x + x^2/2 + x^3/6 + x^4 \mathrm{e}^{\eta x}/24$ for some $\eta = \eta_x \in [0, 1]$, respectively, we write

$$h(\underline{t}) = \mathbb{E}\, \mathrm{e}^{Y_q + Y_a} = 1 + \mathbb{E}\, Y_q + \mathbb{E}\, Y_a^2/2 + e(\underline{t}), \tag{25}$$

where

$$\begin{aligned} e(\underline{t}) &= \mathbb{E}[\mathrm{e}^{Y_q} \mathrm{e}^{Y_a} - 1 - Y_q - Y_a^2/2] = \mathbb{E}[(1 + Y_q + Y_q^2 \mathrm{e}^{\zeta Y_q}/2)\mathrm{e}^{Y_a} - 1 - Y_q - Y_a^2/2] \qquad (26) \\ &= \mathbb{E}[(1 + Y_q)(1 + Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4 \mathrm{e}^{\eta Y_a}/24) + Y_q^2 \mathrm{e}^{\zeta Y_q} \mathrm{e}^{Y_a}/2 - 1 - Y_q - Y_a^2/2] \\ &= \mathbb{E}[Y_a + Y_a^3/6 + Y_a^4 \mathrm{e}^{\eta Y_a}/24 + Y_q(Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4 \mathrm{e}^{\eta Y_a}/24) + Y_q^2 \mathrm{e}^{\zeta Y_q} \mathrm{e}^{Y_a}]. \end{aligned}$$

We use $(x + y)^2 \leqslant 2(x^2 + y^2)$ to obtain

$$\begin{aligned} Y_q &\leqslant -\frac{S_R^2}{P + \sigma^2} - \frac{P_0}{4P_R + \sigma^2} + \sigma^2 \left(\frac{S_R^2}{(P + \sigma^2)^2} + \frac{P_0}{(4P_R + \sigma^2)^2} \right) \\ &= -\frac{S_R^2}{P + \sigma^2} \left(1 - \frac{\sigma^2}{P + \sigma^2} \right) - \frac{P_0}{4P_R + \sigma^2} \left(1 - \frac{\sigma^2}{4P_R + \sigma^2} \right) \leqslant 0 \qquad \text{almost surely.} \end{aligned}$$

Using this, together with $\eta Y_a \leqslant |Y_a|$ and $\mathbb{E}\, Y_a = \mathbb{E}\, Y_a^3 = 0$, results in

$$e(\underline{t}^*) \leqslant \mathbb{E}\, Y_a^4 \mathrm{e}^{|Y_a|}/24 + \mathbb{E}\, Y_q(Y_a + Y_a^3/6) + \mathbb{E}\, Y_q^2 \mathrm{e}^{Y_a}.$$

Clearly, from $\mathbb{E}\, S_R^2 = P_R$ and $\mathbb{E}\, S_{R^c} = P_0^{1/2}$,

$$\mathbb{E}\,Y_q = -\frac{P_R}{P+\sigma^2} - \frac{P_0}{4P_R+\sigma^2} + \frac{\sigma^2}{2}\left(\frac{P_R}{(P+\sigma^2)^2} + \frac{P_0}{(4P_R+\sigma^2)^2}\right)$$

and

$$\mathbb{E}\,Y_a^2 = \frac{P_R(P-P_R)}{(P+\sigma^2)^2} + \frac{4P_0P_R}{(4P_R+\sigma^2)^2} + \frac{4P_0^{1/2}}{(4P_R+\sigma^2)(P+\sigma^2)} P_0^{1/2} P_R,$$

so that

$$1+\mathbb{E}\,Y_q + \frac{\mathbb{E}\,Y_a^2}{2} = 1 - \frac{P_0(4P_R+\sigma^2) - \sigma^2 P_0/2 - 2P_0P_R}{(4P_R+\sigma^2)^2} - \frac{P_R(P+\sigma^2) - \sigma^2 P_R/2 - P_R P/2}{(P+\sigma^2)^2}$$

$$- \frac{P_R^2}{(P+\sigma^2)^2} + \frac{2P_0}{4P_R+\sigma^2}\frac{P_R}{P+\sigma^2} = 1 - \mathcal{H} + \mathcal{O}(\mathcal{H}^2). \tag{27}$$

Hölder's inequality and the fact that $\mathbb{E}\,|Z|^p \leqslant (\mathbb{E}\,|Z|^q)^{p/q}$, for $p \leqslant q$ and any random variable Z, yield

$$e(\underline{t}^*)\text{h} \leqslant (\mathbb{E}\,Y_a^6)^{2/3}(\mathbb{E}\,\mathrm{e}^{3|Y_a|})^{1/3} + \mathbb{E}\,Y_qY_a + (\mathbb{E}\,Y_q^4)^{1/4}(\mathbb{E}\,Y_a^6)^{1/2} + (\mathbb{E}\,Y_q^4)^{1/2}(\mathbb{E}\,\mathrm{e}^{2Y_a})^{1/2}. \tag{28}$$

Hence, in order to have $e(\underline{t}^*) \leqslant C\mathcal{H}^2$, it is sufficient to prove that for $\underline{t} = \underline{t}^*$, $\mathbb{E}\,Y_qY_a \leqslant 0$, $\mathbb{E}\,\mathrm{e}^{3|Y_a|}$ and $\mathbb{E}\,\mathrm{e}^{2Y_a}$ are bounded and that

$$\mathbb{E}\,Y_q^4 \leqslant C\mathcal{H}^4 \quad \text{and} \quad \mathbb{E}\,Y_a^6 \leqslant C\mathcal{H}^3.$$

Indeed, it then follows from (28) that $e(\underline{t}^*) \leqslant C\mathcal{H}^2$ and thus, using (25) and (27), it follows that

$$H_{k,R}^{(2)} \geqslant -\log h(\underline{t}^*) \geqslant -\log(1-\mathcal{H}+\mathcal{O}(\mathcal{H}^2)) = \mathcal{H}(1+\mathcal{O}(\mathcal{H})), \tag{29}$$

which is the desired result. Thus, the remainder of this proof is focused on proving these five statements. It is clear that $\mathbb{E}\,Y_qY_a \leqslant 0$, since

$$\mathbb{E}\,S_R = \mathbb{E}\,S_RS_{R^c} = \mathbb{E}\,S_R^3 = \mathbb{E}\,S_R^3S_R^c = 0, \quad \mathbb{E}\,S_R^2 = P_R \geqslant 0, \qquad \mathbb{E}\,S_R^2S_{R^c} = P_RP_0^{1/2} \geqslant 0.$$

By symmetry, we have $\mathbb{E}\,\mathrm{e}^{3|Y_a|} \leqslant 2\mathbb{E}\,\mathrm{e}^{3Y_a}$. Recall the definition of Y_a in (23) and use the Cauchy–Schwarz inequality on $\exp(-3(P+\sigma^2)^{-1}S_RS_{R^c})$ and $\exp(-3(2P_0^{1/2})(4P_R+\sigma^2)^{-1}S_R)$. This results in

$$\mathbb{E}\,\mathrm{e}^{3Y_a} \leqslant \left(\mathbb{E}\exp\left(-6\left(P+\sigma^2\right)^{-1}\right)S_RS_{R^c}\right)^{1/2}\left(\mathbb{E}\exp\left(-12\left(P_0^{1/2}4P_R+\sigma^2\right)^{-1}\right)S_R\right)^{1/2}. \tag{30}$$

In order to prove that the expression above is bounded, the following lemma will be useful. The proof, which is omitted, is an easy extension of the proof of Lemma 3.3 in van der Hofstad and Klok, (2004), and is based on (16), together with a weak convergence argument.

Lemma 5.2. *Let A_1, A_2 be disjoint subsets of $\mathbb{N}\cup\{0\}$. Then $\mathbb{E}\,\mathrm{e}^{(x/P_{A_1})S_{A_1}S_{A_2}}$ is uniformly bounded whenever $x^2P_{A_2}/P_{A_1} \leqslant 1-\varepsilon$, for some fixed $\varepsilon \in (0, 1)$.*

We can apply Lemma 5.2 to (30), since both $P_R/(P+\sigma^2)$ and $P_0/(4P_R+\sigma^2)$ are $o(1)$

and thus $6^2 P_R^2/(P+\sigma^2)^2$ and $12^2 P_R P_0/(4P_R+\sigma^2)^2$ are clearly less than or equal to $1-\varepsilon$ when $\mathcal{H}$ is sufficiently small, so that $\mathbb{E}\,e^{3|Y_a|}$ is indeed uniformly bounded. In an identical manner, $\mathbb{E}\,e^{2Y_a}$ is also shown to be uniformly bounded.

Using $|x+y|^l \leqslant 2^{l-1}(|x|^l+|y|^l) \leqslant 2^{l-1}(|x|+|y|)^l$ for $x, y \in \mathbb{R}$, $l = 1, 2, \ldots,$ together with (21), it is straightforward to show that for $\underline{t} = \underline{t}^*$,

$$\begin{aligned}\mathbb{E}\,Y_q^4 &\leqslant C\frac{1}{(P+\sigma^2)^4}\mathbb{E}\,S_R^8 + C\frac{P_0^4}{(4P_R+\sigma^2)^4} + C\sigma^8\left(\frac{\mathbb{E}\,S_R^8}{(P+\sigma^2)^8} + \frac{P_0^4}{(4P_R+\sigma^2)^8}\right) \qquad (31)\\ &\leqslant C\frac{P_R^4}{(P+\sigma^2)^4} + C\frac{P_0^4}{(4P_R+\sigma^2)^4} + C\sigma^8\left(\frac{P_R^4}{(P+\sigma^2)^8} + \frac{P_0^4}{(4P_R+\sigma^2)^8}\right)\\ &\leqslant C\frac{P_R^4(1+\sigma^8/(P+\sigma^2)^4)}{(P+\sigma^2)^4} + C\frac{P_0^4(1+\sigma^8/(4P_R+\sigma^2)^4)}{(4P_R+\sigma^2)^4}\\ &\leqslant C\frac{P_R^4}{(P+\sigma^2)^4} + C\frac{P_0^4}{(4P_R+\sigma^2)^4} = \mathcal{O}(\mathcal{H}^4).\end{aligned}$$

Similarly,

$$\mathbb{E}\,Y_a^6 \leqslant C\frac{1}{(P+\sigma^2)^6}\mathbb{E}\,S_R^6\mathbb{E}\,S_{R^c}^6 + C\frac{P_0^3}{(4P_R+\sigma^2)^6}\mathbb{E}\,S_R^6 \leqslant C\frac{P_R^3 P_{R^c}^3}{(P+\sigma^2)^6} + C\frac{P_0^3 P_R^3}{(4P_R+\sigma^2)^6} \qquad (32)$$

$$\leqslant C\frac{P_R^3}{(P+\sigma^2)^3} + C\frac{P_0^3}{(4P_R+\sigma^2)^3} = \mathcal{O}(\mathcal{H}^3). \qquad (33)$$

Step 2: Upper bound. For the upper bound, we are not allowed to substitute any value for t_{1m}. However, similar to the proof of the upper bound of $H_k^{(1)}$, we can define an ellipse which allows us to derive the desired result. In order to do so, we first observe that

$$\mathbb{E}\left(\prod_{i=1}^{l} S_{A_i}\right) \geqslant 0, \qquad \text{for all } l \in \mathbb{N},\ A_i \subset \mathbb{N}\cup\{0\}. \qquad (34)$$

We consider

$$-\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\max_{m\in R} Z_m^{(1)} \leqslant 0,\ \min_{m\in R^*} Z_m^{(1)} \geqslant 0,\ \overline{Z}_0^{(2)} \leqslant 0\right) = -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\underline{Y}_i \in D\right),$$

where $\underline{Y}_i$ is a k-dimensional vector with elements

$$Y_{1m,i} = \sum_{j=1}^{k-1} P_j^{1/2} X_{ji} X_{mi} + X_{mi} N_i, \qquad 1 \leqslant m \leqslant k-1,$$

$$Y_{20,i} = P_0^{1/2} + 2\sum_{j\in R} P_j^{1/2} X_{ji} + N_i,$$

and where $D = \{\underline{s} = (s_{11}, \ldots, s_{1,k-1}, s_{20}) : s_{1m} \leqslant 0\,,\ m \in R,\ s_{1m} \geqslant 0\,,\ m \in R^*,\ s_{20} \leqslant 0\}$. According to Cramér's theorem,

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} \sum_{i=1}^{n} \underline{Y}_i \in D \right) = \sup_{\underline{t}\in D} \{-\log h(\underline{t})\}, \qquad \text{where } h(\underline{t}) = \mathbb{E}\, \mathrm{e}^{\langle \underline{t}, \underline{Y}_1 \rangle}.$$

The expectation over N_1 is easy, since $\mathbb{E}\, \mathrm{e}^{xN_1} = \mathrm{e}^{x^2\sigma^2/2}$. Thus,

$$h(\underline{t}) = \mathbb{E} \exp\left\{ \sum_{m=1}^{k-1} t_{1m} \left(\sum_{j=1}^{k-1} P_j^{1/2} X_{j1} X_{m1} \right) + t_{20} \left(P_0^{1/2} + 2 \sum_{j\in R} P_j^{1/2} X_{j1} \right). \right.$$

$$\left. + \left(t_{20} + \sum_{m=1}^{k-1} t_{1m} X_{m1} \right)^2 \sigma^2/2 \right\}.$$

We split the exponent into Y_q, the quadratic part, and Y_a, the asymmetric part. This division will reduce the number of calculations of moments. This yields

$$Y_q\mathrm{h} = t_{20} P_0^{1/2} + T_R S_R + \sigma^2 (t_{20} + T_R + T_{R^*})^2/2, \qquad Y_a = (2t_{20} S_R + S_{R^c} T_R) + S_{R_0} T_{R^*}, \tag{35}$$

where we have split Y_a according to the signs of the elements of $\underline{t}$. Similar to (25), we write

$$h(\underline{t}) = 1 + \mathbb{E}\, Y_q + \mathbb{E}\, Y_a^2/2 + e(\underline{t}), \tag{36}$$

where $e(\underline{t})$ is given in (26). Since we can discard all terms that are non-negative almost surely, (26) reduces to

$$e(\underline{t}) \geqslant \mathbb{E}[Y_a + Y_a^3/6 + Y_q(Y_a + Y_a^2/2 + Y_a^3/6 + Y_a^4 \mathrm{e}^{\eta Y_a})/24].$$

A straightforward calculation gives

$$\mathbb{E}\, Y_q = t_{20} P_0^{1/2} + \sum_{m\in R} P_m^{1/2} t_{1m} + \frac{\sigma^2}{2} \left(t_{20}^2 + \sum_{m\in R} t_{1m}^2 + \sum_{m\in R^*} t_{1m}^2 \right)$$

and (use $\mathbb{E}\, S_{R_0}^2 T_{R^*}^2 = P \sum_{m\in R^*} t_{1m}^2 + \sum_{m\in R^*} \sum_{t\in R^*, l\neq m} P_m^{1/2} P_l^{1/2} t_{1m} t_{1l}$)

$$\mathbb{E}\, Y_a^2 = 4P_R t_{20}^2 + P_{R^c} \sum_{m\in R} t_{1m}^2 + 4t_{20} P_0^{1/2} \sum_{m\in R} P_m^{1/2} t_{1m}$$

$$+ P \sum_{m\in R^*} t_{1m}^2 + \sum_{\substack{l\in R^* \\ l\neq m}}^{m\in R^*} P_m^{1/2} P_l^{1/2} t_{1m} t_{1l} + \left(\sum_{m\in R} P_m^{1/2} t_{1m} \right) \left(\sum_{m\in R^*} P_m^{1/2} t_{1m} \right)$$

$$\geqslant 4P_R t_{20}^2 + P_{R^c} \sum_{m\in R} t_{1m}^2 + P \sum_{m\in R^*} t_{1m}^2 + \left(\sum_{m\in R^*} P_m^{1/2} t_{1m} \right) \left(\sum_{m\in R} P_m^{1/2} t_{1m} \right),$$

where we have used $\underline{t} \in D$ to obtain the inequality. The next goal is to write the lower bound of h as a sum of squares. It is convenient to introduce some more abbreviations. First of all, we introduce t^*_{1m} and t^*_{20} as

$$t^*_{1m} = \begin{cases} -\dfrac{P_m^{1/2}}{P+\sigma^2} & m \in R, \\ 0 & m \in R^*, \end{cases} \qquad t^*_{20} = -\frac{P_0^{1/2}}{4P_R+\sigma^2}.$$

We will prove later on that the minimizers of $\sup_{\underline{t}\in D}\{-\log h(\underline{t})\}$ converge to $\underline{t}^*$ as $\mathcal{H} \to 0$.

In order to obtain a lower bound for $h(\underline{t})$, we observe that

$$\begin{aligned}\mathbb{E}\, Y_q + \frac{1}{2}\mathbb{E}\, Y_a^2 &\geqslant \sum_{m\in R} P_m^{1/2} t_{1m} + \frac{P+\sigma^2-P_R}{2}\sum_{m\in R} t_{1m}^2 + \frac{P+\sigma^2}{2}\sum_{m\in R^*} t_{1m}^2 + P_0^{1/2} t_{20} \qquad (37)\\ &\quad + \frac{4P_R+\sigma^2}{2} t_{20}^2 + \left(\sum_{m\in R^*} P_m^{1/2} t_{1m}\right)\left(\sum_{m\in R} P_m^{1/2} t_{1m}\right)\\ &= \frac{P+\sigma^2}{2}\sum_{m\in R}(t_{1m}-t^*_{1m})^2 - \sum_{m\in R}\frac{P_m}{2(P+\sigma^2)} - \frac{P_R}{2}\sum_{m\in R} t_{1m}^2 + \frac{P+\sigma^2}{2}\sum_{m\in R^*} t_{1m}^2\\ &\quad + \frac{4P_R+\sigma^2}{2}(t_{20}-t^*_{20})^2 - \frac{P_0}{2(4P_R+\sigma^2)} + \left(\sum_{m\in R^*} P_m^{1/2} t_{1m}\right)\left(\sum_{m\in R} P_m^{1/2} t_{1m}\right)\end{aligned}$$

(the first inequality is only due to the lower bound of $\mathbb{E}\, Y_a^2$). When we substitute this in (36) and use $\mathcal{H} = P_0/(2(4P_R+\sigma^2)) + P_R/(2(P+\sigma^2))$, we can rearrange terms to arrive at

$$\begin{aligned}h(\underline{t}) &\geqslant 1 - \mathcal{H} + \frac{P+\sigma^2}{2}\sum_{m\in R}(t_{1m}-t^*_{1m})^2 + \frac{P+\sigma^2}{4}\sum_{m\in R^*}(t_{1m}-t^*_{1m})^2\\ &\quad + \frac{4P_R+\sigma^2}{2}(t_{20}-t^*_{20})^2 + e_1(\underline{t}), \qquad (38)\end{aligned}$$

where

$$\begin{aligned}e_1(\underline{t}) = &-\frac{P_R}{2}\sum_{m\in R} t_{1m}^2 \qquad (39)\\ &+ \frac{P+\sigma^2}{8}\sum_{m\in R^*} t_{1m}^2 + \mathbb{E}\, Y_a + \frac{1}{2}\left(\sum_{m\in R^*} P_m^{1/2} t_{1m}\right)\left(\sum_{m\in R} P_m^{1/2} t_{1m}\right) + \mathbb{E}\, Y_q Y_a\\ &+ \frac{P+\sigma^2}{8}\sum_{m\in R^*} t_{1m}^2 + \mathbb{E}\, Y_a^3/6\\ &+ \mathbb{E}\, Y_q(Y_a^2/2 + Y_a^3/6 + Y_a^4 \mathrm{e}^{\eta Y_a}/24).\end{aligned}$$

This can be seen by comparing all terms with t_{ji} on the right-hand side of (37) and (38) together with (39). Note that we have split $\frac{1}{2}(P+\sigma^2)\sum_{m\in R^*} t_{1m}^2$ into three parts, for technical reasons.

The next step is to introduce an appropriate ellipse $\mathcal{E}$, similarly to the proof of Proposition 4.1. When we can prove that on $(\partial\mathcal{E})\cap D$, $h(\underline{t})>1$, we can conclude that on $\mathcal{E}^c\cap D$ the minimum is never attained. Indeed, for every $\underline{x}\in\mathcal{E}^c\cap D$, we can find a unique $0<\alpha<1$ such that $\alpha\underline{x}\in(\partial\mathcal{E})\cap D$. But then, since $h(\underline{0})=1$ and h is convex, (18) leads to $h(\underline{x})>1$. Clearly the minimum is at most 1 ($h(\underline{0})=1$), so $\underline{x}$ is never a minimizer. We define $\mathcal{E}$ as

$$\mathcal{E}=\left\{\underline{t}\colon \sum_{m\in R}\frac{P+\sigma^2}{2}(t_{1m}-t_{1m}^*)^2+\sum_{m\in R^*}\frac{P+\sigma^2}{4}(t_{1m}-t_{1m}^*)^2+\frac{4P_R+\sigma^2}{2}(t_{20}-t_{20}^*)^2\leqslant 2\mathcal{H}\right\}.$$

It is straightforward to prove that for $\underline{t}\in\mathcal{E}$,

$$|t_{20}|\leqslant C\sqrt{\frac{\mathcal{H}}{4P_R+\sigma^2}},\qquad \sum_{m\in R\cup R^*} t_{1m}^2\leqslant C\frac{\mathcal{H}}{P+\sigma^2}. \tag{40}$$

We can next bound $e_1(\underline{t})$. Since we do this using the same techniques as in this proof, and the derivation will distract the reader from more interesting issues, the proof is postponed to Appendix A.

Lemma 5.3. *There exists a constant C, not depending on k or $\underline{R}$, such that for $\underline{t}\in\mathcal{E}$,*

$$e_1(\underline{t})\geqslant -C\mathcal{H}^2.$$

Thus, for $\underline{t}\in\mathcal{E}\cap D$ (recall (38)),

$$\begin{aligned} h(\underline{t})\geqslant{}& 1-\mathcal{H}+\frac{P+\sigma^2}{2}\sum_{m\in R}(t_{1m}-t_{1m}^*)^2+\frac{P+\sigma^2}{4}\sum_{m\in R^*}(t_{1m}-t_{1m}^*)^2\\ &+\frac{4P_R+\sigma^2}{2}(t_{20}-t_{20}^*)^2-C\mathcal{H}^2. \end{aligned}\tag{41}$$

This implies that for $\underline{t}\in(\partial\mathcal{E})\cap D$, and when $\mathcal{H}$ is small enough,

$$h(\underline{t})\geqslant 1-\mathcal{H}+2\mathcal{H}-C\mathcal{H}^2>1,$$

so that the minimum is never attained outside the ellipse. Finally, when $\underline{t}\in\mathcal{E}\cap D$, it is clear that the minimum of the right-hand side of (41) is attained at $\underline{t}=\underline{t}^*$, implying $H_{k,R}^{(2)}\leqslant\mathcal{H}(1+\mathcal{O}(\mathcal{H}))$. □

6. Proof of Theorem 3.1

According to (20), we have to minimize $H_{k,R}^{(2)}$ over subsets $R\subseteq\{1,\dots,k-1\}$. If we followed the naive approach, we would minimize the asymptotic rate $P_0/2(4P_R+\sigma^2)+P_R/2(P+\sigma^2)$ over P_R, with the result

$$H_k^{(2)} \approx \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)}, \qquad \text{for } \frac{P_R}{P_0} = \frac{1}{2}\sqrt{\frac{P+\sigma^2}{P_0}} - \frac{\sigma^2}{4P_0}.$$

Note that the last expression precisely equals ρ. Since P_R attains values on some grid, depending on the individual P_js, it is not clear that P_R/P_0 can attain the value ρ. We show that under the condition in (i), ρ can be attained with the right order deviation.

We will split the proof into three steps. In the first step, we show that when $\mathcal{H} = o(1)$,

$$H_k^{(2)} \geqslant \begin{cases} \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right), & \rho \geqslant 0, \\ \frac{P_0}{2\sigma^2} + \mathcal{O}\left(\frac{P_0^2}{\sigma^4}\right), & \rho \leqslant 0. \end{cases} \tag{42}$$

In step 2, we prove that $\mathcal{H} \geqslant \varepsilon$ does not give a smaller lower bound if $P_0/(P+\sigma^2)$ is sufficiently small. Finally, we show that the upper bound of $H_k^{(2)}$ equals the asymptotic lower bound, whenever we assume $(*)$ for $\rho \geqslant 0$.

Step 1: Lower bound when $\mathcal{H} = o(1)$. According to Theorem 5.1, there exists an $M > 0$ such that when $P_0/(P+\sigma^2)$ is sufficiently small,

$$\min_{R\subseteq\{1,\ldots,k-1\}} H_{k,R}^{(2)} \geqslant \min_{R\subseteq\{1,\ldots,k-1\}} \mathcal{H} - M\mathcal{H}^2 \geqslant \min_{P_R\geqslant 0} \mathcal{H} - M\mathcal{H}^2.$$

Taking the derivative of the right-hand side with respect to P_R gives the sufficient and necessary condition

$$\left(\frac{-4P_0}{(4P_R+\sigma^2)^2} + \frac{1}{P+\sigma^2}\right)(1-2M\mathcal{H}) = 0.$$

When $\mathcal{H}$ is sufficiently small, $1 - 2M\mathcal{H} > 0$, so that the optimal P_R obeys

$$\frac{P_R}{P_0} = \max\left\{\frac{1}{2}\sqrt{\frac{P+\sigma^2}{P_0}} - \frac{\sigma^2}{4P_0}, 0\right\} = \max\{\rho, 0\}. \tag{43}$$

The condition of Theorem 5.1 is fulfilled, since for $\rho \geqslant 0$,

$$\mathcal{H} = \frac{P_0}{2(4P_R+\sigma^2)} + \frac{P_R}{2(P+\sigma^2)} = \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)} = o(1),$$

while, for $\rho \leqslant 0$, we obtain from (43) that $P_R = 0$ and thus

$$\mathcal{H} = \frac{P_0}{2\sigma^2} = \frac{1}{2}\sqrt{\frac{P_0(P+\sigma^2)}{\sigma^4}}\sqrt{\frac{P_0}{P+\sigma^2}} \leqslant \frac{1}{4}\sqrt{\frac{P_0}{P+\sigma^2}} = o(1), \tag{44}$$

because $\rho \leqslant 0 \Leftrightarrow (P_0(P+\sigma^2)/\sigma^4)^{1/2} \leqslant \frac{1}{2}$. This results in the following lower bound:

$$H_k^{(2)} \geqslant \begin{cases} \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)} + \mathcal{O}\left(\left(\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{4(P+\sigma^2)}\right)^2\right), & \rho \geqslant 0, \\ \frac{P_0}{2\sigma^2} + \mathcal{O}\left(\frac{P_0^2}{\sigma^4}\right), & \rho \leqslant 0. \end{cases}$$

Step 1 is completed once we have proven that for $\rho \geqslant 0$,

$$\mathcal{O}\left(\left(\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{4(P+\sigma^2)}\right)^2\right) = \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right).$$

This is easy, since we assumed $\rho \geqslant 0$, so that

$$\frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} \leqslant \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} + \rho\frac{P_0}{P+\sigma^2} = \sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{4(P+\sigma^2)} \leqslant \sqrt{\frac{P_0}{P+\sigma^2}}. \tag{45}$$

We have proven the lower bounds in (42) when $\mathcal{H} = o(1)$.

Step 2: $\mathcal{H} \geqslant \varepsilon$ does not give smaller lower bound. The goal of this step is to prove that, for an arbitrary $\varepsilon > 0$, $\mathcal{H} \geqslant \varepsilon$ implies the lower bounds in (42). This allows us to conclude that the lower bounds are as desired. Since $\mathcal{H} \geqslant \varepsilon$ implies that either $P_0/(4P_R + \sigma^2) \geqslant \varepsilon$ or $P_R/(P+\sigma^2) \geqslant \varepsilon$, we can focus on the two cases separately.

We first assume that $P_0/(4P_R + \sigma^2) \geqslant \varepsilon$. Since $\mathbb{P}(A \cap B) \leqslant \mathbb{P}(A)$, we can focus on the second stage alone:

$$H_{k,R}^{(2)} \geqslant -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(P_0^{1/2} + \sum_{j\in R} 2P_j^{1/2} \frac{1}{n}\sum_{i=1}^{n} X_{ji} \leqslant 0\right). \tag{46}$$

Because this is in fact the situation where a user with power $P_0^{1/2}$ suffers from interfering users in the set R with powers $4P_j$ and from external noise with intensity σ^2, the rate above is, according to Proposition 4.1, bounded from below by

$$\frac{P_0}{2(\sum_{j\in R} 4P_j + \sigma^2)} = \frac{P_0}{2(4P_R + \sigma^2)} \geqslant \varepsilon/2 \geqslant \sqrt{\frac{P_0}{P+\sigma^2}},$$

for $P_0/(P+\sigma^2)$ sufficiently small, because ε is fixed.

We will next treat the case $P_R/(P+\sigma^2) \geqslant \varepsilon$. To do so, we split according to the largest power in P_R. Suppose we can choose an $\tilde{m} \in R$ such that $P_{\tilde{m}} \geqslant 2\sqrt{P_0(P+\sigma^2)}$. Then, taking only the event $\{Z_{\tilde{m}}^{(1)} \leqslant 0\}$ into account, the rate is bounded from below by

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(P_{\tilde{m}}^{1/2} + \sum_{\substack{j=0 \\ j\neq\tilde{m}}}^{k-1} P_j^{1/2} \frac{1}{n}\sum_{i=1}^{n} X_{ji}X_{\tilde{m}i} \leqslant 0\right) \geqslant \frac{P_{\tilde{m}}}{2(P+\sigma^2)} \geqslant \sqrt{\frac{P_0}{P+\sigma^2}},$$

where we have used Proposition 4.1 to obtain the first inequality and $P_{\tilde{m}} \geqslant 2\sqrt{P_0(P+\sigma^2)}$ to obtain the second.

When we *cannot* choose an $\tilde{m} \in R$ such that $P_m \geqslant 2\sqrt{P_0(P+\sigma^2)}$, all powers P_m, $m \in R$, must obey $P_m \leqslant 2\sqrt{P_0(P+\sigma^2)}$ for $m \in R$. In this case, we can choose an $\tilde{R} \subset R$ such that

$$4\sqrt{P_0(P+\sigma^2)} \leqslant P_{\tilde{R}} \leqslant 6\sqrt{P_0(P+\sigma^2)}.$$

We observe that, with our usual definition $R^* = \{1, \ldots, k-1\}\backslash R$,

$$\begin{aligned}
&-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\max_{m\in R} \mathrm{sgnr}_m(Z_m^{(1)}) < 0,\ \min_{m\in R^*} \mathrm{sgnr}_m(Z_m^{(1)}) > 0,\ \mathrm{sgnr}_0(Z_0^{(2,H)}) < 0 \right) \\
&\geqslant -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\max_{m\in \tilde{R}} \mathrm{sgnr}_m(Z_m^{(1)}) < 0 \right) \geqslant -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\max_{m\in \tilde{R}} Z_m^{(1)} \leqslant 0 \right).
\end{aligned}$$

We next use the lower bound in Theorem 5.1 on the scenario with $k+1$ users with powers $0, P_2, \ldots, P_k, P_0$ (user 0 has power 0, so $\{\mathrm{sgnr}_0(\overline{Z}_0^{(2,\mathrm{H})}) < 0\}$ does not contribute to the rate), with the following result:

$$\begin{aligned}
-\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\max_{m\in \tilde{R}} Z_m^{(1)} \leqslant 0 \right) &\geqslant \frac{P_{\tilde{R}}}{2(P+\sigma^2)} + \mathcal{O}\left(\frac{P_{\tilde{R}}^2}{(P+\sigma^2)^2} \right) \geqslant \frac{P_{\tilde{R}}}{4(P+\sigma^2)} \\
&\geqslant \frac{4\sqrt{P_0(P+\sigma^2)}}{4(P+\sigma^2)} = \sqrt{\frac{P_0}{P+\sigma^2}},
\end{aligned}$$

when $P_{\tilde{R}}/(P+\sigma^2)$ is sufficiently small. This is guaranteed for $P_0/(P+\sigma^2) \to 0$, because $P_{\tilde{R}}/(P+\sigma^2) \leqslant 6\sqrt{P_0/(P+\sigma^2)} = o(1)$.

The result of step 2 is that when $\mathcal{H} \geqslant \varepsilon$, we have $H_k^{(2)} \geqslant \sqrt{P_0/(P+\sigma^2)}$, which is larger than the lower bounds in (42), where we use the fact that for $\rho \leqslant 0$, $P_0/(2\sigma^2) \leqslant \frac{1}{4}\sqrt{P_0/(P+\sigma^2)}$, according to (44).

Step 3: Upper bound. For the upper bound, we can substitute the optimal P_R obtained above. It is sufficient to prove that this optimal P_R can be approximated with small enough deviation. For $\rho \leqslant 0$, the claim is trivial, since $R = \varnothing$ suffices. Therefore, assume $\rho \geqslant 0$.

We take R such that $P_R/P_0 - \rho = \varepsilon$, where, according to $(*)$, $|\varepsilon| \leqslant C(P+\sigma^2)^{1/4}/P_0^{1/4}$, so that $\varepsilon(P_0/(P+\sigma^2))^{1/2} = o(1)$. When we substitute $P_R = \sqrt{P_0(P+\sigma^2)}/2 - \sigma^2/4 + \varepsilon P_0$ in $\mathcal{H}$ and use the above fact for ε, we arrive at

$$\begin{aligned}
\mathcal{H} &= \frac{P_0}{2(4P_R+\sigma^2)} + \frac{P_R}{2(P+\sigma^2)} \\
&= \frac{1}{4}\sqrt{\frac{P_0}{P+\sigma^2}}\left(\frac{1}{1+2\varepsilon\sqrt{P_0/(P+\sigma^2)}}\right) + \frac{1}{4}\sqrt{\frac{P_0}{P+\sigma^2}}\left(1+2\varepsilon\sqrt{\frac{P_0}{P+\sigma^2}}\right) - \frac{\sigma^2}{8(P+\sigma^2)} \\
&= \frac{1}{4}\sqrt{\frac{P_0}{P+\sigma^2}}\left(1-2\varepsilon\sqrt{\frac{P_0}{P+\sigma^2}} + \mathcal{O}\left(\varepsilon^2\frac{P_0}{P+\sigma^2}\right)\right) \\
&\quad + \frac{1}{4}\sqrt{\frac{P_0}{P+\sigma^2}}\left(1+2\varepsilon\sqrt{\frac{P_0}{P+\sigma^2}}\right) - \frac{\sigma^2}{8(P+\sigma^2)} \\
&= \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)} + \sqrt{\frac{P_0}{P+\sigma^2}}\mathcal{O}\left(\varepsilon^2\frac{P_0}{P+\sigma^2}\right) \\
&= \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right).
\end{aligned}$$

Thus, substitution of R such that $P_R/P_0 = \rho + \varepsilon$ gives

$$\begin{aligned}
H_k^{(2)}\mathrm{h} \leqslant \mathrm{h}\mathcal{H} + \mathcal{O}(\mathcal{H}^2) = {} & \frac{1}{2}\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{8(P+\sigma^2)} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right) \\
& + \mathcal{O}\left(\left(\sqrt{\frac{P_0}{P+\sigma^2}} - \frac{\sigma^2}{4(P+\sigma^2)} + \mathcal{O}\left(\frac{P_0}{P+\sigma^2}\right)\right)^2\right).
\end{aligned}$$

Finally, use (45) on the last order term to obtain the desired equality.

Appendix: Proof of Lemma 5.3

The proof is split into 4 steps. In every step one line of the right-hand side of (39) is treated and is proved to be non-negative, $\mathcal{O}(\mathcal{H}^2)$ or a combination of those two.

Line 1. This term is bounded as

$$\left| -\frac{P_R}{2}\sum_{m\in R} t_{1m}^2 \right| \leqslant \mathrm{C}\frac{P_R}{P+\sigma^2}\mathcal{H} \leqslant \mathrm{C}\mathcal{H}^2, \qquad t \in \mathcal{E},$$

by (40) and $P_R/(P+\sigma^2) \leqslant 2\mathcal{H}$.

Line 2. Observe that

$$\mathbb{E}\, Y_a = \sum_{m\in R^*} P_m^{1/2} t_{1m} \quad \text{and} \quad \mathbb{E}\, Y_q Y_a = t_{20} P_0^{1/2} \mathbb{E}\, Y_a + \mathbb{E}\, T_R S_R Y_a + \frac{\sigma^2}{2} \mathbb{E}(t_{20} + T_R + T_{R^*})^2 Y_a. \tag{47}$$

It is straightforward to prove that

$$t_{20} P_0^{1/2} \mathbb{E}\, Y_a + \mathbb{E}\, T_R S_R Y_a = t_{20} P_0^{1/2} \left(\sum_{m\in R^*} P_m^{1/2} t_{1m} \right) + \left(\sum_{m\in R} P_m^{1/2} t_{1m} \right) \left(\sum_{m\in R^*} P_m^{1/2} t_{1m} \right) \tag{48}$$

and

$$\begin{aligned} \frac{\sigma^2}{2} \mathbb{E}(t_{20} + T_R + T_{R^*})^2 Y_a = {} & \frac{\sigma^2}{2} \mathbb{E}(t_{20} + T_R)^2 (2t_{20} S_R + S_{R^c} T_R + S_{R_0} T_{R^*}) \\ & + \sigma^2 \mathbb{E}(t_{20} + T_R) T_{R^*} (2t_{20} S_R + S_{R^c} T_R + S_{R_0} T_{R^*}) \\ & + \frac{\sigma^2}{2} \mathbb{E}\, T_{R^*}^2 (2t_{20} S_R + S_{R^c} T_R + S_{R_0} T_{R^*}). \end{aligned}$$

Using $t \in D$ and (34), it follows that $\mathbb{E}(t_{20} + T_R)^2 S_{R_0} T_{R^*} \geqslant 0$. Similarly, other terms with such combinations of ts can be shown to be non-negative. This leads to

$$\begin{aligned} & \frac{\sigma^2}{2} \mathbb{E}(t_{20} + T_R + T_{R^*})^2 Y_a \qquad (49) \\ & \geqslant \frac{\sigma^2}{2} \mathbb{E}(t_{20} + T_R)^2 (2t_{20} S_R + S_{R^c} T_R) + \sigma^2 \mathbb{E}(t_{20} + T_R) T_{R^*} S_{R_0} T_{R^*} + \frac{\sigma^2}{2} \mathbb{E}\, T_{R^*}^2 (2t_{20} S_R + S_{R^c} T_R) \\ & = 2\sigma^2 t_{20}^2 \sum_{m\in R} P_m^{1/2} t_{1m} + \sigma^2 t_{20} P_0^{1/2} \sum_{m\in R} t_{1m}^2 + \sigma^2 t_{20} P_0^{1/2} \sum_{m\in R^*} t_{1m}^2 \\ & \quad + \frac{\sigma^2}{2} \left(\sum_{m\in R} P_m^{1/2} t_{1m} \right) \left(\sum_{m\in R^*} t_{1m}^2 \right), \end{aligned}$$

where the equality can be proven using straightforward calculation of moments. The first two terms in the line above are of order $\mathcal{H}^2$. Indeed, by (40), and $|x, y| \leqslant \|x\| \, \|y\|$,

$$\begin{aligned} \left| 2\sigma^2 t_{20}^2 \sum_{m\in R} P_m^{1/2} t_{1m} + \sigma^2 t_{20} P_0^{1/2} \sum_{m\in R} t_{1m}^2 \right| & \leqslant 2\sigma^2 t_{20}^2 \left(P_R \sum_{m\in R} t_{1m}^2 \right)^{1/2} + \sigma^2 |t_{20}| P_0^{1/2} \sum_{m\in R} t_{1m}^2 \\ & \leqslant C \Bigg(\sigma^2 \frac{\mathcal{H}}{4P_R + \sigma^2} \left(P_R \frac{\mathcal{H}}{P + \sigma^2} \right)^{1/2} \\ & \quad + \sigma^2 \frac{\mathcal{H}^{1/2}}{(4P_R + \sigma^2)^{1/2}} P_0^{1/2} \frac{\mathcal{H}}{P + \sigma^2} \Bigg) \leqslant C\mathcal{H}^2, \end{aligned}$$

since $\sigma^2/(4P_R+\sigma^2)$ and $\sigma^2/(P+\sigma^2)$ are bounded by 1. Thus, according to (47), (48) and (49), $\mathbb{E}\,Y_a+\mathbb{E}\,Y_qY_a$ is bounded from above by

$$\left(\sum_{m\in R^*} t_{1m}^2\right)\left(\sigma^2 t_{20}P_0^{1/2}+\frac{\sigma^2}{2}\sum_{m\in R} t_{1m}^2\right)+\left(\sum_{m\in R^*} P_m^{1/2}t_{1m}\right)\left(1+t_{20}P_0^{1/2}+\sum_{m\in R} P_m^{1/2}t_{1m}\right)-C\mathcal{H}^2.$$

When $\mathcal{H}$ is sufficiently small, we bound the second line of (39) as

$$\begin{aligned}
&\frac{P+\sigma^2}{8}\sum_{m\in R^*} t_{1m}^2+\left(\sum_{m\in R^*} P_m^{1/2}t_{1m}\right)\left(\sum_{m\in R} P_m^{1/2}t_{1m}\right)+\mathbb{E}\,Y_a+\mathbb{E}\,Y_qY_a\\
&\geqslant (P+\sigma^2)\left(\sum_{m\in R^*} t_{1m}^2\right)\left(\frac{1}{8}+\frac{\sigma^2 t_{20}P_0^{1/2}}{P+\sigma^2}+\frac{\sigma^2}{2(P+\sigma^2)}\sum_{m\in R} P_m^{1/2}t_{1m}\right)\\
&\quad+\left(\sum_{m\in R^*} P_m^{1/2}t_{1m}\right)\left(1+t_{20}P_0^{1/2}+2\sum_{m\in R} P_m^{1/2}t_{1m}\right)-C\mathcal{H}^2\\
&\geqslant (P+\sigma^2)\left(\sum_{m\in R^*} t_{1m}^2\right)\left(\frac{1}{8}-C\frac{\sigma^2}{P+\sigma^2}\mathcal{H}^{1/2}\frac{P_0^{1/2}}{(4P_R+\sigma^2)^{1/2}}-\frac{\sigma^2}{2(P+\sigma^2)}\left(P_R\sum_{m\in R} t_{1m}^2\right)^{1/2}\right)\\
&\quad+\left(\sum_{m\in R^*} P_m^{1/2}t_{1m}\right)\left(1-C\mathcal{H}-2\left(P_R\sum_{m\in R} t_{1m}^2\right)^{1/2}\right)-C\mathcal{H}^2\\
&\geqslant (P+\sigma^2)\left(\sum_{m\in R^*} t_{1m}^2\right)\left(\frac{1}{8}-C\mathcal{H}\right)+\left(\sum_{m\in R^*} P_m^{1/2}t_{1m}\right)(1-C\mathcal{H})-C\mathcal{H}^2\geqslant -C\mathcal{H}^2,
\end{aligned}$$

where we have again used $|\langle x,\, y\rangle|\leqslant \|x\|\,\|y\|$ and $t_{1m}\geqslant 0$ for $m\in R^*$.

Line 3. Working out the terms of Y_a^3, and observing that $\mathbb{E}(2t_{20}S_R+S_{R^c}T_R)^2S_{R_0}T_{R^*}\geqslant 0$ and $\mathbb{E}\,S_{R_0}^3T_{R^*}^3\geqslant 0$ by (34), (35) and the fact that $\underline{t}\in D$, leads to

$$\mathbb{E}\,Y_a^3\geqslant \mathbb{E}(2t_{20}S_R+S_{R^c}T_R)^3+3\mathbb{E}(2t_{20}S_R+S_{R^c}T_R)S_{R_0}^2T_{R^*}^2.$$

By symmetry arguments, the first term on the right-hand side above equals 0. When $\mathcal{H}$ is sufficiently small, the second term, together with the remaining term on the third line of (39), is bounded using Cauchy–Schwarz, yielding

$$\frac{P+\sigma^2}{8}\sum_{m\in R^*} t_{1m}^2 + \frac{1}{2}\mathbb{E}(2t_{20}S_R + S_{R^c}T_R)S_{R_0}^2 T_{R^*}^2$$

$$\geqslant \frac{P+\sigma^2}{8}\sum_{m\in R^*} t_{1m}^2 - \frac{1}{2}\left(\mathbb{E}(2t_{20}S_R + S_{R^c}T_R)^2\right)^{1/2}\left(\mathbb{E}\, S_{R_0}^8\right)^{1/4}\left(\mathbb{E}\, T_{R^*}^8\right)^{1/4}$$

$$\geqslant \frac{P+\sigma^2}{8}\sum_{m\in R^*} t_{1m}^2 - C\left(4P_R t_{20}^2 + P\sum_{m\in R} t_{1m}^2 + 4t_{20}P_0^{1/2}P_R^{1/2}\left(\sum_{m\in R} t_{1m}^2\right)^{1/2}\right)^{1/2} P\sum_{m\in R^*} t_{1m}^2$$

$$\geqslant \frac{P+\sigma^2}{8}\sum_{m\in R^*} t_{1m}^2 - C\mathcal{H}^{1/2}P\sum_{m\in R^*} t_{1m}^2 \geqslant 0,$$

where we have used (40) to obtain the third inequality.

Line 4. Finally, whenever we can prove $\mathbb{E}\, Y_q^2 \leqslant C\mathcal{H}^2$, $\mathbb{E}\, Y_a^6 \leqslant C\mathcal{H}^3$ and $\mathbb{E}\, \mathrm{e}^{\eta 12 Y_a}$ is bounded, it follows from Hölder's inequality that $\mathbb{E}\, Y_q Y_a^2 = \mathcal{O}(\mathcal{H}^2)$, $\mathbb{E}\, Y_q Y_a^3 = \mathcal{O}(\mathcal{H}^2)$ and $\mathbb{E}\, Y_q Y_a^4 \mathrm{e}^{\eta Y_a} = \mathcal{O}(\mathcal{H}^2)$. This can be done easily using Cauchy–Schwarz, $(x+y)^l \leqslant 2^{l-1}(x^l + y^l)$ for Lemma 5.2, and (21) and (40).

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