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Edgeworth-type expansions for transition densities of Markov chains converging to diffusions

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We consider triangular arrays of Markov chains that converge weakly to a diffusion process. We prove Edgeworth-type expansions of order $o(n^{-1-\delta})$, $\delta > 0$, for transition densities. For this purpose we apply the paramatrix method to represent the transition density as a functional of densities of sums of independent and identically distributed variables. Then we apply Edgeworth expansions to the densities. The resulting series gives our Edgeworth-type expansion for the Markov chain transition density.

Keywords: diffusion processes; Edgeworth expansions; Markov chains; transition densities

1. Introduction

In this paper we study triangular arrays of homogeneous Markov chains $X_n(k)$ ($n \geqslant 1$, $0 \leqslant k \leqslant n$) that converge weakly to a diffusion process (for $n \to \infty$). Our main result will give Edgeworth-type expansions for the transition densities. The order of the expansions is $o(n^{-1-\delta})$, $\delta > 0$. The theory of Edgeworth expansions is well developed for sums of independent random variables. For more general models, approaches have been used where the expansion is reduced to models with sums of independent random variables. This is also the basic idea behind our approach. We will make use of the parametrix method. In this approach the transition density is represented as a nested sum of functionals of densities of sums of independent variables. Plugging Edgeworth expansions into this representation will result in an expansion for the transition density.

Weak convergence of the distribution of scaled discrete-time Markov processes to diffusions has been extensively studied in the literature (see Skorohod 1965; Strook and Varadhan 1979). Local limit theorems for Markov chains were given in Konovalov (1981), Konakov and Molchanov (1984) and Konakov and Mammen (2000; 2001). In Konakov and Mammen (2000) it was shown that the transition density of a Markov chain converges at rate $O(n^{-1/2})$ to the transition density in the diffusion model. For the proof there an analytical approach was chosen that made essential use of the parametrix method. This method allows tractable representations of transition densities of diffusions to be obtained

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that are based on Gaussian densities (see Lemma 3.1 below). Similar representations hold for discrete-time Markov chains X_n (see Lemma 5.1 below). For a short exposition of the parametrix method, see Section 3 and Konakov and Mammen (2000). The parametrix method for Markov chains developped in Konakov and Mammen (2000) is described in Section 5.1. Applications to Markov random walks are given in Konakov and Mammen (2001). In Konakov and Mammen (2002) the approach is used to give Edgeworth-type expansions for Euler schemes for differential equations. Standard references for the parametrix method are Friedman (1964) and Ladyzhenskaja *et al*. (1968) on parabolic partial differential equations, and for diffusions McKean and Singer (1967).

This paper is organized as follows. In the next section we present our model for the Markov chain. In Section 3 we give a short introduction to the parametrix method for diffusions. Our main result, which states an Edgeworth-type expansion for Markov chains, is given in Section 4. Some auxiliary results are given in Section 5. In particular, in Section 5.1 we recall the parametrix approach developed in Konakov and Mammen (2000) for Markov chains. The proof of our main result is given in Section 6.

2. Markov chain model

We now give a more detailed description of Markov chains and their diffusion limit. For all $n \geqslant 1$, we consider Markov chains $X_n(k)$ where the time k runs from 0 to n. The Markov chain X_n is assumed to take values in $\mathbb{R}^p$. The dynamics of the chain X_n is described by

$$X_n(k+1) = X_n(k) + n^{-1}m\{X_n(k)\} + n^{-1/2}\varepsilon_n(k+1). \tag{1}$$

Here, m is a function $m : \mathbb{R}^p \to \mathbb{R}^p$. We make the Markov assumption that the conditional distribution of the innovation $\varepsilon_n(k+1)$ given the past $X_n(k)$, $X_n(k-1)$, ... depends only on the last value $X_n(k)$. Given $X_n(i) = x(i)$, for $i = 0, \ldots, k$, the variable $\varepsilon_n(k+1)$ has a conditional density $q\{x(k), \cdot\}$. The conditional covariance matrix of $\varepsilon_n(k+1)$ is denoted by $\Sigma\{x(k)\}$. Here q is a function mapping $\mathbb{R}^p \times \mathbb{R}^p$ into $\mathbb{R}_+$. Furthermore, Σ is a function mapping $\mathbb{R}^p$ into the set of positive definite $p \times p$ matrices. The conditional density of $X_n(n)$, given $X_n(0) = x$, is denoted by $p_n(x, \cdot)$. This paper is concerned with the study of the transition densities $p_n(x, y)$. Conditions on $m\{x(k)\}$, $q\{x(k), \cdot\}$ and $\Sigma\{x(k)\}$ are given below.

After time transformation the Markov chain X_n defines a process Y_n on [0, 1]. More precisely, put $Y_n(t) = X_n(k)$, for $k/n \leqslant t < (k+1)/n$. Under our assumptions (see below), the process Y_n converges weakly to a diffusion $Y(t)$. This follows, for instance, from Theorem 1 in Skorohod (1987, p. 82). The diffusion is defined by $Y(0) = x$ and

$$\mathrm{d}Y(t) = m\{Y(t)\}\mathrm{d}t + \Lambda\{Y(t)\}\mathrm{d}W(t),$$

where W is a p-dimensional Brownian motion. The matrix $\Lambda(z)$ is the symmetric matrix defined by $\Lambda(z)\Lambda(z)^{\mathrm{T}} = \Sigma(z)$. The conditional density of $Y(1)$, given $Y(0) = x$, is denoted by $p(x, \cdot)$. Recall that the conditional density of $Y_n(1)$, given $Y_n(0) = x$, is denoted by $p_n(x, \cdot)$.

For our result we use the following conditions.

(A1) For $x \in \mathbb{R}^p$, let $q\{x, \cdot\}$ be a density in $\mathbb{R}^p$ with $\int q\{x, z\}z\mathrm{d}z = 0$ for all $x \in \mathbb{R}^p$, and $\int q\{x, z\}z_i z_j \mathrm{d}z = \sigma_{ij}(x)$ for all $x \in \mathbb{R}^p$ and $i, j = 1, \ldots, p$. The matrix with elements $\sigma_{ij}(x)$ is denoted by $\Sigma(x)$.

(A2) There exist a positive integer S', a constant $\gamma > 0$ and a function $\psi : \mathbb{R}^p \to \mathbb{R}$ with $\sup_{x \in \mathbb{R}^p} \psi(x) < \infty$ and $\int_{\mathbb{R}^p} \|x\|^S \psi(x)\mathrm{d}x < \infty$ for $S = 2pS' + 4$ such that $|\mathrm{D}_z^\nu q\{x, z\}| \leqslant \psi(z)$ for all $x, z \in \mathbb{R}^p$ and $|\nu| = 0, \ldots, 6$, $|\mathrm{D}_x^\nu q\{x, z\}| \leqslant \psi(z)$ for all $x, z \in \mathbb{R}^p$ and $|\nu| = 0, \ldots, 6$, and $|\mathrm{D}_x^\nu q^{(k)}\{x, z\}| \leqslant k^\gamma \psi(K^{-\gamma} z)$ for all $x, z \in \mathbb{R}^p$, $k \geqslant 1$ and $|\nu| = 0, 1$.

Here $q^{(k)}(x, z)$ denotes the k-fold convolution of q for fixed x as a function of z.

(A3) There exist positive constants c and C such that

$$c \leqslant \theta^{\mathrm{T}} \Sigma(x) \theta \leqslant C$$

for all θ, $\|\theta\| = 1$ and x.

(A4) The functions $m(x)$ and $\Sigma(x)$ and their derivatives up to order 6 are bounded (uniformly in x) and Lipschitz continuous with respect to x.

3. The parametrix method

Our approach makes use of the parametrix method. This approach allows series expansions to be stated for the transition densities of the limiting diffusion and for the Markov chain. The series only depend on transition densities of 'frozen' processes. The 'frozen' diffusion is a Gaussian process that has a Gaussian density as transition density. For the 'frozen' Markov chain we obtain transition densities that are densities of sums of independent variables. In this section we will give an overview on the method for diffusions and Markov chains.

We now discuss the parametrix method for diffusions. This gives an infinite series expansion of the transition density p of the limiting diffusion process Y (see Lemma 3.1). We give a similar expansion for the Markov chain in the next subsection (see Lemma 3.3). Our proof of Theorem 4.1 is based on the comparison of these two series. The series for the transition densities is derived by the parametrix method. We give a description of the parametrix method below.

For the statement of the expansion of p in Lemma 3.1 we have to introduce additional diffusion processes. For $0 < s < 1$ and $x, y \in \mathbb{R}^p$, we define diffusions $\tilde{Y} = \tilde{Y}_{s,x,y}$ that are defined for $s \leqslant t \leqslant 1$ by

$$\tilde{Y}(s) = x$$

and

$$\mathrm{d}\tilde{Y}(t) = m\{y\}\mathrm{d}t + \Lambda\{y\}\mathrm{d}W(t).$$

The processes $\tilde{Y}$ are called 'frozen' diffusions. We define $\tilde{p}(s, t, x, y)$ as the conditional density of $\tilde{Y}(t)$ [$= \tilde{Y}_{s,x,y}(t)$] at the point y, given $\tilde{Y}(s) = x$. Note that the variable y acts here twice: as the argument of the density and as a defining quantity of the process $\tilde{Y} = \tilde{Y}_{s,x,y}$. Furthermore, we denote by $\tilde{p}_j^y(x, z)$ the conditional density of $\tilde{Y}((j+1)/n)$

[$= \tilde{Y}_{j/n,x,y}((j+1)/n)$] at the point z, given $\tilde{Y}(j/n) = x$. The process $\tilde{Y}$ is a simple Gaussian process. Its transition densities $\tilde{p}$ are given explicitly. By definition, we have that

$$\tilde{p}(s, t, x, y) = (2\pi(t-s))^{-p/2}(\det \Sigma(y))^{-1/2} \times \exp\left[-\frac{1}{2}(t-s)^{-1}\{y - x - (t-s)m(y)\}^{\mathrm{T}}\Sigma(y)^{-1}\{y - x - (t-s)m(y)\}\right]. \quad (2)$$

Let us introduce the following differential operators L and $\tilde{L}$:

$$Lf(s, t, x, y) = m(x)^{\mathrm{T}}\frac{\partial f(s, t, x, y)}{\partial x} + \frac{1}{2}\mathrm{tr}\left[\Lambda(x)^{\mathrm{T}}\frac{\partial^2 f(s, t, x, y)}{(\partial x)^2}\Lambda(x)\right],$$

$$\tilde{L}f(s, t, x, y) = m(y)^{\mathrm{T}}\frac{\partial f(s, t, x, y)}{\partial x} + \frac{1}{2}\mathrm{tr}\left[\Lambda(y)^{\mathrm{T}}\frac{\partial^2 f(s, t, x, y)}{\partial x^2}\Lambda(y)\right].$$

Note that L and $\tilde{L}$ correspond to the infinitesimal operators of Y and of the frozen process $\tilde{Y}_{s,x,y}$, respectively, that is,

$$Lf(s, t, x, y) = \lim_{h\to 0} h^{-1}\{\mathrm{E}[f(s, t, Y(s+h), y)|Y(s) = x] - f(s, t, x, y)\}, \quad (3)$$

$$\tilde{L}f(s, t, x, y) = \lim_{h\to 0} h^{-1}\{\mathrm{E}[f(s, t, \tilde{Y}_{s,x,y}(s+h), y)] - f(s, t, x, y)\}. \quad (4)$$

We put

$$H = (L - \tilde{L})\tilde{p}.$$

Then

$$H(s, t, x, y) = \frac{1}{2}\sum_{i,j=1}^{p}(\sigma_{ij}(x) - \sigma_{ij}(y))\frac{\partial^2 \tilde{p}(s, t, x, y)}{\partial x_i \partial x_j} + \sum_{i=1}^{p}(m_i(x) - m_i(y))\frac{\partial \tilde{p}(s, t, x, y)}{\partial x_i}. \quad (5)$$

Now we define the following convolution-type binary operation $\otimes$:

$$(f \otimes g)(s, t, x, y) = \int_s^t \mathrm{d}u \int_{\mathbb{R}^p} f(s, u, x, z)g(u, t, z, y)\mathrm{d}z.$$

We write $g \otimes H^{(0)}$ for g, and for $r = 1, 2, \ldots$ we denote the r-fold 'convolution' $(g \otimes H^{(r-1)}) \otimes H$ by $g \otimes H^{(r)}$.

With the foregoing notation we can state our expansion for p.

Lemma 3.1. *For* $0 \leqslant s < t \leqslant 1$,

$$p(s, t, x, y) = \sum_{r=0}^{\infty}\left(\tilde{p} \otimes H^{(r)}\right)(s, t, x, y).$$

A proof of Lemma 3.1 can be found in McKean and Singer (1967). We will make use of the bounds on H and $\tilde{p} \otimes H^{(r)}$ that are stated in the following lemma. Proofs of these bounds can again be found in McKean and Singer (1967). For a more detailed proof of Lemma 3.2, see also Ladyzhenskaja *et al*. (1968).

Lemma 3.2. *There exist constants C and C_1 (not depending on x and y) such that*

$$|H(s, t, x, y)| \leqslant C_1 \rho^{-1} \phi_{C,\rho}(y - x)$$

and

$$|\tilde{p} \otimes H^{(r)}(s, t, x, y)| \leqslant C_1^{r+1} \frac{\rho^r}{\Gamma(1 + r/2)} \phi_{C,\rho}(y - x),$$

where $\rho^2 = t - s$, $\phi_{C,\rho}(u) = \rho^{-p} \phi_C(u/\rho)$ and

$$\phi_C(u) = \frac{\exp(-C\|u\|^2)}{\int \exp(-C\|v\|^2)\mathrm{d}v}.$$

4. Edgeworth-type expansions for Markov chains

The following theorem contains our main result. It gives Edgeworth-type expansions for p_n. For the statement of the theorem we introduce the following differential operators:

$$\mathcal{F}_1[f](s, t, x, y) = \sum_{|\nu|=3} \frac{\mu_\nu(x)}{\nu!} D_x^\nu f(s, t, x, y),$$

$$\mathcal{F}_2[f](s, t, x, y) = \sum_{|\nu|=4} \frac{\chi_\nu(x)}{\nu!} D_x^\nu f(s, t, x, y).$$

Furthermore,

$$\mu_\nu(x) = \int z^\nu q(x, z)\mathrm{d}z,$$

$$\tilde{\pi}_1(s, t, x, y) = (t - s) \sum_{|\nu|=3} \frac{\chi_\nu(y)}{\nu!} D_x^\nu \tilde{p}(s, t, x, y),$$

$$\tilde{\pi}_2(s, t, x, y) = (t - s) \sum_{|\nu|=4} \frac{\chi_\nu(y)}{\nu!} D_x^\nu \tilde{p}(s, t, x, y) + \frac{1}{2}(t - s)^2 \left\{ \sum_{|\nu|=3} \frac{\chi_\nu(y)}{\nu!} D_x^\nu \right\}^2 \tilde{p}(s, t, x, y),$$

where $\chi_\nu(x)$ are the cumulants of the density $q(x, \cdot)$.

Theorem 4.1. *Assume* (A1)–(A4). *Then there exists a constant $\delta > 0$ such that*

$$\sup_{x,y\in\mathbb{R}^p}\big(1+\|y-x\|^{2(S'-1)}\big)|p_n(x, y)-p(x, y)-n^{-1/2}\pi_1(x, y)-n^{-1}\pi_2(x, y)|=O(n^{-1-\delta}),$$

where S' is defined in (A2) *and where*

$$\pi_1(x, y)=(p\otimes\mathcal{F}_1[p])(0, 1, x, y), \tag{6}$$

$$\begin{aligned}\pi_2(x, y)=\tfrac{1}{2}\Big(p\otimes(L_*^2-L^2)p\Big)(0, 1, x, y)+(p\otimes\mathcal{F}_2[p])(0, 1, x, y)\\ +(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p]])(0, 1, x, y).\end{aligned} \tag{7}$$

Here $p(s, t, x, y)$ is the transition density of the diffusion $Y(t)$, and L_* is defined analogously to $\tilde{L}$ but with the coefficients 'frozen' at the point x. The norm $\|\cdot\|$ is the usual Euclidean norm.

The proof of Theorem 4.1 will be given in Section 6. We now make some remarks concerning the approximating terms $\pi_1(x, y)$ and $\pi_2(x, y)$.

First, it can be shown that the term $\pi_1(x, y)$ and each term on the right-hand side of (7) have sub-Gaussian tails. This means that these terms can be bounded from above by $C_1\exp[-C_2(y-x)^2]$ with some positive constants C_1 and C_2.

Secondly, if the innovation density $q(x, \cdot)$ does not depend on x then one obtains that $L_*=L$ and that $p(s, t, x, y)=\tilde{p}(s, t, x, y)$, where $\tilde{p}$ is defined in (2) with $\Sigma(y)=\Sigma$ and $m(y)=m$. This gives

$$\begin{aligned}\pi_1(x, y)&=\int_0^1 \mathrm{d}s\int\tilde{p}(0, s, x, v)\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_v^\nu\tilde{p}(s, 1, v, y)\mathrm{d}v\\
&=-\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_y^\nu\int_0^1\mathrm{d}s\int\tilde{p}(0, s, x, v)\tilde{p}(s, 1, v, y)\mathrm{d}v\\
&=\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_x^\nu\tilde{p}(0, 1, x, y)\\
&=\tilde{\pi}_1(0, 1, x, y),\\
(\tilde{p}\otimes\mathcal{F}_1[\tilde{p}])(s, 1, z, y)&=\int_s^1\mathrm{d}u\int\tilde{p}(s, u, z, w)-\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_w^\nu\tilde{p}(u, 1, w, y)\mathrm{d}w\\
&=-\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_y^\nu\int_s^1\mathrm{d}u\int\tilde{p}(s, u, z, w)\tilde{p}(u, 1, w, y)\mathrm{d}w\\
&=(1-s)\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_z^\nu\tilde{p}(s, 1, z, y),\end{aligned}$$

$$(\mathcal{F}_1[\tilde{p}\otimes\mathcal{F}_1[\tilde{p}]])(s,\,1,\,z,\,y)=(1-s)\left\{\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_z^\nu\right\}^2\tilde{p}(s,\,1,\,z,\,y),$$

$$\begin{aligned}(\tilde{p}\otimes\mathcal{F}_2[\tilde{p}])&(0,\,1,\,x,\,y)+(\tilde{p}\otimes\mathcal{F}_1[\tilde{p}\otimes\mathcal{F}_1[\tilde{p}]])(0,\,1,\,x,\,y)\\ &=\int_0^1\mathrm{d}s\int\tilde{p}(0,\,s,\,x,\,v)\left(\sum_{|\nu|=4}\frac{\chi_\nu}{\nu!}D_v^\nu\tilde{p}(s,\,1,\,v,\,y)\right.\\ &\qquad\left.+(1-s)\left\{\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_v^\nu\right\}^2\tilde{p}(s,\,1,\,v,\,y)\right)\mathrm{d}v\\ &=\sum_{|\nu|=4}\frac{\chi_\nu}{\nu!}D_y^\nu\int_0^1\mathrm{d}s\int\tilde{p}(0,\,s,\,x,\,v)\tilde{p}(s,\,1,\,v,\,y)\mathrm{d}v\\ &\qquad+\left\{\sum_{|\nu|=3}\frac{\mu_\nu}{\nu!}D_y^\nu\right\}^2\int_0^1(1-s)\mathrm{d}s\int\tilde{p}(0,\,s,\,x,\,v)\tilde{p}(s,\,1,\,v,\,y)\mathrm{d}v\\ &=\tilde{\pi}_2(0,\,1,\,x,\,y).\end{aligned}$$

Thus from Theorem 4.1 for this case we just obtain the first two terms of the classical Edgeworth expansion $n^{-1/2}\tilde{\pi}_1(0,\,1,\,x,\,y)+n^{-1}\tilde{\pi}_2(0,\,1,\,x,\,y)$.

Thirdly, if $\mu_\nu(x)=0$ for $|\nu|=3$ and for $x\in\mathbb{R}^p$, then $\mathcal{F}_1=0$. This gives that the expansion of Theorem 4.1 holds with

$$\pi_1(x,\,y)=0,$$

$$\pi_2(x,\,y)=(p\otimes\mathcal{F}_2)[p](0,\,1,\,x,\,y)+\tfrac{1}{2}\Big(p\otimes(L_*^2-L^2)p\Big)(0,\,1,\,x,\,y).$$

If in addition we have that $\chi_\nu(x)=0$ for $|\nu|=4$, then the first four moments of the innovations coincide with the first four moments of a normal distribution with zero mean and covariance matrix $\Sigma(x)$. In this case we have $\mathcal{F}_2=0$ and

$$\pi_1(x,\,y)=0, \tag{8}$$

$$\pi_2(x,\,y)=\tfrac{1}{2}\Big(p\otimes(L_*^2-L^2)p\Big)(0,\,1,\,x,\,y). \tag{9}$$

Fourthly, our expansion can be applied to study the performance of discrete approximations of diffusions. An Euler approximating scheme is defined by putting

$$Y_n([k+1]/n)=Y_n(k/n)+n^{-1}m(Y_n(k/n))+\Lambda(Y_n(k/n))[W([k+1]/n)-W(k/n)].$$

It has been shown that $Y_n(1)=Y(1)+O_P(n^{-1/2})$. For a discussion of Euler approximations, see Kloeden and Platen (1992). For this scheme we have that $\mathcal{F}_1=\mathcal{F}_2=0$. Thus the expansion of Theorem 4.1 holds with (8) and (9). This result was obtained by Bally and Talay

(1996a; 1996b). Higher-order asymptotic expansions for Euler schemes are given in Konakov and Mammen (2002).

Fifthly, a more refined approximating scheme for stochastic differential equations was introduced by Mil'shtein (1974). Mil'shtein's scheme is based on higher-order stochastic approximations of the stochastic differential equation. Mil'shtein (1974) proved that for his scheme $Y_n(1) = Y(1) + O_P(n^{-1})$. Thus this scheme has a better strong approximation rate than Euler schemes. We now apply Theorem 4.1 to this approximating scheme. We compare the approximations of the transition densities for these two schemes. It turns out that the rate is not improved for Mil'shtein's schemes, in contrast to the rates of strong approximation mentioned. However, we argue that Mil'shtein schemes lead to more stable approximations. For simplicity we consider only the one-dimensional case. For Mil'shtein schemes we have that

$$\mu_{2,n}(x) = \sigma^2(x) + \frac{1}{2n}(\sigma(x)\sigma'(x))^2, \qquad \mu_{3,n}(x) = \frac{3\sigma(x)\sigma'(x)}{\sqrt{n}},$$

$$\chi_{4,n}(x) = \mu_{4,n}(x) - 3\sigma^4(x) = \frac{15}{2n}\sigma^5(x)(\sigma'(x))^3.$$

Hence,

$$\pi_1(x, y) = 0,$$

$$\pi_2(x, y) = \tfrac{1}{2}\Big(p \otimes (L_*^2 - L^2)p\Big)(0, 1, x, y) + (p \otimes M)(0, 1, x, y),$$

where

$$M(s, t, x, y) = \frac{1}{2}\sigma(x)\sigma'(x)\frac{\partial^3 p(s, t, x, y)}{\partial x^3}.$$

The last expression for $\pi_2(x, y)$ allows us to compare Mil'shtein and Euler schemes. In the one-dimensional case the function $\frac{1}{2}(L_*^2 - L^2)p(s, 1, z, y)$ is equal to

$$\tfrac{1}{2}(L_*^2 - L^2)p(s, 1, z, y) = R(s, 1, z, y) - M(s, 1, z, y),$$

where

$$\begin{aligned} R(s, 1, z, y) = &- \left[\frac{1}{2}m(z)m'(z) + \frac{1}{4}m''(z)\sigma^2(z)\right]\frac{\partial p(s, 1, z, y)}{\partial z} \\ &- \left[\frac{1}{2}m(z)\sigma(z)\sigma'(z) + \frac{1}{2}m'(z)\sigma^2(z) + \frac{1}{4}(\sigma(z)\sigma'(z))^2 + \frac{1}{4}\sigma^3(z)\sigma''(z)\right] \\ &\times \frac{\partial^2 p(s, 1, z, y)}{\partial z^2}. \end{aligned}$$

By linearity of $\otimes$ we obtain that, for Mil'shtein schemes,

$$\pi_2(x, y) = (p \otimes R)(0, 1, x, y).$$

Thus Mil'shtein schemes are constructed such that in the expansion the third derivative of the diffusion density p is eliminated from the expansion of the Euler scheme. This derivative is the most unstable and singular summand near the point $s = 1$. This suggests that Mil'shtein schemes lead to more stable approximations of transition densities of diffusions. At this stage this discussion is purely heuristic. The conditions of Theorem 4.1 are not fulfilled for two reasons. The densities of the innovations depend on n and do not satisfy the required smoothness conditions. The first point is not crucial. For Mil'shtein schemes the innovation density $q_n(x, \cdot)$ depends on n, but asymptotically this dependence vanishes. Theorem 4.1 can be extended to cover this case. We also conjecture that Theorem 4.1 can be extended to non-smooth innovation densities. This could be done by introducing a new Markov process X_n^* where one step consists of l subsequent steps of X_n, that is $X_n^*(k) = X_n(kl)$. Typically, X_n^* has smoother innovation densities as X_n. We conjecture that with l large enough under appropriate conditions the assumptions of our theorem are satisfied. A detailed discussion of this will be given elsewhere.

Finally, Theorem 4.1 states an expansion of the Markov transition density with terms of order $n^{-1/2}$ and n^{-1}. The coefficients of these terms do not depend on n. This is desirable for many theoretical applications. If one is interested in numerical approximations, dependence of coefficients on n does not matter. Then other expansions may be preferable that are less computer-intensive. In particular, calculation of the diffusion transition density p is highly non-trivial. An expansion of p_n that avoids the calculation of p can be derived from Lemmas 5.1 and 5.2. For this purpose the expansion of $\tilde{p}_n$, given in Lemma 5.2, could be plugged into the formula of Lemma 5.1. Another modification of Theorem 4.1 would be helpful for numerical approximations of the diffusion transition density p based on numerical calculations of the Markov transition density p_n. If we could justify the (formal) differentiation of the expansion of Theorem 4.1, then by iterative use of these expansions we can obtain an expansion for p with coefficients depending only on p_n and on the derivatives of p_n (and not on p and its derivatives). Discussion of the accuracy of these expansions and their usefulness for numerical approximations is outside the scope of this paper.

5. Some auxiliary results

This section contains some auxiliary results that will be used in the proof of Theorem 4.1. In Section 3 we represented the transition densities of the diffusion by nested sums of functionals of densities of 'frozen' processes. The difference between the densities of 'frozen' Markov chains and Gaussian densities can be treated by Edgeworth expansions. This is done in Section 5.2. In contrast to Konakov and Mammen (2000a), we now use higher-order Edgeworth expansions. These are the main steps of the proof of Theorem 4.1. In Section 5.3 we give some bounds for the kernels and their differences used in the expansions of the parametrix method.

5.1. Application of the parametrix method to Markov chains

In this subsection we derive a finite series expansion of the transition density $p_n(s, t, x, y)$ of the Markov chain (see Lemma 5.1). Here, $p_n(s, t, x, \cdot)$ denotes the conditional density of

$Y_n(t)$, given $Y_n(s) = x$ (in particular, $p_n(0, 1, x, y) = p_n(x, y)$). Similarly to Section 3, we again apply the parametrix method and for this purpose we introduce additional 'frozen' Markov chains. These are defined as follows. For all $0 \leqslant j \leqslant n$ and $x, y \in \mathbb{R}^p$, we define the Markov chains $\tilde{X}_n = \tilde{X}_{n,j,x,y}$. For fixed j, x and y, the chain is defined for i with $j \leqslant i \leqslant n$. The dynamics of the chain is described by

$$\tilde{X}_n(j) = x$$

and

$$\tilde{X}_n(i+1) = \tilde{X}_n(i) + n^{-1}m\{y\} + n^{-1/2}\tilde{\varepsilon}_n(i+1).$$

The stochastic structure of the $\mathbb{R}^p$-valued innovations $\tilde{\varepsilon}_n(i)$ is described as follows. Given $\tilde{X}_n(l) = x(l)$ for $l = j, \ldots, i$, the variable $\tilde{\varepsilon}_n(i+1)$ has a conditional density $q\{y, \cdot\}$. Note that the conditional distribution of $\tilde{X}_n(i+1) - \tilde{X}_n(i)$ does not depend on the past $\tilde{X}_n(l)$ for $l = j, \ldots, i$. Let us call $\tilde{X}_n$ the Markov chain *frozen at* y. We put $\tilde{Y}_n(t) = \tilde{X}_n\{k\}$ for $k/n \leqslant t < (k+1)/n$, and write $\tilde{p}_n(j/n, k/n, x, y)$ for the conditional density of $\tilde{X}_n(k)$ $[= \tilde{X}_{n,j,x,y}(k)]$ at the point y, given $\tilde{X}_n(j) = x$. Note that, as in the case of a 'frozen' diffusion, the variable y acts here twice: as the argument of the density and as a defining quantity of the process $\tilde{X}_n = \tilde{X}_{n,j,x,y}$. Let us introduce the following infinitesimal operators L_n and $\tilde{L}_n$:

$$L_n f(j/n, k/n, x, y) = n\left[\int p_{n,j}(x, z) f((j+1)/n, k/n, z, y)\mathrm{d}z - f((j+1)/n, k/n, x, y)\right],$$

$$\tilde{L}_n f(j/n, k/n, x, y) = n\left[\int \tilde{p}^{y}_{n,j}(x, z) f((j+1)/n, k/n, z, y)\mathrm{d}z - f((j+1)/n, k/n, x, y)\right],$$

where we write

$$p_{n,j}(x, z) = p_n(j/n, (j+1)/n, x, z)$$

and where $\tilde{p}^{y}_{n,j}(x, \cdot)$ denotes the conditional density of $\tilde{X}_n(j+1)$ $[= \tilde{X}_{n,j,x,y}(j+1)]$ given $\tilde{X}_n(j) = x$. Note that L_n and $\tilde{L}_n$ are defined by analogy with the definition of L and $\tilde{L}$ (see (3)–(4)). We remark that for technical reasons the terms $f((j+1)/n, \ldots)$ appear on the right-hand side of the definitions of $L_n f$ and $\tilde{L}_n f$ instead of $f(j/n, \ldots)$. The reasons will become apparent in the development of the proof of Theorem 4.1. For $k > j$ we put, by analogy with the definition of H,

$$H_n = \{L_n - \tilde{L}_n\}\tilde{p}_n.$$

The next lemma, from Konakov and Mammen (2000), gives the 'parametrix' expansion of p_n.

Lemma 5.1. *For* $0 \leqslant j < k \leqslant n$,

$$p_n(j/n, k/n, x, y) = \sum_{r=0}^{k-j} \left(\tilde{p}_n \otimes_n H_n^{(r)}\right)(j/n, k/n, x).$$

Here $\otimes_n$ *denotes the following binary 'convolution-type' operator:*

$$(f \otimes_n g)(i/n, k/n, x, y) = \frac{1}{n}\sum_{j=i}^{k-1} \int_{\mathbb{R}^p} f(i/n, j/n, x, z) g(j/n, k/n, z, y) \mathrm{d}z,$$

where in the sum on the right-hand side the term $f(i/n, i/n, x, y)$ is equal to the Dirac function $\delta(x-y)$ at $x-y$.

5.2. Bounds on $\tilde{p}_n - \tilde{p}$ based on Edgeworth expansions

In this subsection we will develop some tools that are helpful in the comparison of the expansion of p (see Lemma 3.1) and the expansion of p_n (see Lemma 5.1). These expansions are simple expressions in $\tilde{p}$ or $\tilde{p}_n$, respectively. Recall that $\tilde{p}$ is a Gaussian density (see (2)), and that $\tilde{p}_n$ is the density of a sum of independent variables. The densities $\tilde{p}$ and $\tilde{p}_n$ can be compared by application of Edgeworth expansions. This is done in Lemma 5.2. This is the essential step for the comparison of the expansions of p and p_n. In the lemma bounds are given for derivatives of $\tilde{p}_n$. The proof of the lemma also makes essential use of Edgeworth expansions. The lemma is a higher-order extension of the results in Section 3.3 in Konakov and Mammen (2000).

Lemma 5.2. *The following bound holds with a constant C for $\nu = (\nu_1, \ldots, \nu_p)^{\mathrm{T}}$, with $0 \leqslant |\nu| \leqslant 6$:*

$$\begin{aligned}
&|D_z^\nu \tilde{p}_n(j/n, k/n, x, y) - D_z^\nu \tilde{p}(j/n, k/n, x, y) \\
&\quad - n^{-1/2} D_z^\nu \tilde{\pi}_1(j/n, k/n, x, y) - n^{-1} D_z^\nu \tilde{\pi}_2(j/n, k/n, x, y)| \\
&\leqslant C n^{-3/2} \rho^{-3} \zeta_\rho^{S-|\nu|}(y-x),
\end{aligned}$$

for all $j < k$, x and y. Here D_z^ν denotes the partial differential operator of order ν with respect to $z = \rho^{-1}\Sigma(y)^{-1/2}(y - x - \rho^2 m(y))$. The quantity ρ again denotes the term $\rho = [(k-j)/n]^{1/2}$. We write $\zeta_\rho^k(\cdot) = \rho^{-p}\zeta^k(\cdot/\rho)$, where

$$\zeta^k(z) = \frac{[1 + \|z\|^k]^{-1}}{\int [1 + \|z'\|^k]^{-1} \mathrm{d}z'}.$$

Proof. We note first that $\tilde{p}_n(j/n, k/n, x, \cdot)$ is the density of the vector

$$x + \rho^2 m(y) + n^{-1/2} \sum_{i=j}^{k-1} \tilde{\varepsilon}_n(i+1),$$

where, as above in the definition of the 'frozen' Markov chain $\tilde{Y}_n$, $\tilde{\varepsilon}_n(i+1)$ is a sequence of independent variables with densities $q(y, \cdot)$. Let $f_n(\cdot)$ be the density of the normalized sum

$$n^{-1/2}\big[n^{-1}(k-j)\Sigma(y)\big]^{-1/2}\sum_{i=j}^{k-1}\tilde{\varepsilon}_n(i+1) = n^{-1/2}\rho^{-1}\Sigma(y)^{-1/2}\sum_{i=j}^{k-1}\tilde{\varepsilon}_n(i+1).$$

Clearly, we have

$$\tilde{p}_n(j/n,\ k/n,\ x,\ \cdot) = \rho^{-p}\det\Sigma(y)^{-1/2}f_n\{\rho^{-1}\Sigma(y)^{-1/2}[\cdot - x - \rho^2 m(y)]\}.$$

We now argue that an Edgeworth expansion holds for f_n. This implies the following expansion for $\tilde{p}_n(j/n,\ k/n,\ x,\ \cdot)$

$$\tilde{p}_n(j/n,\ k/n,\ x,\ \cdot)$$
$$= \rho^{-p}\det\Sigma(y)^{-1/2}\left[\sum_{r=0}^{S-3}(k-j)^{-r/2}P_r(-\phi\colon\{\overline{\chi}_{\beta,r}\})\{\rho^{-1}\Sigma(y)^{-1/2}[\cdot - x - \rho^2 m(y)]\}.\right.$$
$$\left. + O([k-j]^{-(S-2)/2}[1+|\{\rho^{-1}\Sigma(y)^{-1/2}[\cdot - x-\rho^2 m(y)]\}|^S]^{-1})\right] \tag{10}$$

with standard notation (see Bhattacharya and Rao, 1976, p. 53). In particular, P_r denotes a product of a standard normal density ϕ with a polynomial that has coefficients depending only on cumulants of order up to $r+2$. Expansion (10) follows from Theorem 19.3 in Bhattacharya and Rao (1976). This can be seen as in the proof of Lemma 3.7 in Konakov and Mammen (2000).

It follows from (10) and (A3) that

$$|\tilde{p}_n(j/n,\ k/n,\ x,\ y) - \tilde{p}(j/n,\ k/n,\ x,\ y) - n^{-1/2}\tilde{\pi}_1(j/n,\ k/n,\ x,\ y) - n^{-1}\tilde{\pi}_2(j/n,\ k/n,\ x,\ y)|$$
$$\leqslant Cn^{-3/2}\rho^{-3}\zeta_\rho^{S-|\nu|}(y-x), \tag{11}$$

where

$$\tilde{p}(j/n,\ k/n,\ x,\ y) = \rho^{-p}\det\Sigma(y)^{-1/2}(2\pi)^{-p/2}$$
$$\exp\{-\tfrac{1}{2}(y-x-\rho^2 m(y))^{\mathrm{T}}\rho^{-2}\Sigma(y)^{-1}(y-x-\rho^2 m(y))\},$$
$$\tilde{\pi}_1(j/n,\ k/n,\ x,\ y) = -\rho^{-1-p}\det\Sigma(y)^{-1/2}\sum_{|\nu|=3}\frac{\tilde{\chi}_\nu(y)}{\nu!}D_z^\nu\phi\Big\{\rho^{-1}\Sigma(y)^{-1/2}(y-x-\rho^2 m(y))\Big\},$$
$$\tilde{\pi}_2(j/n,\ k/n,\ x,\ y) = \rho^{-2-p}\det\Sigma(y)^{-1/2}\left[\sum_{|\nu|=4}\frac{\tilde{\chi}_\nu(y)}{\nu!}D_z^\nu\phi\Big\{\rho^{-1}\Sigma(y)^{-1/2}(y-x-\rho^2 m(y))\Big\}.\right.$$
$$\left.+\frac{1}{2}\left\{\sum_{|\nu|=3}\frac{\tilde{\chi}_\nu(y)}{\nu!}D_z^\nu\right\}^2\phi\Big\{\rho^{-1}\Sigma(y)^{-1/2}(y-x-\rho^2 m(y))\Big\}\right],$$

in which $\tilde{\chi}_\nu(y)$ are the cumulants of $\Sigma(y)^{-1/2}\tilde{\varepsilon}_n(i+1)$ and $D_z^\nu\phi$ denotes the derivative of ϕ of order ν. The definitions of $\tilde{\pi}_1$ and $\tilde{\pi}_2$ coincide with the definitions given at the beginning of Section 4. This follows by replacing the differential operator D_z^ν by D_x^ν. For $\nu = 0$ the statement of the lemma immediately follows from (11). For $\nu > 0$ one proceeds similarly. See the remark at the end of the proof of Lemma 3.7 in Konakov and Mammen (2000). □

From Lemma 5.2 we obtain the following corollary. The statement of this lemma is an extension of Lemma 3.7 in Mammen and Konakov (2000), where the result has been shown for $0 \leqslant |b| \leqslant 2,\ a = 0$.

Lemma 5.3. *For all $j < k$, for all x and y, and for all a, b with $0 \leqslant |a| + |b| \leqslant 6$,*

$$|D_y^a D_x^b \tilde{p}_n(j/n,\ k/n,\ x,\ y)| \leqslant C\rho^{-|a|-|b|}\zeta_\rho^{S-|a|}(y - x),$$

where $\rho = [(k-j)/n]^{1/2}$ (for simplicity the indices n, j and k are suppressed in the notation) and the constant S was defined in (A2).

5.3. Bounds on operator kernels used in the parametrix expansions

In this subsection we will present bounds for operator kernels appearing in the expansions based on the parametrix method. In Lemma 5.4 we compare the infinitesimal operators L_n and $\tilde{L}_n$ with the differential operators L and $\tilde{L}$. We give an approximation for the error if, in the definition of $H_n = (L_n - \tilde{L}_n)\tilde{p}_n$, the terms L_n and $\tilde{L}_n$ are replaced by L and $\tilde{L}$, respectively. We show that this term can be approximated by $K_n + M_n$, where $K_n = (L - \tilde{L})\tilde{p}_n$ and M_n is defined in Lemma 5.4. Bounds on H_n, K_n and M_n are given in Lemma 5.5. These bounds will be used in the proof of our theorem to show that in the expansion of p_n the terms $\tilde{p}_n \otimes_n H_n^{(r)}$ can be replaced by $\tilde{p}_n \otimes_n (M_n + K_n)^{(r)}$.

Lemma 5.4. *For some constant C,*

$$|H_n(j/n,\ k/n,\ x,\ y) - K_n(j/n,\ k/n,\ x,\ y) - M_n(j/n,\ k/n,\ x,\ y)| \leqslant Cn^{-3/2}\rho^{-1}\zeta_\rho^S(y - x),$$

where

$$K_n(j/n,\ k/n,\ x,\ y) = (L - \tilde{L})\tilde{p}_n(j/n,\ k/n,\ x,\ y),$$

$$M_n(j/n,\ k/n,\ x,\ y) = \sum_{l=1}^{3} M_{n,l}(j/n,\ k/n,\ x,\ y),$$

$$M_{n,1}(j/n,\ k/n,\ x,\ y) = M_{n,11}(j/n,\ k/n,\ x,\ y) + M_{n,12}(j/n,\ k/n,\ x,\ y) + M_{n,13}(j/n,\ k/n,\ x,\ y),$$

$$M_{n,11}(j/n,\ k/n,\ x,\ y) = n^{-1/2} \sum_{|\nu|=3} \frac{D_x^\nu \tilde{p}_n(j/n,\ k/n,\ x,\ y)}{\nu!} (\mu_\nu(x) - \mu_\nu(y)),$$

$$M_{n,12}(j/n,\ k/n,\ x,\ y) = n^{-1} \sum_{|\nu|=4} \frac{D_x^\nu \tilde{p}_n(j/n,\ k/n,\ x,\ y)}{\nu!}$$

$$\times \left\{ \mu_\nu(x) - \mu_\nu(y) - \sum_{|\nu'|=2} \nu! N(\nu,\ \nu') \mu_{\nu'}(y) [\mu_{\nu-\nu'}(x) - \mu_{\nu-\nu'}(y)] \right\},$$

$$M_{n,13}(j/n,\ k/n,\ x,\ y)$$

$$= n^{-3/2} \left\{ \sum_{|\nu|=5} \frac{5}{\nu!} \int\int_0^1 [q(x,\ \theta) - q(y,\ \theta)] \theta^\nu D^\nu \lambda(x + \delta \tilde{h}(\theta))(1-\delta)^4 \mathrm{d}\delta \mathrm{d}\theta \right.$$

$$+ \frac{1}{2} \sum_{i,l} \sigma_{il}(y) \sum_{|\nu|=1} \int\int_0^1 q(y,\ \theta)(\theta + n^{-1/2} m(y))^\nu (L - \tilde{L}) D^{\nu + e_i + e_l} \lambda(x + \delta \tilde{h}(\theta)) \mathrm{d}\delta\, \mathrm{d}\theta$$

$$- 3 \sum_{|\nu|=3} \frac{1}{\nu!} \int\int_0^1 q(y,\ \theta)(\theta + n^{-1/2} m(y))^\nu (1-\delta)^2 (L - \tilde{L}) D^\nu \lambda(x + \delta \tilde{h}(\theta)) \mathrm{d}\delta\, \mathrm{d}\theta$$

$$- 2 \sum_{|\nu|=3} \frac{\mu_\nu(x) - \mu_\nu(y)}{\nu!} \sum_{|\nu'|=2} \frac{1}{\nu'!} \int\int_0^1 q(y,\ \theta)(\theta + n^{-1/2} m(y))^{\nu'} (1-\delta) D^{\nu+\nu'} \lambda(x + \delta \tilde{h}(\theta)) \mathrm{d}\delta\, \mathrm{d}\theta$$

$$\left. - \sum_{|\nu|=4} \frac{\mu_\nu(x) - \mu_\nu(y)}{\nu!} \sum_{|\nu'|=1} \int\int_0^1 q(y,\ \theta)(\theta + n^{-1/2} m(y))^{\nu'} D^{\nu+\nu'} \lambda(x + \delta \tilde{h}(\theta)) \mathrm{d}\delta\, \mathrm{d}\theta \right\},$$

$$M_{n,2}(j/n,\ k/n,\ x,\ y) = M_{n,21}(j/n,\ k/n,\ x,\ y) + \ldots + M_{n,25}(j/n,\ k/n,\ x,\ y),$$

$$M_{n,21}(j/n,\ k/n,\ x,\ y)$$

$$= n^{-1} \sum_{|\nu|=3} \frac{D_x^\nu \tilde{p}_n(j/n,\ k/n,\ x,\ y)}{\nu!} \sum_{i=1}^p \nu_i (m_i(x) - m_i(y)) \cdot (\mu_{\nu - e_i}(x) - \mu_{\nu - e_i}(y)),$$

$$M_{n,22}(j/n,\ k/n,\ x,\ y) = n^{-3/2}\Bigg\{\sum_{|\nu|=4}\frac{(D^\nu\lambda)(x)}{\nu!}\sum_{r=1}^{p}\nu_r[m_r(x)\mu_{\nu-e_r}(x) - m_r(y)\mu_{\nu-e_r}(y)]$$

$$+\sum_{i=1}^{p} m_i(y)\cdot\sum_{|\nu'|=1}\iint_0^1 q(y,\ \theta)(\theta + n^{-1/2}m(y))^{\nu'}(L-\tilde{L})D_x^{\nu'+e_i}\lambda(x+\delta\tilde{h}(\theta))\mathrm{d}\delta\,\mathrm{d}\theta$$

$$-\sum_{|\nu|=2}\frac{m(x)^\nu - m(y)^\nu}{\nu!}\sum_{|\nu'|=1}\iint_0^1 q(y,\ \theta)(\theta + n^{-1/2}m(y))^{\nu'}D_x^{\nu+\nu'}\lambda(x+\delta\tilde{h}(\theta))\mathrm{d}\delta\,\mathrm{d}\theta$$

$$-\sum_{|\nu|=3}\frac{1}{\nu!}\sum_{i=1}^{p}\nu_i[m_i(x)\mu_{\nu-e_i}(x) - m_i(y)\mu_{\nu-e_i}(y)]$$

$$\cdot\sum_{|\nu'|=1}\iint_0^1 q(y,\ \theta)(\theta + n^{-1/2}m(y))^{\nu'}D_x^{\nu+\nu'}\lambda(x+\delta\tilde{h}(\theta))\mathrm{d}\delta\,\mathrm{d}\theta$$

$$-\sum_{|\nu|=3}\frac{\mu_\nu(x)-\mu_\nu(y)}{\nu!}\cdot\sum_{|\nu'|=1} m(y)^{\nu'}D_x^{\nu+\nu'}\lambda(x)\Bigg\}$$

$$M_{n,23}(j/n,\ k/n,\ x,\ y) = 5n^{-2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^{p}(m_i(x)-m_i(y))\iint_0^1 q(x,\ \theta)$$

$$\cdot\,\theta^{\nu-e_i}D_x^\nu\lambda(x+\delta\tilde{h}(\theta))(1-\delta)^4\mathrm{d}\delta\,\mathrm{d}\theta,$$

$$M_{n,24}(j/n,\ k/n,\ x,\ y) = 5n^{-2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^{p} m_i(y)\iint_0^1[q(x,\ \theta)-q(y,\ \theta)]$$

$$\cdot\,\theta^{\nu-e_i}D^\nu\lambda(x+\delta\tilde{h}(\theta))(1-\delta)^4\mathrm{d}\delta\,\mathrm{d}\theta,$$

$$M_{n,25}(j/n,\ k/n,\ x,\ y) = 5n^{-5/2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^{p}\iint_0^1 q(x,\ \theta)\theta^\nu\sum_{|\mu|=1}D^{\nu+\mu}\lambda(x+\delta\tilde{h}(\theta))$$

$$\cdot\,(m_i(x)-m_i(y))^\mu\delta(1-\delta)^4\mathrm{d}\delta\,\mathrm{d}\theta,$$

$$M_{n,3}(j/n,\ k/n,\ x,\ y) = M_{n,31}(j/n,\ k/n,\ x,\ y) + \ldots + M_{n,34}(j/n,\ k/n,\ x,\ y),$$

$$M_{n,31}(j/n,\ k/n,\ x,\ y) = n^{-1}\sum_{|\nu|=2}\frac{D_x^\nu\tilde{p}_n(j/n,\ k/n,\ x,\ y)}{\nu!}\Bigg\{[m(x)^\nu - m(y)^\nu].$$

$$\cdot\sum_{i=1}^{p}\nu_i m^{\nu-e_i}(y)[m_i(y)-m_i(x)]\Bigg\},$$

$$M_{n,32}(j/n,\ k/n,\ x,\ y) = 5n^{-5/2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i,i'=1}^{p}[m_i(x)m_{i'}(x) - m_i(y)m_{i'}(y)]$$

$$\cdot \int\int_0^1 q(x,\ \theta)\theta^{\nu-e_i-e_{i'}} D^\nu\lambda(x+\delta\tilde{h}(\theta))(1-\delta)^4 \mathrm{d}\delta\,\mathrm{d}\theta,$$

$$M_{n,33}(j/n,\ k/n,\ x,\ y) = 5n^{-5/2}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i,i'=1}^{p} m_i(y)m_{i'}(y)\int\int_0^1[q(x,\ \theta) - q(y,\ \theta)]$$

$$\cdot\ \theta^{\nu-e_i-e_{i'}} D^\nu\lambda(x+\delta\tilde{h}(\theta))(1-\delta)^4\mathrm{d}\delta\,\mathrm{d}\theta,$$

$$M_{n,34}(j/n,\ k/n,\ x,\ y) = 5n^{-3}\sum_{|\nu|=5}\frac{1}{\nu!}\sum_{i=1}^{p} m_i(x)\int\int_0^1 q(x,\ \theta)\theta^{\nu-e_i}$$

$$\cdot\sum_{|\mu|=1} D^{\nu+\mu}\lambda(x+\delta\tilde{h}(\theta))(m_i(x)-m_i(y))^\mu\delta[1-\delta]^4\mathrm{d}\delta\,\mathrm{d}\theta.$$

Here e_r denotes a p-dimensional vector with rth element equal to 1 and with all other elements equal to 0. Furthermore, for $|\nu| = 4$, $|\nu'| = 2$, we define

$$N(\nu,\ \nu') = 2^{\chi[\nu'!=1]+\chi[(\nu-\nu')!=1]-2}$$

where $\chi[\cdot]$ denotes an indicator function. We put $m(x)^\nu = m_1(x)^{\nu_1}\cdot\ldots\cdot m_p(x)^{\nu_p}$ and $m(x)^\nu = 0$ and $\mu_\nu(x) = 0$ if at least one of the coordinates of ν is negative. We also define the functions

$$\lambda(z) = \tilde{p}_n((j+1)/n,\ k/n,\ z,\ y),$$
$$\tilde{h}(\theta) = n^{-1}m(y) + n^{-1/2}\theta.$$

For all $j < k$, x and y, the function ζ_ρ is defined as in Lemma 5.2. Here again ρ denotes the term $\rho = [(k-j)/n]^{1/2}$. For $j = k-1$ and $l = 1, \ldots, 3$, we define

$$M_{n,l}(j/n,\ k/n,\ x,\ y) = 0.$$

The proof of Lemma 5.4 is based on some lengthy calculations. It follows the lines of the proof of Lemma 3.9 in Konakov and Mammen (2000). The difference is that this time we use higher-order Taylor expansions. Then we replace $\lambda(x) = \tilde{p}_n((j+1)/n,\ k/n,\ x,\ y)$ by $\tilde{p}_n(j/n,\ k/n,\ x,\ y)$ in $(L-\tilde{L})\lambda(x)$ and in the expressions for $M_{n,11}$, $M_{n,12}$, $M_{n,21}$ and $M_{n,31}$. To this end we use the Taylor expansion for λ in the formula

$$D_x^\nu\tilde{p}_n(j/n,\ k/n,\ x,\ y) = \int q(y,\ \theta)D_x^\nu\lambda(x+\tilde{h}(\theta))\mathrm{d}\theta.$$

Lemma 5.5. *For some constant C,*

$$|M_{n,l}(j/n,\ k/n,\ x,\ y)| \leqslant Cn^{-(l-1)/2}\rho^{-1}\zeta_\rho^{S-1}(y-x), \qquad \textit{for } l = 1, 2, 3, \tag{12}$$

$$|D_x^a D_y^b M_{n,l}(j/n,\ k/n,\ x,\ y)| \leqslant C\rho^{-|a|-|b|-1}\zeta_\rho^{S-|a|-1}(y-x), \qquad \textit{for } l = 1, 2, 3, \tag{13}$$

$$|D_x^a D_y^b H_n(j/n,\ k/n,\ x,\ y)| \leqslant C\rho^{-|a|-|b|-1}\zeta_\rho^{S-|a|-1}(y-x), \tag{14}$$

$$|D_x^a D_y^b K_n(j/n,\ k/n,\ x,\ y)| \leqslant C\rho^{-|a|-|b|-1}\zeta_\rho^{S-|a|-1}(y-x), \tag{15}$$

$$\begin{aligned} &|D_x^a D_y^b[\tilde{p} \otimes_n (K_n + M_n)^{(r)}](j/n,\ k/n,\ x,\ y)| \\ &\quad \leqslant C^r B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right)\rho^{r-1-|a|-|b|}\zeta_\rho^{S-|a|-2}(y-x), \end{aligned} \tag{16}$$

for all $j < k$ and y. Here again $\rho = [(k-j)/n]^{1/2}$.

Proof. For $a, b = 0$ claims (14) and (15) were shown in Konakov and Mammen (2000, Lemma 3.10). For a proof of these claims for $|a| > 0$ or $|b| > 0$ and for the proofs of (12) and (16), one proceeds similarly. □

6. Proof of Theorem 4.1

The main tools for the proof of Theorem 4.1 were given in Sections 5.1–5.3. From Lemmas 3.1 and 3.2 we obtain that

$$p(0,\ 1,\ x,\ y) = \sum_{r=0}^{n} \tilde{p} \otimes H^{(r)}(0,\ 1,\ x,\ y) + \frac{1}{n^2} R_n(x,\ y),$$

where $R_n(x, y)$ is a function with sub-Gaussian tails, that is, for constants C, C',

$$|R_n(x,\ y)| \leqslant C\exp[-C'(x-y)^2].$$

With Lemma 5.1, this gives

$$p(0,\ 1,\ x,\ y) - p_n(0,\ 1,\ x,\ y) = T_1 + \ldots + T_5 + n^{-2}R_n(x,\ y), \tag{17}$$

where

$$T_1 = \sum_{r=0}^{n} \tilde{p} \otimes H^{(r)}(0, 1, x, y) - \sum_{r=0}^{n} \tilde{p} \otimes_n H^{(r)}(0, 1, x, y),$$

$$T_2 = \sum_{r=0}^{n} \tilde{p} \otimes_n H^{(r)}(0, 1, x, y) - \sum_{r=0}^{n} \tilde{p} \otimes_n (H + M_n + n^{-1/2} N_1)^{(r)}(0, 1, x, y),$$

$$T_3 = \sum_{r=0}^{n} \tilde{p} \otimes_n (H + M_n + n^{-1/2} N_1)^{(r)}(0, 1, x, y) - \sum_{r=0}^{n} \tilde{p} \otimes_n (K_n + M_n)^{(r)}(0, 1, x, y),$$

$$T_4 = \sum_{r=0}^{n} \tilde{p} \otimes_n (K_n + M_n)^{(r)}(0, 1, x, y) - \sum_{r=0}^{n} \tilde{p}_n \otimes_n (K_n + M_n)^{(r)}(0, 1, x, y),$$

$$T_5 = \sum_{r=0}^{n} \tilde{p}_n \otimes_n (K_n + M_n)^{(r)}(0, 1, x, y) - \sum_{r=0}^{n} \tilde{p}_n \otimes_n H_n^{(r)}(0, 1, x, y).$$

Here we put $N_1(s, t, x, y) = (L - \tilde{L})\tilde{\pi}_1(s, t, x, y)$. We now discuss the asymptotic behaviour of the terms $T_1, \ldots, T_5$.

Asymptotic treatment of T_1. Using Theorem 2.1 and the remark following Theorem 1.1 in Konakov and Mammen (2002), we obtain that

$$T_1 = \frac{1}{2n} p \otimes_n (L - \tilde{L})^2 \tilde{p} \otimes_n \Phi(0, 1, x, y) + \frac{1}{n^2} R_n(x, y),$$

where $R_n(x, y)$ is a function with sub-Gaussian tails, that is, for constants C, C',

$$|R_n(x, y)| \leqslant C \exp[-C'(x - y)^2]$$

and where $\Phi(s, t, x, y) = \sum_{r=0}^{\infty} H^{(r)}(s, t, x, y)$.

Asymptotic treatment of T_2. We will show the following expansion of T_2 for a constant $C > 0$ and for $\delta > 0$ small enough:

$$\Bigg| T_2 - 4 \sum_{r=0}^{\infty} \tilde{p} \otimes_n H^{(r)}(0, 1, x, y) + \sum_{r=0}^{\infty} \tilde{p} \otimes_n (H + M_{n,11} + n^{-1/2} N_1)^{(r)}(0, 1, x, y) + \sum_{r=0}^{\infty} \tilde{p} \otimes_n (H + M_{n,12})^{(r)}(0, 1, x, y) + \sum_{r=0}^{\infty} \tilde{p} \otimes_n (H + M_{n,21})^{(r)}(0, 1, x, y) + \sum_{r=0}^{\infty} \tilde{p} \otimes_n (H + M_{n,31})^{(r)}(0, 1, x, y) \Bigg| \leqslant C n^{-1-\delta} \zeta(y - x). \quad (18)$$

For the terms on the left-hand side of (18) we will show the following bounds with a constant $C > 0$:

$$|\tilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1 + H)^{(r)}(0, k/n, x, y) - \tilde{p} \otimes_n H^{(r)}(0, k/n, x, y)|$$

$$\leqslant \frac{C^r}{\sqrt{n}} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right)\left(\frac{k}{n}\right)^{(r-1)/2} \zeta^{r+1,0,k}(y-x), \tag{19}$$

$$|\tilde{p} \otimes_n (M_{n,12} + H)^{(r)}(0, k/n, x, y) - \tilde{p} \otimes_n H^{(r)}(0, k/n, x, y)|$$

$$\leqslant \frac{C^r}{n} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right)\left(\frac{k}{n}\right)^{(r-2)/2} \zeta^{r+1,0,k}(y-x), \tag{20}$$

$$|\tilde{p} \otimes_n (H + M_{n,21})^{(r)}(0, k/n, x, y) - \tilde{p} \otimes_n H^{(r)}(0, k/n, x, y)|$$

$$\leqslant \frac{C^r}{n} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right)\left(\frac{k}{n}\right)^{(r-1)/2} \zeta^{r+1,0,k}(y-x), \tag{21}$$

$$|\tilde{p} \otimes_n (H + M_{n,31})^{(r)}(0, k/n, x, y) - \tilde{p} \otimes_n H^{r}(0, k/n, x, y)|$$

$$\leqslant \frac{C^r}{n} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right)\left(\frac{k}{n}\right)^{(r-1)/2} \zeta^{r+1,0,k}(y-x), \tag{22}$$

where

$$\zeta^{l,j,k}(x) = \max\{\zeta_{\rho_1} * \ldots * \zeta_{\rho_l}(x)\colon \rho_1 \geqslant 0, \ldots, \rho_l \geqslant 0, \rho_1^2 + \ldots + \rho_l^2 = (k-j)/n\},$$

$$\zeta(x) = [1 + \|x\|^{2s'}]^{-1}\left\{\int [1 + \|u\|^{2s'}]^{-1} \mathrm{d}u\right\}^{-1}.$$

We now give the proofs of (18), (19) and (20). The proofs of claims (21) and (22) are omitted. These claims follow by arguments similar to those used to prove (19) and (20). All claims are proved iteratively by induction. In the induction steps the bounds given in (19)–(22) are used for $1 \leqslant k \leqslant n-1$. Note that in (18) the terms only appear for $k = n$. We start by proving (19) and (20). The proof of (18) will be given afterwards.

We first prove (19) for $r = 1$. For this purpose we write

$$\tilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1)(0, k/n, x, y) = n^{-1/2} \sum_{|\nu|=3} \frac{1}{\nu!} S_\nu,$$

where

$$S_\nu = \frac{1}{n}\sum_{j=0}^{k-1}\left\{\int \tilde{p}(0, j/n, x, u) D_u^\nu \tilde{p}_n(j/n, k/n, u, y)[\mu_\nu(u) - \mu_\nu(y)]\mathrm{d}u\right.$$

$$+ \chi_\nu(y)\rho^2 \int \tilde{p}(0, j/n, x, u)\left[\sum_{i=1}^{p} D_u^{\nu+e_i}\tilde{p}(j/n, k/n, u, y)(m_i(u) - m_i(y))\right.$$

$$\left.\left. + \frac{1}{2}\sum_{i,l=1}^{p} D_u^{\nu+e_i+e_l}\tilde{p}(j/n, k/n, u, y)(\sigma_{il}(u) - \sigma_{il}(y))\right]\mathrm{d}u\right\}. \tag{23}$$

We now decompose the summand

$$S_\nu = \frac{1}{n}\sum_{j\in J_1}\int \ldots \mathrm{d}u + \frac{1}{n}\sum_{j\in J_2}\int \ldots \mathrm{d}u,$$

where $J_1 = \{j : 0 \leqslant j \leqslant (k-1)/2\}$ and $J_2 = \{j : (k-1)/2 < j \leqslant k-1\}$. With $\rho = \sqrt{k/n - j/n}$ and $\kappa = \sqrt{k/n}$ we obtain from Lemma 5.3, (23) and (A2), for constants C and C_1,

$$\left|n^{-1}\sum_{j\in J_1}\int \ldots \mathrm{d}u\right| \leqslant Cn^{-1}\sum_{j\in J_1}\rho^{-2}\zeta_\kappa(y-x)$$

$$\leqslant C\zeta_\kappa(y-x)\int_0^{k/(2n)}\frac{\mathrm{d}t}{(k/n - t)}$$

$$= C\zeta_\kappa(y-x)\ln(2)$$

$$\leqslant C_1\zeta^{2,0,k}(y-x)B\left(\frac{1}{2}, \frac{1}{2}\right).$$

Furthermore, with $e_r + e_s \leqslant \nu$ (componentwise),

$$\left|n^{-1}\sum_{j\in J_2}\int \ldots \mathrm{d}u\right| = \left|n^{-1}\sum_{j\in J_2}\left\{\int D_u^{e_r+e_s}[\tilde{p}(0, j/n, x, u)\{\mu_\nu(u) - \mu_\nu(y)\}]\right.\right.$$

$$\cdot D_u^{\nu-e_r-e_s}\tilde{p}_n(j/n, k/n, u, y)\mathrm{d}u.$$

$$+ \chi_\nu(y)\rho^2\sum_{i=1}^{p}\int D_u^{e_r+e_s}[\tilde{p}(0, j/n, x, u)\{m_i(u) - m_i(y)\}] \cdot D_u^{\nu+e_i-e_r-e_s}\tilde{p}(j/n, k/n, u, y)\mathrm{d}u$$

$$+\chi_\nu(y)\rho^2\frac{1}{2}\sum_{i,l=1}^{p}\int D_u^{e_r+e_s}[\tilde{p}(0, j/n, x, u)\{\sigma_{il}(u)-\sigma_{il}(y)\}]$$

$$\cdot D_u^{\nu+e_i+e_l-e_r-e_s}\tilde{p}(j/n, k/n, u, y)\mathrm{d}u\Bigg\}\Bigg|$$

$$\leqslant C_2 n^{-1}\sum_{j\in J_2}\left[\frac{1}{j/n}+\frac{1}{\sqrt{j/n}\sqrt{k/n-j/n}}\right]\zeta^{2,0,k}(y-x)$$

$$\leqslant C_3\zeta^{2,0,k}(y-x)$$

for constants $C_2,\ C_3>0$. Combining the last two estimates, we obtain that

$$|S_\nu|\leqslant C_4\zeta^{2,0,k}(y-x)B\left(\frac{1}{2},\frac{1}{2}\right),$$

for some constant C_4. This shows claim (19) for $r=1$.

We now check the claim for $r=2$. We have

$$\tilde{p}\otimes_n(M_{n,11}+n^{-1/2}N_1+H)^{(2)}-\tilde{p}\otimes_n H^{(2)}$$

$$=\tilde{p}\otimes_n(M_{n,11}+n^{-1/2}N_1)\otimes_n(M_{n,11}+n^{-1/2}N_1+H)$$

$$+\tilde{p}\otimes_n H\otimes_n(M_{n,11}+n^{-1/2}N_1).$$

The first term on the right-hand side can be bounded as follows

$$|\tilde{p}\otimes_n(M_{n,11}+n^{-1/2}N_1)\otimes_n(M_{n,11}+n^{-1/2}N_1+H)(0, k/n, x, y)|$$

$$\leqslant\frac{C}{\sqrt{n}}\frac{1}{n}\sum_{i=0}^{k-1}\int\zeta^{2,0,i}(z-x)B\left(\frac{1}{2},\frac{1}{2}\right)\frac{1}{\sqrt{k/n-i/n}}\zeta^{1,i,k}(y-z)\mathrm{d}z$$

$$\leqslant\frac{C}{\sqrt{n}}\zeta^{3,0,k}(y-x)B\left(\frac{1}{2},\frac{1}{2}\right)\frac{1}{n}\sum_{i=0}^{k-1}\frac{1}{\sqrt{k/n-i/n}}$$

$$\leqslant\frac{C}{\sqrt{n}}\left(\frac{k}{n}\right)^{1/2}\zeta^{3,0,k}(y-x)B\left(\frac{1}{2},\frac{1}{2}\right)B\left(1,\frac{1}{2}\right).$$

For the second term we obtain

$$|\tilde{p} \otimes_n H \otimes_n (M_{n,11} + n^{-1/2} N_1)(0, k/n, x, y)|$$

$$= \Bigg| n^{-1/2} \sum_{|\nu|=3} \frac{1}{\nu!} \frac{1}{n} \sum_{j=0}^{k-1} \Bigg\{ \int (\tilde{p} \otimes_n H)(0, j/n, x, u) D_u^\nu \tilde{p}_n(j/n, k/n, u, y)[\mu_\nu(u) - \mu_\nu(y)] \mathrm{d}u$$

$$+ \chi_\nu(y)\rho^2 \int (\tilde{p} \otimes_n H)(0, j/n, x, u) \Bigg[\sum_{i=1}^{p} D_u^{\nu+e_i} \tilde{p}(j/n, k/n, u, y)(m_i(u) - m_i(y))$$

$$+ \frac{1}{2} \sum_{i,l=1}^{p} D_u^{\nu+e_i+e_l} \tilde{p}(j/n, k/n, u, y)(\sigma_{il}(u) - \sigma_{il}(y)) \Bigg] \mathrm{d}u \Bigg\} \Bigg|$$

$$\leqslant n^{-1/2} \sum_{|\nu|=3} \frac{1}{\nu!} \frac{1}{n} \sum_{j \in J_1} \left| \int \ldots \mathrm{d}u \right| + n^{-1/2} \sum_{|\nu|=3} \frac{1}{\nu!} \frac{1}{n} \sum_{j \in J_2} \left| \int \ldots \mathrm{d}u \right|.$$

These two terms can be treated as in the proof for $r = 1$. The first term can be bounded by use of direct estimates. The second term can be easily bounded after two applications of partial integrations.

The proof for $r \geqslant 2$ follows by iteration and use of similar methods.

The proof of claim (20) follows along similar lines to the proof of (19). Again the region of the summation is split into two regions, J_1 and J_2. Again, for the treatment of the second sum partial integration is used.

Turning to the proof of (18), this expansion immediately follows from the following bounds:

$$|\tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + M_{n,12} + M_{n,13} + H)^r(0, k/n, x, y)$$
$$- \tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + M_{n,12} + H)^{(r)}(0, k/n, x, y)|$$
$$\leqslant \frac{C^r}{n^{3/2}} \log(n) B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right) \left(\frac{k}{n}\right)^{(r-3)/2} \zeta^{r+1,0,k}(y - x), \tag{24}$$

$$|\tilde{p} \otimes_n (H + M_{n,1} + n^{-1/2} N_1 + M_{n,2})^{(r)}(0, k/n, x, y)$$
$$- \tilde{p} \otimes_n (H + M_{n,1} + n^{-1/2} N_1 + M_{n,21})^{(r)}(0, k/n, x, y)|$$
$$\leqslant \frac{C^r}{n^{3/2}} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right) \left(\frac{k}{n}\right)^{(r-1)/2} \zeta^{r+1,0,k}(y - x), \tag{25}$$

$$|\tilde{p} \otimes_n (H + M_{n,1} + n^{-1/2} N_1 + M_{n,2} + M_{n,3})^{(r)}(0, k/n, x, y)$$
$$- \tilde{p} \otimes_n (H + M_{n,1} + n^{-1/2} N_1 + M_{n,2} + M_{n,31})^{(r)}(0, k/n, x, y)|$$
$$\leqslant \frac{C^r}{n^{3/2}} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right) \left(\frac{k}{n}\right)^{(r-1)/2} \zeta^{r+1,0,k}(y - x), \tag{26}$$

$$
\begin{aligned}
&|\tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + M_{n,12} + H)^{(r)}(0, 1, x, y) \\
&\quad - \tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + H)^{(r)}(0, 1, x, y) \\
&\quad - [\tilde{p} \otimes_n (M_{n,12} + H)^{(r)} - \tilde{p} \otimes_n H^{(r)}](0, 1, x, y)| \\
&\leqslant \frac{C^r}{n^{3/2-\varepsilon}} B(\varepsilon, \varepsilon) \cdot \ldots \cdot B\left(\frac{1}{2} + r\varepsilon, \varepsilon\right) \zeta(y - x),
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&|\tilde{p} \otimes_n (M_{n,1} + n^{-1/2} N_1 + M_{n,21} + H)^{(r)}(0, 1, x, y) \\
&\quad - \tilde{p} \otimes_n (M_{n,1} + n^{-1/2} N_1 + H)^{(r)}(0, 1, x, y) \\
&\quad - [\tilde{p} \otimes_n (H + M_{n,21})^{(r)} - \tilde{p} \otimes_n H^{(r)}](0, 1, x, y)| \\
&\leqslant \frac{C^r}{n^{3/2-\varepsilon}} B(1, \varepsilon) \cdot \ldots \cdot B(1 + r\varepsilon, \varepsilon) \zeta(y - x),
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
&|\tilde{p} \otimes_n (M_{n,1} + n^{-1/2} N_1 + M_{n,2} + M_{n,31} + H)^{(r)}(0, 1, x, y) \\
&\quad - \tilde{p} \otimes_n (M_{n,1} + n^{-1/2} N_1 + M_{n,2} + H)^{(r)}(0, 1, x, y) \\
&\quad - [\tilde{p} \otimes_n (H + M_{n,31})^{(r)} - \tilde{p} \otimes_n H^{(r)}](0, 1, x, y)| \\
&\leqslant \frac{C^r}{n^{3/2-\varepsilon}} B(1, \varepsilon) \cdot \ldots \cdot B(1 + r\varepsilon, \varepsilon) \zeta(y - x).
\end{aligned} \tag{29}
$$

These estimates are valid for any $\varepsilon \in (0, \frac{1}{2})$ with a constant $C(\varepsilon) < \infty$ depending on ε. We will prove (24) and (27). The proofs of the other claims are quite similar to the proofs of (19), (24) and (27) and will be omitted.

We prove (24) in two steps. For $r = 1$, we obtain by use of direct bounds,

$$
|\tilde{p} \otimes_n M_{n,13}(0, k/n, x, y)| \leqslant \frac{C}{n^{3/2}} \left(\frac{k}{n}\right)^{-1} B\left(\frac{1}{2}, \frac{1}{2}\right) \zeta^{2,0,k}(y - x).
$$

For $r = 2$ we use the bound

$$
\sum_{j=1}^{k/2} \frac{1}{n} \left(\frac{j}{n}\right)^{-1} \leqslant C \log n.
$$

This gives the additional log factor in (24). The rest of the proof proceeds along the lines of the proof of (19).

Turning to the proof of (27), denote the expression inside the modulus bars in that inequality by Γ_r. Then we have the following reccurrence formula:

$$\Gamma_r = \Gamma_{r-1} \otimes_n H + [\tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + M_{n,12} + H)^{(r-1)}.$$
$$- \tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + H)^{(r-1)}] \otimes_n (M_{n,11} + n^{-1/2} N_1)$$
$$+ [\tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + M_{n,12} + H)^{(r-1)}.$$
$$- \tilde{p} \otimes_n (M_{n,12} + H)^{(r-1)}] \otimes_n M_{n,12} = I + II + III. \tag{30}$$

Note that $\Gamma_0 = \Gamma_1 = 0$. We start with the estimation of the second summand II in (30). Let

$$II = \frac{1}{n} \sum_{j \in J_1} \int \ldots \, \mathrm{d}u + \frac{1}{n} \sum_{j \in J_2} \int \ldots \, \mathrm{d}u.$$

The following bound is a modification of (20):

$$\Bigg| \tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + M_{n,12} + H)^{(r)} \left(0, \frac{k}{n}, x, y\right)$$
$$- \tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + H)^{(r)} \left(0, \frac{k}{n}, x, y\right) \Bigg|$$
$$\leqslant \frac{C^r}{n} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right) \left(\frac{k}{n}\right)^{(r-2)/2} \zeta^{r+1,0,k}(y - x). \tag{31}$$

This claim can be proved similarly to (19). Again the sum is split into two regions, J_1 and J_2, and for the second sum partial integration is used. For the partial integration we make use of the following bounds that easily follow from the definition of $M_{n,11}$:

$$|D_y^a D_x^b (M_{n,11} + n^{-1/2} N_1)(j/n, k/n, x, y)| \leqslant \frac{\mathrm{C}_1}{n^{1/2}} \left(\frac{k-j}{n}\right)^{-(2+|a|+|b|)/2} \zeta_\rho(y - x),$$
$$|D_x^b (M_{n,11} + n^{-1/2} N_1)(j/n, k/n, x, x + v)| \leqslant \frac{C_2}{n^{1/2}} \left(\frac{k-j}{n}\right)^{-1} \zeta_\rho(v),$$

for some constants C_1, $C_2 > 0$. We also have that for some constants C_3 and C_4,

$$|D_y^a D_x^b (\tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1)^{(r)})(0, k/n, x, y)|$$
$$\leqslant \frac{C_3^r}{n^{1/2}} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right) \left(\frac{k}{n}\right)^{-(|a|+|b|+1-r)/2} \zeta_\rho(y - x), \tag{32}$$
$$|D_x^b (\tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1)^{(r)})(0, k/n, x, x + v)|$$
$$\leqslant \frac{C_4^r}{n^{1/2}} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right) \left(\frac{k}{n}\right)^{(r-1)/2} \zeta_\rho(v). \tag{33}$$

Inequalities (32) and (33) can be shown as in the proof of (5.7) and (5.8) in Konakov and Mammen (2002).

Using (31) we obtain, for $r \geqslant 2$,

$$\left|\frac{1}{n}\sum_{j\in J_1}\int \ldots \mathrm{d}u\right| \leqslant \frac{C^{r-1}}{n} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r-1}{2}, \frac{1}{2}\right)$$

$$\times \frac{1}{n}\sum_{j\in J_1}\left(\frac{j}{n}\right)^{(r-3)/2} \frac{1}{n^{1/2}(k/n - j/n)}\int \zeta^{r,0,j}(u-x)\zeta(y-u)\mathrm{d}u$$

$$\leqslant \frac{C_1^r}{n^{3/2}} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right)\left(\frac{k}{n}\right)^{(r-3)/2} \zeta^{r+1,0,k}(y-x). \tag{34}$$

For $j \in J_2$ we have, with $\varepsilon \in (0, \frac{1}{2})$,

$$\left|\frac{1}{n}\sum_{j\in J_2}\int \ldots \mathrm{d}u\right| \leqslant \frac{C^{r-1}}{n} B(\varepsilon, \varepsilon) B\left(\frac{1}{2}+\varepsilon, \varepsilon\right) \cdot \ldots \cdot B\left(\frac{1}{2}+(r-2)\varepsilon, \varepsilon\right)$$

$$\times \frac{1}{n}\sum_{j\in J_2}\left(\frac{j}{n}\right)^{(r-2)\varepsilon} \frac{1}{n^{1/2-\varepsilon}(k/n - j/n)^{1-\varepsilon}}\int \zeta^{r,0,j}(u-x)\zeta(y-u)\mathrm{d}u$$

$$\leqslant \frac{C_1^r}{n^{3/2-\varepsilon}} B(\varepsilon, \varepsilon) B\left(\frac{1}{2}+\varepsilon, \varepsilon\right) \cdot \ldots \cdot B\left(\frac{1}{2}+(r-1)\varepsilon, \varepsilon\right)\left(\frac{k}{n}\right)^{(r-1)\varepsilon} \zeta^{r+1,0,k}(y-x). \tag{35}$$

To estimate the term *III* in (30) we use the following estimate for the derivatives:

$$|D_y^{e_i}[\tilde{p} \otimes_n (M_{n,11} + n^{-1/2}N_1 + M_{n,12} + H)^{(r)} - \tilde{p} \otimes_n (M_{n,12} + H)^{(r)}](0, k/n, x, y)|$$

$$\leqslant \frac{C^r}{n^{1/2}} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r}{2}, \frac{1}{2}\right)\left(\frac{k}{n}\right)^{(r-2)/2} \zeta^{r+1,0,k}(y-x). \tag{36}$$

Inequality (36) can be shown by induction on r. The basic tools are integration by parts and the following estimates for H and $\tilde{p}_n$:

$$\left|D_y^{e_i} H\left(\frac{j}{n}, \frac{k}{n}, x, y\right) + D_x^{e_i} H\left(\frac{j}{n}, \frac{k}{n}, x, y\right)\right| \leqslant \frac{C}{\rho}\phi_{C,\rho}(y-x), \tag{37}$$

$$\left|D_y^{e_i} D_x^{\nu}\tilde{p}_n\left(\frac{j}{n}, \frac{k}{n}, x, y\right) + D_x^{e_i} D_x^{\nu}\tilde{p}_n\left(\frac{j}{n}, \frac{k}{n}, x, y\right)\right| \leqslant \frac{C}{\rho}\zeta_\rho(y-x). \tag{38}$$

Inequality (37) is contained in Lemma 3.4 in Konakov and Mammen (2000). Inequality (38) can be shown by direct calculations. The proof uses the representation of $\tilde{p}_n$ and of its derivatives (with respect to covariance and mean) from Lemma 3.7 in Konakov and Mammen (2000). To estimate the derivatives we also use Lemma 4 from Konakov and Molchanov (1984). We omit the details. For estimating the term *III* in (30) we again split the summation region

$$III = \frac{1}{n}\sum_{j\in J_1}\int \ldots \mathrm{d}u + \frac{1}{n}\sum_{j\in J_2}\int \ldots \mathrm{d}u.$$

To estimate $(1/n)\sum_{j\in J_1}\int \ldots \mathrm{d}u$ we use the direct estimate

$$\Bigg|\tilde{p} \otimes_n (M_{n,11} + n^{-1/2} N_1 + M_{n,12} + H)^{(r-1)}\left(0, \frac{k}{n}, x, y\right)$$

$$- \tilde{p} \otimes_n (M_{n,12} + H)^{(r-1)}\left(0, \frac{k}{n}, x, y\right)\Bigg|$$

$$\leqslant \frac{C^{r-1}}{n^{1/2}} B\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{r-1}{2}, \frac{1}{2}\right)\left(\frac{k}{n}\right)^{(r-2)/2} \zeta^{r,0,k}(y-x).$$

To estimate $(1/n)\sum_{j \in J_2} \int \ldots \mathrm{d}u$ we apply integration by parts and (36) several times. This completes the proof of (27).

Asymptotic treatment of T_3. We will show that

$$T_3 = \sum_{r=1}^{n} \tilde{p} \otimes_n H^{(r)}(0, 1, x, y) - \sum_{r=1}^{n} \tilde{p} \otimes_n \left[H + \frac{1}{n} N_2\right]^{(r)}(0, 1, x, y) + R_n^*(x, y), \tag{39}$$

with $N_2(s, t, x, y) = (L - \tilde{L})\tilde{\pi}_2(s, t, x, y)$, $|R_n^*(x, y)| \leqslant C n^{-1-\delta} \xi(y-x)$ for $\delta > 0$ small enough, and a constant C depending on δ. For the proof of (39) it suffices to show that, for δ small enough,

$$\left|\sum_{r=1}^{n} \tilde{p} \otimes_n (H + M_n + n^{-1/2} N_1 + n^{-1} N_2)^{(r)}(0, 1, x, y) - \sum_{r=1}^{n} \tilde{p} \otimes_n (K_n + M_n)^{(r)}(0, 1, x, y)\right|$$

$$\leqslant C n^{-1-\delta} \xi(y-x), \tag{40}$$

$$\Bigg|\sum_{r=1}^{n} \tilde{p} \otimes_n (H + M_n + n^{-1/2} N_1)^{(r)}(0, 1, x, y).$$

$$- \sum_{r=1}^{n} \tilde{p} \otimes_n (H + M_n + n^{-1/2} N_1 + n^{-1} N_2)^{(r)}(0, 1, x, y)$$

$$- \left[\sum_{r=1}^{n} \tilde{p} \otimes_n H^{(r)}(0, 1, x, y) - \sum_{r=1}^{n} \tilde{p} \otimes_n (H + n^{-1} N_2)^{(r)}(0, 1, x, y)\right]\Bigg|$$

$$\leqslant C n^{-1-\delta} \xi(y-x). \tag{41}$$

Let us prove (40). Denote, for $0 \leqslant m \leqslant n$,

$$D_{3,m}(0, j/n, x, y) = \sum_{r=0}^{m} [\tilde{p} \otimes_n (K_n + M_n)^{(r)}(0, j/n, x, y)$$

$$- \tilde{p} \otimes_n (H + M_n + n^{-1/2} N_1 + n^{-1} N_2)^{(r)}(0, j/n, x, y)].$$

Then we have to show that

$$|D_{3,n}(0, 1, x, y)| \leqslant C n^{-1-\delta} \xi(y-x).$$

We now make iterative use of

$$D_{3,m} = D_{3,m-1} \otimes_n (H + M_n + n^{-1/2}N_1 + n^{-1}N_2) + h_{m-1},$$

for $m = 1, 2, \ldots,$ where

$$\begin{aligned} h_m(0, j/n, x, y) &= -\sum_{r=0}^{m} \tilde{p} \otimes_n (K_n + M_n)^{(r)} \otimes_n (H - K_n + n^{-1/2}N_1 + n^{-1}N_2)(0, j/n, x, y) \\ &= S_{n,m} \otimes_n (L - \tilde{L})\mathrm{d}_n(0, j/n, x, y) \end{aligned}$$

with

$$\mathrm{d}_n = \tilde{p}_n - \tilde{p} - n^{-1/2}\tilde{\pi}_1 - n^{-1}\tilde{\pi}_2,$$
$$S_{n,m}(0, i/n, x, y) = \sum_{r=0}^{m} \tilde{p} \otimes_n (K_n + M_n)^{(r)}(0, i/n, x, y).$$

Iterative application of this equation gives

$$D_{3,n}(0, 1, x, y) = \sum_{r=0}^{n-1} h_r \otimes_n (H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(0, 1, x, y). \tag{42}$$

To prove (40) we will show that

$$\begin{aligned} &|h_r \otimes_n (H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}(0, 1, x, y)| \\ &\leqslant n^{-1-\delta}C^{n-r-1}B\left(1, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{n-r-1}{2}, \frac{1}{2}\right)\zeta(y - x). \end{aligned} \tag{43}$$

For this purpose we decompose the left-hand side of (43) into four terms:

$$a_{r,1} = \sum_{0\leqslant i\leqslant n/2} \frac{1}{n}\int h_r\left(0, \frac{i}{n}, x, u\right)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}\left(\frac{i}{n}, 1, u, y\right)\mathrm{d}u,$$

$$a_{r,2} = \sum_{n/2\leqslant i\leqslant n} \frac{1}{n^2}\sum_{0\leqslant k\leqslant i/2}\int S_{n,r}\left(0, \frac{k}{n}, x, v\right)$$

$$\times (L-\tilde{L})\mathrm{d}_n\left(\frac{k}{n}, \frac{i}{n}, v, u\right)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}\left(\frac{i}{n}, 1, u, y\right)\mathrm{d}v\,\mathrm{d}u,$$

$$a_{r,3} = \sum_{n/2< i\leqslant n} \frac{1}{n^2}\sum_{i/2<k\leqslant i-n^{\delta'}}\int (L^t-\tilde{L}^t)S_{n,r}\left(0, \frac{k}{n}, x, v\right)$$

$$\times \mathrm{d}_n\left(\frac{k}{n}, \frac{i}{n}, v, u\right)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}\left(\frac{i}{n}, 1, u, y\right)\mathrm{d}v\,\mathrm{d}u,$$

$$a_{r,4} = \sum_{n/2< i\leqslant n} \frac{1}{n^2}\sum_{i-n^{\delta'}<k\leqslant i-1}\int (L^t-\tilde{L}^t)S_{n,r}\left(0, \frac{k}{n}, x, v\right)$$

$$\times \mathrm{d}_n\left(\frac{k}{n}, \frac{i}{n}, v, u\right)(H + M_n + n^{-1/2}N_1 + n^{-1}N_2)^{(n-r-1)}\left(\frac{i}{n}, 1, v, y\right)\mathrm{d}v\,\mathrm{d}u.$$

Here L^t and $\tilde{L}^t$ denote the adjoint operators of L and $\tilde{L}$. Note that

$$(L^t-\tilde{L}^t)f(s, t, x, u) = \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial u_i\partial u_j}[f(s, t, x, u)$$

$$\times (\sigma_{ij}(u)-\sigma_{ij}(y))] - \sum_i \frac{\partial}{\partial u_i}[f(s, t, x, u)(m_i(u)-m_i(y))].$$

In particular, we have

$$h_r = (L^t-\tilde{L}^t)S_{n,r}\otimes_n \mathrm{d}_n.$$

For the proof of (43) it suffices to show, for $j = 1, \ldots, 4$, that

$$|a_{r,j}| \leqslant n^{-1-\delta}C^{n-r-1}B\left(1, \frac{1}{2}\right)\cdot\ldots\cdot B\left(\frac{n-r-1}{2}, \frac{1}{2}\right)\zeta(y-x). \tag{44}$$

For $j = 2$, claim (44) follows from (12) and (15) by noting that, for $k \leqslant i/2$, $n/2 < i$,

$$\left|S_{n,r}\left(0, \frac{k}{n}, x, v\right)\right| \leqslant C\zeta^{S-2}_{(k/n)^{1/2}}(v-x),$$

$$\left|(H+M_n+n^{-1/2}N_1+n^{-1}N_2)^{(n-r-1)}\left(\frac{i}{n}, 1, u, y\right)\right|$$

$$\leqslant C^{n-r-1}\left(1-\frac{i}{n}\right)^{-1/2}B\left(1, \frac{1}{2}\right)\cdot\ldots\cdot B\left(\frac{n-r-1}{2}, \frac{1}{2}\right)\zeta_{((n-i)/n)^{1/2}}(y-u),$$

$$\left|(L-\tilde{L})\mathrm{d}_n\left(\frac{k}{n}, \frac{i}{n}, v, u\right)\right| \leqslant Cn^{-3/2}\zeta^{S-8}_{((i-k)/n)^{1/2}}(v-u). \tag{45}$$

Let $j=3$. We apply the fact that $i/2<k\leqslant i-n^{\delta'}$, $n/2<i$. It follows from (11), (13), (15) and (45) that

$$|(L^t-\tilde{L}^t)S_{n,r}(0, k/n, x, v)| \leqslant C\zeta^{S-4}_{(k/n)^{1/2}}(v-x),$$
$$|\mathrm{d}_n(k/n, i/n, v, u)| \leqslant Cn^{-3/2}\rho^{-3}\zeta^{S-6}_{\rho}(u-v)$$

with $\rho=\sqrt{(i-k)/n}$. Note that

$$\left|\frac{1}{n}\sum_{i/2<k\leqslant i-n^{\delta'}}\int(L^t-\tilde{L}^t)S_{n,r}\left(0, \frac{k}{n}, x, v\right)\mathrm{d}_n\left(\frac{k}{n}, \frac{i}{n}, v, u\right)\mathrm{d}v\right|$$

$$\leqslant \frac{1}{n}\sum_{i/2<k\leqslant i-n^{\delta'}} n^{-3/2}\rho^{-3}\zeta(u-x)$$

$$\leqslant n^{-1-(\delta')^2/2}\cdot\frac{1}{n}\sum_{i/2<k\leqslant i-n^{\delta'}}\rho^{-2+\delta'}\zeta(u-x)$$

$$\leqslant Cn^{-1-\delta''}\zeta(u-x),$$

for δ'' small enough.

Now let $j=4$. For $i-n^{\delta'}<k\leqslant i-1$, $n/2<i$, we have

$$\int (L^t - \tilde{L}^t) S_{n,r}\left(0, \frac{k}{n}, x, v\right) \tilde{p}_n\left(\frac{k}{n}, \frac{i}{n}, v, u\right) \mathrm{d}v$$

$$= \int (L^t - \tilde{L}^t) S_{n,r}\left(0, \frac{k}{n}, x, u - \frac{w}{\sqrt{n}} - \frac{m(u)}{n}\right) q^{(i-k)}(u, w) \mathrm{d}w$$

$$= (L^t - \tilde{L}^t) S_{n,r}\left(0, \frac{k}{n}, x, u - \frac{m(u)}{n}\right)$$

$$+ \frac{1}{2} \sum_{j,l=1}^{p} D_{w_j w_l} \left[(L^t - \tilde{L}^t) S_{n,r}\left(0, \frac{k}{n}, x, u - \frac{m(u)}{n}\right) \right] \frac{i-k}{n} \sigma_{j,l}(u)$$

$$+ O(n^{-1-\delta}) \zeta^{S-2}(u - x),$$

for δ small enough. Here (A2) has been applied and the equalities

$$\int q^{(i-k)}(u, w) \mathrm{d}w = 1, \qquad \int w_l q^{(i-k)}(u, w) \mathrm{d}w = 0, \qquad \int w_l w_j q^{(i-k)}(u, w) \mathrm{d}w = (i-k) \sigma_{j,l}(u)$$

used. The same expansion holds with $\tilde{p}_n$ replaced by $\tilde{p}$. Furthermore, one can show by partial integration that

$$\int (L^t - \tilde{L}^t) S_{n,j}\left(0, \frac{k}{n}, x, v\right) \left[\frac{1}{\sqrt{n}} \tilde{\pi}_1 + \frac{1}{n} \tilde{\pi}_2 \right] \left(\frac{k}{n}, \frac{i}{n}, v, u\right) \mathrm{d}v = O(n^{-1-\delta}) \zeta_{(i/n)^{1/2}}(u - x),$$

for δ small enough. Hence, (44) holds for $j = 4$.

Finally, let $j = 1$. We define

$$a_{r,5} = \sum_{0 \leqslant i \leqslant n/2} \frac{1}{n^2} \sum_{i - n^{\delta'} < k \leqslant i-1} \int (L^t - \tilde{L}^t) S_{n,r}\left(0, \frac{k}{n}, x, v\right)$$

$$\cdot \mathrm{d}_n\left(\frac{k}{n}, \frac{i}{n}, v, u\right) (H + M_n + n^{-1/2} N_1 + n^{-1} N_2)^{(n-r-1)}\left(\frac{i}{n}, 1, u, y\right) \mathrm{d}v \, \mathrm{d}u.$$

By integrating by parts with respect to v and by using (45) and arguments as in the proof of (44) for $j = 4$, one can show that

$$|a_{r,5}| \leqslant O(n^{-1-\delta}) B\left(1, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{n-r-1}{2}, \frac{1}{2}\right) \zeta(y - x),$$

for δ small enough. Now, by using arguments as in the proof of (44) for $j = 3$, one can show that

$$|a_{r,4} - a_{r,5}| \leqslant O(n^{-1-\delta}) B\left(1, \frac{1}{2}\right) \cdot \ldots \cdot B\left(\frac{n-r-1}{2}, \frac{1}{2}\right) \xi(y - x),$$

for δ small enough. This shows (44) for $j = 1$.

Now for (39) it remains to show (41). This can be done by arguments as in the proof of (27).

Asymptotic treatment of T_4. We will show that

$$T_4 = -n^{-1/2}\sum_{r=0}^{\infty}\tilde{\pi}_1 \otimes_n (H + M_{n,11} + n^{-1/2}N_1)^{(r)}(0, 1, x, y)$$

$$- n^{-1}\sum_{r=0}^{\infty}\tilde{\pi}_2 \otimes_n H^{(r)}(0, 1, x, y) + R_n(x, y) \tag{46}$$

with $|R_n(x, y)| = o(n^{-1-\delta}) \cdot \zeta(y - x)$. Note, first, that with $S_n(s, t, x, y) = \sum_{r=1}^{n}(K_n + M_n)^{(r)}(s, t, x, y)$, the term T_4 can be rewritten as

$$T_4 = (\tilde{p} - \tilde{p}_n)(0, 1, x, y) + (\tilde{p} - \tilde{p}_n) \otimes_n S_n(0, 1, x, y).$$

To prove (46), we start by showing that, for $\delta > 0$ small enough (uniformly for x, $y \in \mathbb{R}^p$),

$$\left|\frac{1}{n}\sum_{1\leqslant j\leqslant n^{\delta}}\int(\tilde{p}_n - \tilde{p})\left(0, \frac{j}{n}, x, u\right)S_n\left(\frac{j}{n}, 1, u, y\right)\mathrm{d}u\right| \leqslant Cn^{-1-\delta'}\zeta(y - x), \tag{47}$$

for δ' small enough. For the proof of (47) we will show that, uniformly for $1 \leqslant j \leqslant n^{\delta}$ and for x, $y \in \mathbb{R}^p$,

$$\int\tilde{p}_n(0, j/n, x, u)S_n(j/n, 1, u, y)\mathrm{d}u = S_n(j/n, 1, x, y) + o(n^{-\delta}\zeta(y - x)), \tag{48}$$

$$\int\tilde{p}(0, j/n, x, u)S_n(j/n, 1, u, y)\mathrm{d}u = S_n(j/n, 1, x, y) + o(n^{-\delta}\zeta(y - x)). \tag{49}$$

Claim (47) immediately follows from (48) and (49).

We now show (48) for $j = 1$. The proof for $j \geqslant 1$ and for (49) follows along the same lines and, in particular, makes use of the last condition in (A2). For the proof we will make use of the fact that, for all $1 \leqslant j \leqslant n$ and all x, $y \in \mathbb{R}^p$ and $|\nu| = 1$,

$$|D_x^{\nu}S_n(j/n, 1, x, y)| \leqslant C(1 - j/n)^{-1}\zeta_{\rho}(y - x) \tag{50}$$

for some constant $C > 0$. Claim (50) can be shown with the same arguments as in the proof of (5.7) in Konakov and Mammen (2002). Note that the function Φ in that paper has a similar structure to S_n. For $1 \leqslant j \leqslant n^{\delta}$, the bound (50) immediately implies, for a constant $C' > 0$,

$$|D_x^{\nu}S_n(j/n, 1, x, y)| \leqslant C'\zeta(y - x). \tag{51}$$

We have $\tilde{p}_n(0, 1/n, x, u) = n^{p/2}q[u, \sqrt{n}(u - x - n^{-1}m(u))]$. Denote the determinant of the Jacobian matrix of $u - n^{-1}m(u)$ by Δ_n. So, because of (A2) and (51), and substituting $w = \sqrt{n}(u - x - n^{-1}m(u))$,

$$\int \tilde{p}_n(0, 1/n, x, u) S_n\left(\frac{1}{n}, 1, u, y\right) \mathrm{d}u$$

$$= \int n^{p/2} q\left(u, \sqrt{n}\left(u - x - \frac{1}{n} m(u)\right)\right) S_n\left(\frac{1}{n}, 1, u, y\right) \mathrm{d}u$$

$$= \int q\left(x + \frac{w}{\sqrt{n}} + \frac{m(u)}{n}, w\right) \Delta_n^{-1} S_n\left(\frac{1}{n}, 1, x + \frac{w}{\sqrt{n}} + \frac{m(u)}{n}, y\right) \mathrm{d}w$$

$$= \int [q(x, w) + o(n^{-1/2})(w+1)\psi(w)][1 + o(n^{-\delta})] S_n\left(\frac{1}{n}, 1, x + \frac{w}{\sqrt{n}} + \frac{m(u)}{n}, y\right) \mathrm{d}w$$

$$= S_n\left(\frac{1}{n}, 1, x, y\right) + o(n^{-\delta})\zeta(y - x).$$

From (47) we obtain that, for some $\delta > 0$,

$$T_4 = (\tilde{p} - \tilde{p}_n)(0, 1, x, y) + \frac{1}{n} \sum_{n^\delta < j \leqslant n} \int (\tilde{p} - \tilde{p}_n)\left(0, \frac{j}{n}, x, u\right) S_n\left(\frac{j}{n}, 1, u, y\right) \mathrm{d}u + R_n(x, y)$$

with $|R_n(x, y)| \leqslant C n^{-1-\delta'} \zeta(y - x)$.

We now make use of the expansion of $\tilde{p}_n - \tilde{p}$ given in Lemma 5.2 . We have, with $\rho = (j/n)^{1/2} \geqslant n^{\delta/2 - 1/2}$,

$$\left| \frac{1}{n} \sum_{j=n^\delta}^{n} n^{-3/2} \rho^{-3} \int \zeta_\rho(u - x) S_n\left(\frac{j}{n}, 1, u, y\right) \mathrm{d}u \right|$$

$$\leqslant C n^{-1-\delta''} \left| n^{-1} \sum_{j=n^\delta}^{n} \rho^{-2+\delta'} \int \zeta_\rho(u - x) S_n\left(\frac{j}{n}, 1, u, y\right) \mathrm{d}u \right|,$$

where $\delta' < \delta(1-\delta)^{-1}$, and $\delta'' > 0$ is small enough. Now using arguments similar to the proof of (19), we obtain that

$$\left| n^{-1} \sum_{j=1}^{n} \rho^{-2+\delta'} \int \zeta_\rho(u - x) S_n\left(\frac{j}{n}, 1, u, y\right) \mathrm{d}u \right| \leqslant C\zeta(y - x),$$

for a constant C. This shows that

$$T_4 = -\left[n^{-1/2}\tilde{\pi}_1 + n^{-1}\tilde{\pi}_2\right](0, 1, x, y)$$

$$-\frac{1}{n} \sum_{j=n^\delta}^{n} \int \left[n^{-1/2}\tilde{\pi}_1 + n^{-1}\tilde{\pi}_2\right]\left(0, \frac{j}{n}, x, u\right) S_n\left(\frac{j}{n}, 1, u, y\right) \mathrm{d}u + R_n{}'(x, y),$$

with $|R_n{}'(x, y)| \leqslant C n^{-1-\delta''} \zeta(y - x)$.

Claim (46) now follows from (47) by application of the expansions of K_n, used above.

Asymptotic treatment of T_5. From Lemma 5.4 we immediately obtain that

$$|T_5| \leqslant O(n^{-3/2}\zeta(y-x)).$$

Plugging in the asymptotic expansions of $T_1, \ldots, T_5$. We now plug the asymptotic expansions of $T_1, \ldots, T_5$ into (17). This gives

$$\begin{aligned} p_n(x, y) - p(x, y) \cong n^{-1/2} \sum_{r=0}^{\infty} \tilde{\pi}_1 \otimes_n (H + M_{n,11} + n^{-1/2} N_1)^{(r)}(0, 1, x, y) \\ + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + M_{n,11} + n^{-1/2} N_1)^{(r)} - H^{(r)}](0, 1, x, y) \\ + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + M_{n,12})^{(r)} - H^{(r)}](0, 1, x, y) \\ + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + M_{n,21})^{(r)} - H^{(r)}](0, 1, x, y) \\ + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + M_{n,31})^{(r)} - H^{(r)}](0, 1, x, y) \\ + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + n^{-1} N_2)^{(r)} - H^{(r)}](0, 1, x, y) \\ + \frac{1}{n} \sum_{r=0}^{\infty} \tilde{\pi}_2 \otimes_n H^{(r)}(0, 1, x, y) - \frac{1}{2n} p \otimes_n (L - \tilde{L})^2 \tilde{p} \otimes_n \Phi(0, 1, x, y), \end{aligned} \tag{52}$$

where $\cong$ denotes an equality up to terms that are smaller in absolute value than $Cn^{-1-\delta} \exp[-C'(x-y)^2]$ for positive constants C, C' and δ.

To prove Theorem 4.1 it remains to show that the right-hand side of (52) can be approximated by $n^{-1/2}\pi_1(x, y) + n^{-1}\pi_2(x, y)$. We prove this claim in three steps. First, we prove that $\tilde{p}_n(j/n, k/n, x, y)$ can be replaced by $\tilde{p}(j/n, k/n, x, y)$ in $M_{n,11}$, $M_{n,12}$, $M_{n,21}$ and $M_{n,31}$. Then, we show that the convolution operator $\otimes_n$ can be replaced by the operator $\otimes$ in (52). Finally, we show that the resulting expression is asymptotically equivalent to $n^{-1/2}\pi_1(x, y) + n^{-1}\pi_2(x, y)$.

Asymptotic replacement of $\tilde{p}_n$ by $\tilde{p}$. We now show that

$$\begin{aligned} p_n(x, y) - p(x, y) \cong n^{-1/2}[\tilde{\pi}_1 + p^{\mathrm{d}} \otimes_n \mathfrak{R}_1] \otimes_n \Phi(0, 1, x, y) \\ + n^{-1}\{[\tilde{\pi}_2 + \tilde{\pi}_1 \otimes_n \Phi \otimes_n \mathfrak{R}_1 + p^{\mathrm{d}} \otimes_n \mathfrak{R}_2 + p^{\mathrm{d}} \otimes_n \mathfrak{R}_3] \otimes_n \Phi(0, 1, x, y). \\ + p^{\mathrm{d}} \otimes_n (\mathfrak{R}_1 \otimes_n \Phi)^{(2)}(0, 1, x, y) + \tfrac{1}{2} p^{\mathrm{d}} \otimes_n (L_*^2 - L^2) p^{\mathrm{d}}(0, 1, x, y)\}, \end{aligned} \tag{53}$$

where

$$\mathfrak{R}_1(s, t, x, y) = (L - \tilde{L})\tilde{\pi}_1(s, t, x, y) + M_3(s, t, x, y) - \tilde{M}_3(s, t, x, y),$$

$$\mathfrak{R}_2(s, t, x, y) = (L - \tilde{L})\tilde{\pi}_2(s, t, x, y) + M_4(s, t, x, y) - \tilde{M}_4(s, t, x, y),$$

$$\mathfrak{R}_3(s, t, x, y) = \sum_{|\nu|=4} \frac{1}{\nu!} D_x^\nu \tilde{p}(s, t, x, y)(\chi_\nu(x) - \chi_\nu(y)),$$

$$M_3(s, t, x, y) = \sum_{|\nu|=3} \frac{\mu_\nu(x)}{\nu!} D_x^\nu \tilde{p}(s, t, x, y),$$

$$\tilde{M}_3(s, t, x, y) = \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} D_x^\nu \tilde{p}(s, t, x, y),$$

$$M_4(s, t, x, y) = \sum_{|\nu|=3} \frac{\mu_\nu(x)}{\nu!} D_x^\nu \tilde{\pi}_1(s, t, x, y),$$

$$\tilde{M}_4(s, t, x, y) = \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} D_x^\nu \tilde{\pi}_1(s, t, x, y),$$

$$p^{\mathrm{d}}(s, t, x, y) = (\tilde{p} \otimes_n \Phi)(s, t, x, y).$$

We will make use of the approximation

$$\begin{aligned} p_n(x, y) - p(x, y) \cong\ & n^{-1/2} \sum_{r=0}^{\infty} \tilde{\pi}_1 \otimes_n (H + N_{n,11} + n^{-1/2} N_1)^{(r)}(0, 1, x, y) \\ & + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + N_{n,11} + n^{-1/2} N_1)^{(r)} - H^{(r)}](0, 1, x, y) \\ & + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + N_{n,12})^{(r)} - H^{(r)}](0, 1, x, y) \\ & + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + N_{n,21})^{(r)} - H^{(r)}](0, 1, x, y) \\ & + \sum_{r=0}^{\infty} \tilde{p} \otimes_n [(H + N_{n,31})^{(r)} - H^{(r)}](0, 1, x, y) \end{aligned}$$

$$+\sum_{r=0}^{\infty}\tilde{p}\otimes_n[(H+n^{-1}N_2)^{(r)}-H^{(r)}](0,1,x,y)$$

$$+\sum_{r=0}^{\infty}\tilde{p}\otimes_n[(H+n^{-1}N_3)^{(r)}-H^{(r)}](0,1,x,y)$$

$$+\frac{1}{n}\sum_{r=0}^{\infty}\tilde{\pi}_2\otimes_n H^{(r)}(0,1,x,y)-\frac{1}{2n}p\otimes_n(L-\tilde{L})^2\tilde{p}\otimes_n\Phi(0,1,x,y), \tag{54}$$

where $N_3(s,t,x,y)=M_4(s,t,x,y)-\tilde{M}_4(s,t,x,y)$ and where $N_{n,ij}$ is defined as $M_{n,ij}$ but with $\tilde{p}_n$ replaced by $\tilde{p}$.

A proof of (54) is given below. To simplify (54) we make use of the identities

$$\sum_{r=0}^{\infty}\tilde{p}\otimes_n[(H+n^{-1}N)^{(r)}-H^{(r)}](0,1,x,y)$$

$$=\sum_{r=1}^{\infty}\frac{1}{n^r}p^{\mathrm{d}}\otimes_n(N\otimes_n\Phi)^{(r)}(0,1,x,y), \tag{55}$$

$$\sum_{r=0}^{\infty}\tilde{p}\otimes_n[(H+n^{-1/2}N')^{(r)}-H^{(r)}](0,1,x,y)$$

$$=\sum_{r=1}^{\infty}\frac{1}{n^{r/2}}p^{\mathrm{d}}\otimes_n(N'\otimes_n\Phi)^{(r)}(0,1,x,y), \tag{56}$$

where N is one of the functions $nN_{n,12}$, $nN_{n,21}$, $nN_{n,31}$, N_2 or N_3 and where $N'=n^{1/2}N_{n,11}+N_1$. These identities follow from linearity of $\otimes_n$ by simple calculations. We will show the following approximations for the right-hand sides of (55) and (56):

$$\sum_{r=1}^{\infty}\frac{1}{n^r}p^{\mathrm{d}}\otimes_n(N\otimes_n\Phi)^{(r)}(0,1,x,y)\cong\frac{1}{n}p^d\otimes_n N\otimes_n\Phi(0,1,x,y), \tag{57}$$

$$\sum_{r=1}^{\infty}\frac{1}{n^{r/2}}p^{\mathrm{d}}\otimes_n(N'\otimes_n\Phi)^{(r)}(0,1,x,y)\cong\frac{1}{n^{1/2}}p^d\otimes_n N\otimes_n\Phi(0,1,x,y)$$

$$+\frac{1}{n}p^d\otimes_n(N'\otimes_n\Phi)^{(2)}(0,1,x,y). \tag{58}$$

We prove (57) for $N=nN_{n,12}$. The proofs for the other cases are essentially the same. For $r=1$ and $|\nu|=4$ it is sufficient to estimate

$$\sum_{j=0}^{k-1}\frac{1}{n}\int p^{\mathrm{d}}\left(0,\frac{j}{n},x,z\right)D_z^{\nu}\tilde{p}\left(\frac{j}{n},\frac{k}{n},z,y\right)(\mu_\nu(z)-\mu_\nu(y))\mathrm{d}z.$$

Splitting the last sum into two sums

$$\sum_{j=0}^{k-1} \cdots = \sum_{\{j: j/n \leqslant k/2n\}} \cdots + \sum_{\{j: j/n > k/2n\}} \cdots,$$

we obtain by applying integration by parts and using Theorem 2.3 in Konakov and Mammen (2002) that

$$\left| (p^{\mathrm{d}} \otimes_n N \otimes_n \Phi)\left(0, \frac{k}{n}, x, y\right) \right| \leqslant C\left(\frac{k}{n}\right)^{-1/2} \phi_{C,(k/n)^{1/2}}(y-x).$$

This bound implies

$$\left| \frac{1}{n^2} p^{\mathrm{d}} \otimes_n (N \otimes_n \Phi)^{(2)}\left(0, \frac{k}{n}, x, y\right) \right| = \left| \frac{1}{n^{1+\delta}} (p^{\mathrm{d}} \otimes_n N \otimes_n \Phi) \otimes_n (n^{-1+\delta} N \otimes_n \Phi) \right|$$

$$\leqslant \frac{C^2}{n^{1+\delta}} B\left(\frac{1}{2}, \varepsilon\right)\left(\frac{k}{n}\right)^{-1/2+\varepsilon} \phi_{C,(k/n)^{1/2}}(y-x),$$

for ε small enough. Iterative application of similar arguments for $r \geqslant 2$ gives

$$\left| \frac{1}{n^r} p^{\mathrm{d}} \otimes_n (N \otimes_n \Phi)^{(r)}(0, \frac{k}{n}, x, y) \right|$$

$$\leqslant \frac{C^r}{n^{1+\delta}} B\left(\frac{1}{2}, \varepsilon\right) B\left(\frac{1}{2}+\varepsilon, \varepsilon\right) \ldots B\left(\frac{1}{2}+(r-2)\varepsilon, \varepsilon\right)\left(\frac{k}{n}\right)^{-1/2+(r-1)\varepsilon} \phi_{C,(k/n)^{1/2}}(y-x). \tag{59}$$

The bound (59) immediately implies (57) for $N = nN_{n,12}$.

By plugging (57) and (58) into (54) and taking into account the relation

$$\frac{1}{2n}(L_* - \tilde{L})^2 \tilde{p}\left(\frac{j}{n}, \frac{k}{n}, x, y\right) + \frac{1}{n} \mathfrak{R}_3\left(\frac{j}{n}, \frac{k}{n}, x, y\right)$$

$$= (N_{n,12} + N_{n,21} + N_{n,31})\left(\frac{j}{n}, \frac{k}{n}, x, y\right),$$

we obtain (53) by collecting similar terms. So it remains to show (54).

To do so, we shall use the recurrence relation

$$\left[\sum_{r=0}^{n} \tilde{p} \otimes_n (H + M_{n,12})^{(r)} - \sum_{r=0}^{n} \tilde{p} \otimes_n (H + N_{n,12})^{(r)}\right](0, 1, x, y)$$

$$= \left[\sum_{r=0}^{n-1} \tilde{p} \otimes_n (H + M_{n,12})^{(r)} - \sum_{r=0}^{n-1} \tilde{p} \otimes_n (H + N_{n,12})^{(r)}\right]$$

$$\otimes_n (H + M_{n,12})(0, 1, x, y) + S_n \otimes_n (M_{n,12} - N_{n,12})(0, 1, x, y), \tag{60}$$

where $S_n(s, t, x, y) = \sum_{r=0}^{n-1} \tilde{p} \otimes_n (H + N_{n,12})^{(r)}(s, t, x, y)$. For fixed ν, $|\nu| = 4$, we consider

$$\frac{1}{n^2}\sum_{j=1}^{n-1}\int S_n\left(0,\frac{j}{n},x,u\right)G_\nu(u,y)\left(D_u^\nu\tilde{p}_n\left(\frac{j}{n},1,u,y\right)-D_u^\nu\tilde{p}\left(\frac{j}{n},1,u,y\right)\right)\mathrm{d}u$$

$$=\frac{1}{n^2}\sum_{n-n^\delta\leqslant j<n}\ldots+\frac{1}{n^2}\sum_{0\leqslant j<n-n^\delta}\ldots=I+II, \tag{61}$$

where

$$G_\nu(u,y)=\mu_\nu(u)-\mu_\nu(y)-\sum_{|\nu'|=2}\nu!N(\nu,\nu')\mu_{\nu'}(y)(\mu_{\nu-\nu'}(u)-\mu_{\nu-\nu'}(y))$$

with $N(\nu,\nu')$ defined as in the statement of Lemma 5.4.

We start by estimating I. For the summand in I for $j=n-1$ we have to consider $\tilde{p}_n(n-1/n,1,u,y)=n^{p/2}q[y,\sqrt{n}(y-u-m(y)/n)]$. With the substitution $w=\sqrt{n}(y-u-m(y)/n)$ and using integration by parts, one obtains

$$\int S_n\left(0,\frac{n-1}{n},x,u\right)G_\nu(u,y)D_u^\nu\tilde{p}_n\left(\frac{n-1}{n},1,u,y\right)\mathrm{d}u$$

$$=n^{p/2}\int D_u^\nu\left[S_n\left(0,\frac{n-1}{n},x,u\right)G_\nu(u,y)\right]q\left(y,\sqrt{n}\left(y-u-\frac{1}{n}m(y)\right)\right)\mathrm{d}u$$

$$=\int D_u^\nu\left[S_n\left(0,\frac{n-1}{n},x,y-\frac{w}{\sqrt{n}}-\frac{m(y)}{n}\right)G_\nu\left(y-\frac{w}{\sqrt{n}}-\frac{m(y)}{n},y\right)\right]q(y,w)\mathrm{d}w$$

$$=D_u^\nu\left[S_n\left(0,\frac{n-1}{n},x,u\right)G_\nu(u,y)\right]\Big|_{u=y}+o(n^{-\delta})\zeta(y-x).$$

Analogously,

$$\int S_n\left(0,\frac{n-1}{n},x,u\right)G_\nu(u,y)D_u^\nu\tilde{p}\left(\frac{n-1}{n},1,u,y\right)\mathrm{d}u$$

$$=D_u^\nu\left[S_n\left(0,\frac{n-1}{n},x,u\right)G_\nu(u,y)\right]\Big|_{u=y}+o(n^{-\delta})\zeta(y-x).$$

These two expansions imply that the summand in I for $j=n-1$ is smaller than $Cn^{-2-\delta'}\zeta(y-x)$, for δ' small enough. For $n-n^\delta\leqslant j\leqslant n-2$, one can show the same bound. This implies that

$$|I|\leqslant Cn^{-1-\delta'}\zeta(y-x), \tag{62}$$

for δ' small enough.

We now treat the sum II. For $0\leqslant j<n-n^\delta$ we have that $\rho_2=\sqrt{1-j/n}\geqslant n^{(\delta-1)/2}$. By applying integration by parts we obtain

$$II = \frac{1}{n^2} \sum_{0 \leqslant j < n-n^\delta} \int D_u^{e_l} \left[S_n\left(0, \frac{j}{n}, x, u\right) G_\nu(u, y) \right] D_u^{\nu - e_l} \left[\tilde{p}_n\left(\frac{j}{n}, 1, u, y\right) - \tilde{p}\left(\frac{j}{n}, 1, u, y\right) \right] \mathrm{d}u$$

$$= \frac{1}{n^2} \sum_{0 \leqslant j < n-n^\delta} \int D_u^{e_l} \left[S_n\left(0, \frac{j}{n}, x, u\right) \right] G_\nu(u, y) D_u^{\nu - e_l} \left[\tilde{p}_n\left(\frac{j}{n}, 1, u, y\right) - \tilde{p}\left(\frac{j}{n}, 1, u, y\right) \right] \mathrm{d}u$$

$$+ \frac{1}{n^2} \sum_{0 \leqslant j < n-n^\delta} \int D_u^{e_q} \left[S_n\left(0, \frac{j}{n}, x, u\right) \right] D_u^{e_l} G_\nu(u, y) D_u^{\nu - e_l - e_q} \left[\tilde{p}_n\left(\frac{j}{n}, 1, u, y\right) - \tilde{p}\left(\frac{j}{n}, 1, u, y\right) \right] \mathrm{d}u$$

$$+ \frac{1}{n^2} \sum_{0 \leqslant j < n-n^\delta} \int S_n\left(0, \frac{j}{n}, x, u\right) D_u^{e_l + e_q} G_\nu(u, y) D_u^{\nu - e_l - e_q} \left[\tilde{p}_n\left(\frac{j}{n}, 1, u, y\right) - \tilde{p}\left(\frac{j}{n}, 1, u, y\right) \right] \mathrm{d}u$$

$$= I' + II' + III'.$$

For I' one can show that $|I'| \leqslant Cn^{-1-\delta}\zeta(y-x)$, with δ small enough. The summands II' and III' can be bounded similarly. Because of the expansion given in Lemma 5.2, this only requires application of the estimate

$$\frac{1}{n^2} \sum_{0 \leqslant j < n-n^\delta} n^{-3/2} \rho_2^{-5} \int \left| D_u^{e_l} \left[S_n\left(0, \frac{j}{n}, x, u\right) \right] \right| \zeta_{\rho_2}(y-u) \mathrm{d}u$$

$$\leqslant Cn^{-1-\delta'} \frac{1}{n} \sum_{j=1}^{n} \rho_1^{-1} \rho_2^{-2+\delta''} \int \zeta_{\rho_1}(u-x) \zeta_{\rho_2}(y-u) \mathrm{d}u$$

$$\leqslant C_1 n^{-1-\delta'} \zeta(y-x),$$

with $\rho_1 = \sqrt{j/n}$ and δ' and δ'' small enough. With the resulting bound on II and with (61) and (62) we obtain

$$|S_n \otimes_n (M_{n,12} - N_{n,12})(0, 1, x, y)| \leqslant C_n n^{-1-\delta'} \zeta(y-x), \tag{63}$$

where $\sum_{n=1}^{\infty} C_n < \infty$. From iterations of (60) and (63) we obtain

$$\left| \left[\sum_{r=0}^{\infty} \tilde{p} \otimes_n (H + M_{n,12})^{(r)} - \sum_{r=0}^{\infty} \tilde{p} \otimes_n (H + N_{n,12})^{(r)} \right] (0, 1, x, y) \right| \leqslant Cn^{-1-\delta'} \zeta(y-x).$$

For the terms in (52) that contain $M_{n,21}$ and $M_{n,31}$ analogous estimates can be obtained for the errors if $M_{n,21}$ and $M_{n,31}$ are replaced by $N_{n,21}$ and $N_{n,31}$, respectively. In $M_{n,11}$ we replace $\tilde{p}_n$ by $\tilde{p} + n^{1/2}\tilde{\pi}_1$ and we obtain a similar bound for the resulting error. By collecting these bounds we obtain (54).

In the next step of the proof of Theorem 4.1 we will replace p^d by p in our expansion of $p_n - p$.

Asymptotic replacement of p^d by p. We now show that in (53) p^d can be replaced by p. This gives the expansion

$$
\begin{aligned}
p_n(x, y) - p(x, y) \cong{}& n^{-1/2}[\tilde{\pi}_1 + p \otimes_n \mathfrak{R}_1] \otimes_n \Phi(0, 1, x, y)\\
&+ n^{-1}\{[\tilde{\pi}_2 + \tilde{\pi}_1 \otimes_n \Phi \otimes_n \mathfrak{R}_1 + p \otimes_n \mathfrak{R}_2 + p \otimes_n \mathfrak{R}_3] \otimes_n \Phi(0, 1, x, y)\\
&+ p \otimes_n (\mathfrak{R}_1 \otimes_n \Phi)^{(2)}(0, 1, x, y) + \tfrac{1}{2} p \otimes_n (L_*^2 - L^2) p(0, 1, x, y)\}. \qquad (64)
\end{aligned}
$$

The proof of claim (64) immediately follows from the formula

$$
\begin{aligned}
&\left| p\left(0, \frac{l}{n}, x, y\right) - p^d\left(0, \frac{l}{n}, x, y\right)\right|\\
&\leqslant \frac{C}{n^{1-\varepsilon}} B\left(\frac{1}{2}, \frac{\varepsilon}{2}\right) B\left(\frac{\varepsilon+1}{2}, \frac{\varepsilon}{2}\right)\left(\frac{l}{n}\right)^{\varepsilon - 1/2} \phi_{C,(l/n)^{1/2}}(y - x), \qquad \varepsilon \in \left(0, \frac{1}{2}\right). \qquad (65)
\end{aligned}
$$

To prove (65) we proceed as in the proof of Theorem 2.1 in Konakov and Mammen (2002). This gives the relation

$$
\begin{aligned}
&p\left(0, \frac{l}{n}, x, y\right) - p^d\left(0, \frac{l}{n}, x, y\right)\\
&= \frac{1}{2n} p \otimes_n (L - \tilde{L})^2 \tilde{p} \otimes_n \Phi\left(0, \frac{l}{n}, x, y\right) + \frac{1}{n^2} R\left(0, \frac{l}{n}, x, y\right), \qquad (66)
\end{aligned}
$$

where

$$
\begin{aligned}
R\left(0, \frac{l}{n}, x, y\right) ={}& \frac{1}{2}\sum_{i=0}^{l-1} \frac{1}{n} \sum_{j=0}^{i-1} \int_{j/n}^{(j+1)/n} \left[n\left(u - \frac{j}{n}\right)\right]^2 \mathrm{d}u \int_0^1 (1 - \delta)\mathrm{d}\delta \int \mathrm{d}z\\
&\cdot \left[\int p(0, s_j, x, v)(L - \tilde{L})^3 \tilde{p}\left(s_j, \frac{i}{n}, v, z\right)\mathrm{d}v \Phi\left(\frac{i}{n}, \frac{l}{n}, z, y\right)\right] \qquad (67)
\end{aligned}
$$

with $s_j = s_j(u, \delta) = j/n + \delta(u - j/n)$. By iteratively using integration by parts in (67), a derivative operator of order 3 can be transferred from $\tilde{p}$ to Φ and a derivative operator of order 1 can be transferred from $\tilde{p}$ to p. We also make use of the inequality

$$
|D_\xi^b \tilde{p}(s, t, \xi, \xi + x)| \leqslant C_1 (t - s)^{-p/2} \exp\left\{-C_2 \frac{\|x\|^2}{(t - s)}\right\}.
$$

This enables us to pass from derivatives with respect to v to derivatives with respect to z. Using beta functions to bound the integrals appearing in the definition of $R(0, \frac{l}{n}, x, y)$, one can show that $n^{-2} R(0, \frac{l}{n}, x, y)$ is bounded by the right-hand side of (65). The first summand in the right-hand side of (66) can be estimated analogously. For the proof of this claim with the help of integration by parts a derivative operator of order 1 is transferred from $\tilde{p}$ to Φ and a derivative operator of order 2 from $\tilde{p}$ to p. By using linearity of $\otimes_n$ and by applying (65) we easily obtain that p^d can be replaced by p. To prove that $\otimes_n$ can be replaced by $\otimes$ in the last summand of (53), we proceed as in the proof of Theorem 2.1 in Konakov and Mammen (2002). This gives the inequality

$$|p \otimes_n (L_*^2 - L^2)p(0, 1, x, y) - p \otimes (L_*^2 - L^2)p(0, 1, x, y)| \leqslant \frac{C}{n}\phi_{C,1}(y - x).$$

We now come to the next modification of our expansion of $p_n - p$.

Asymptotic replacement of $p \otimes_n (\mathfrak{R}_1 \otimes_n \Phi)^{(2)}(0, 1, x, y)$ *by* $p \otimes (\mathfrak{R}_1 \otimes_n \Phi)^{(2)}(0, 1, x, y)$ *and of* $p \otimes_n \mathfrak{R}_i \otimes_n \Phi(0, 1, x, y)$ *by* $p \otimes \mathfrak{R}_i \otimes_n \Phi(0, 1, x, y)$, $i = 1, 2, 3$. We now show the expansion

$$\begin{aligned} p_n(x, y) - p(x, y) &\cong n^{-1/2}[\tilde{\pi}_1 + p \otimes \mathfrak{R}_1] \otimes_n \Phi(0, 1, x, y) \\ &+ n^{-1}\{[\tilde{\pi}_2 + \tilde{\pi}_1 \otimes_n \Phi \otimes_n \mathfrak{R}_1 + p \otimes \mathfrak{R}_2 + p \otimes \mathfrak{R}_3] \otimes_n \Phi(0, 1, x, y) \\ &+ p \otimes (\mathfrak{R}_1 \otimes_n \Phi)^{(2)}(0, 1, x, y) + \tfrac{1}{2}p \otimes (L_*^2 - L^2)p(0, 1, x, y)\}. \end{aligned} \tag{68}$$

This follows from the estimates

$$|p \otimes \mathfrak{R}_i \otimes_n \Phi(0, 1, x, y) - p \otimes_n \mathfrak{R}_i \otimes_n \Phi(0, 1, x, y)| \leqslant C(\varepsilon)n^{-1+\varepsilon}\phi_{C,1}(y - x) \tag{69}$$

for $i = 1, 2, 3$ and for $\varepsilon \in (0, \frac{1}{2})$. Thus it remains to prove (69), which we do for $i = 1$. The proof for $i = 2, 3$ is quite similar. Because of linearity of $\otimes$ it is sufficient to consider the differences corresponding to the four summands in the definition of $\mathfrak{R}_1(s, t, x, y)$. The proof for the four summands is quite similar. We only consider the difference $p \otimes \tilde{L}\tilde{\pi}_1 \otimes_n \Phi(0, 1, x, y) - p \otimes_n \tilde{L}\tilde{\pi}_1 \otimes_n \Phi(0, 1, x, y)$. As in the proof of Theorem 2.1 in Konakov and Mammen (2002), with H replaced by $\tilde{L}\tilde{\pi}_1 \otimes_n \Phi$, we obtain

$$\begin{aligned} &p \otimes \tilde{L}\tilde{\pi}_1 \otimes_n \Phi(0, 1, x, y) - p \otimes_n \tilde{L}\tilde{\pi}_1 \otimes_n \Phi(0, 1, x, y) \\ &= \frac{1}{2n} p \otimes_n (L\tilde{L} - \tilde{L}L)\tilde{\pi}_1 \otimes_n \Phi(0, 1, x, y) + \frac{1}{n^2} R(0, 1, x, y), \end{aligned}$$

where now

$$\begin{aligned} R(0, 1, x, y) = &\frac{1}{2}\sum_{l=0}^{n-1}\frac{1}{n}\sum_{j=0}^{l-1}\int_{j/n}^{(j+1)/n}\left[n\left(u - \frac{j}{n}\right)\right]^2 \mathrm{d}u\int_0^1(1-\delta)\mathrm{d}\delta\int \mathrm{d}z \\ &\cdot\left[\int p(0, s_j, x, v)(L\tilde{L} - \tilde{L}L)^2\tilde{\pi}_1\left(s_j, \frac{l}{n}, v, z\right)\mathrm{d}v\,\Phi\left(\frac{l}{n}, 1, z, y\right)\right]. \end{aligned}$$

These terms can be bounded by using integration by parts and dividing the sums in the definition of R into appropriate partial sums. This completes the proof of (68).

We now further simplify our expansion of $p_n - p$. We prove the expansion

$$\begin{aligned} p_n(x, y) - p(x, y) \approx\ &n^{-1/2}(p \otimes \mathcal{F}_1[p_\Delta])(0, 1, x, y) \\ &+ n^{-1}\{(p \otimes \mathcal{F}_2[p_\Delta])(0, 1, x, y) + (p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p_\Delta]])(0, 1, x, y). \\ &+ \tfrac{1}{2}p \otimes (L_*^2 - L^2)p(0, 1, x, y)\}, \end{aligned} \tag{70}$$

where for $t \in \{1/n, \ldots, 1\}$, $s \in [0, t - 1/n]$

$$p_\Delta(s, t, v, y) = (\tilde{p} \otimes'_n \Phi)(s, t, v, y)$$
$$= \tilde{p}(s, t, v, y) + \sum_{j \geqslant ns}^{nt-1} \frac{1}{n} \int \tilde{p}\left(s, \frac{j}{n}, v, z\right) \Phi_1\left(\frac{j}{n}, t, z, y\right) \mathrm{d}z.$$

Here $\Phi_1 = \sum_{r=1}^{\infty} H^{(r)}$, $H^{(r)} = H^{(r-1)} \otimes_n H$, and the binary type operation $\otimes'_n$ is defined as follows:

$$(f \otimes'_n g)(s, t, x, y) = \sum_{j \geqslant ns}^{nt-1} \frac{1}{n} \int f\left(s, \frac{j}{n}, x, z\right) g\left(\frac{j}{n}, t, z, y\right) \mathrm{d}z.$$

Note that for $s \in \{1/n, \ldots, 1\}$ the operator $\otimes'_n$ coincides with $\otimes_n$.

The proof of (70), begins by noting that the linearity of $\otimes$ implies

$$(p \otimes \mathfrak{R}_1)(s, t, x, y) = (p \otimes L\tilde{\pi}_1)(s, t, x, y) - (p \otimes \tilde{L}\tilde{\pi}_1)(s, t, x, y)$$
$$+ (p \otimes M_3)(s, t, x, y) - (p \otimes \tilde{M}_3)(s, t, x, y). \tag{71}$$

We now consider the second summand on the right-hand side of (71):

$$(p \otimes \tilde{L}\tilde{\pi}_1)(s, t, x, y) = \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int_s^s \mathrm{d}\tau \int p(s, \tau, x, v)(t - \tau) D_v^\nu(\tilde{L}\tilde{p}(\tau, t, v, y) \mathrm{d}v$$
$$= \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int_s^{(s+t)/2} \cdots + \sum_{|\nu|=3} \frac{\mu_\nu(y)}{\nu!} \int_{(s+t)/2}^{t} \cdots$$
$$= I + II.$$

By application of the Kolmogorov backward and forward equations and by using integration by parts with respect to the time variable, we obtain that

$$
\begin{aligned}
I = & -\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int \mathrm{d}v\int_s^{(s+t)/2}\mathrm{d}\tau(t-\tau)p(s,\,\tau,\,x,\,v)\frac{\partial}{\partial\tau}\left(D_v^\nu\tilde{p}(\tau,\,t,\,v,\,y)\right)\\
& -\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int \mathrm{d}v\Bigg[p(s,\,\tau,\,x,\,v)(t-\tau)D_v^\nu\tilde{p}(\tau,\,t,\,v,\,y)\big|_s^{(s+t)/2}.\\
& -\int_s^{(s+t)/2}D_v^\nu\tilde{p}(\tau,\,t,\,v,\,y)\left(\frac{\partial p(s,\,\tau,\,x,\,v)}{\partial\tau}(t-\tau)-p(s,\,\tau,\,x,\,v)\right)\mathrm{d}\tau\Bigg]\\
= & -\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int\frac{(t-s)}{2}p\left(s,\,\frac{s+t}{2},\,x,\,v\right)D_v^\nu\tilde{p}\left(\frac{s+t}{2},\,t,\,v,\,y\right)\mathrm{d}v+\tilde{\pi}_1(s,\,t,\,x,\,y)\\
& +\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int_s^{(s+t)/2}\mathrm{d}\tau(t-\tau)\int L^t p(s,\,\tau,\,x,\,v)D_v^\nu\tilde{p}(\tau,\,t,\,v,\,y)\mathrm{d}v\\
& -\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int_s^{(s+t)/2}\mathrm{d}\tau\int p(s,\,\tau,\,x,\,v)D_v^\nu\tilde{p}(\tau,\,t,\,v,\,y)\mathrm{d}v. \qquad (72)
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
II = & \sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int\frac{(t-s)}{2}p\left(s,\,\frac{s+t}{2},\,x,\,v\right)D_v^\nu\tilde{p}\left(\frac{s+t}{2},\,t,\,v,\,y\right)\mathrm{d}v\\
& +\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int_{(s+t)/2}^{1}\mathrm{d}\tau(t-\tau)\int L^t p(s,\,\tau,\,x,\,v)D_v^\nu\tilde{p}(\tau,\,t,\,v,\,y)\mathrm{d}v\\
& -\sum_{|\nu|=3}\frac{\mu_\nu(y)}{\nu!}\int_{(s+t)/2}^{t}\mathrm{d}\tau\int p(s,\,\tau,\,x,\,v)D_v^\nu\tilde{p}(\tau,\,t,\,v,\,y)\mathrm{d}v. \qquad (73)
\end{aligned}
$$

Substituting $(p\otimes\tilde{L}\tilde{\pi}_1)(0,\,1,\,x,\,y)=I+II$ into (71), we obtain, after cancellation of some terms,

$$
\tilde{\pi}_1(s,\,t,\,x,\,y)+(p\otimes\mathfrak{R}_1)(s,\,t,\,x,\,y)=(p\otimes M_3)(s,\,t,\,x,\,y). \qquad (74)
$$

Similarly, by using integration by parts with respect to the time variable, we obtain

$$
(\tilde{\pi}_1\otimes_n'\Phi_1)(s,\,t,\,x,\,y)+(p\otimes\mathfrak{R}_1\otimes_n'\Phi_1)(s,\,t,\,x,\,y)=(p\otimes M_3\otimes_n'\Phi_1)(s,\,t,\,x,\,y), \qquad (75)
$$

where $t\in\{1/n,\,\ldots,\,1\}$, $s\in[0,\,t-1/n]$. From (74) and (75) we have

$$
\begin{aligned}
&(\tilde{\pi}_1\otimes_n'\Phi)(s,\,t,\,x,\,y)+(p\otimes\mathfrak{R}_1\otimes_n'\Phi)(s,\,t,\,x,\,y)\\
&\quad=(p\otimes M_3\otimes_n'\Phi)(s,\,t,\,x,\,y)=(p\otimes\mathcal{F}_1[p_\Delta])(s,\,t,\,x,\,y). \qquad (76)
\end{aligned}
$$

Using arguments similar to the proof of (76) one can show that

$$(\tilde{\pi}_2 \otimes_n' \Phi + p \otimes \mathfrak{R}_2 \otimes_n' \Phi + p \otimes \mathfrak{R}_3 \otimes_n' \Phi)(s, t, x, y)$$
$$= (p \otimes M_4 \otimes_n' \Phi + p \otimes \mathcal{F}_2[\tilde{p}] \otimes_n' \Phi)(s, t, x, y)$$
$$= (p \otimes M_4 \otimes_n' \Phi)(s, t, x, y) + (p \otimes \mathcal{F}_2[p_\Delta])(s, t, x, y). \quad (77)$$

By plugging (76) and (77) into (68) we obtain that the right-hand side of (70) is equal to

$$n^{-1/2}(p \otimes \mathcal{F}_1[p_\Delta])(0, 1, x, y) + n^{-1}\{(p \otimes \mathcal{F}_2[p_\Delta] + \tfrac{1}{2} p \otimes (L_*^2 - L^2)p$$
$$+ p \otimes M_4 \otimes_n' \Phi + \tilde{\pi}_1 \otimes_n \Phi \otimes_n \mathfrak{R}_1 \otimes_n \Phi + p \otimes (\mathfrak{R}_1 \otimes_n \Phi)^{(2)})(0, 1, x, y)\}. \quad (78)$$

For the sum of the two last terms in (78) we obtain, from (76),

$$[\tilde{\pi}_1 \otimes_n \Phi + p \otimes \mathfrak{R}_1 \otimes_n \Phi] \otimes_n (\mathfrak{R}_1 \otimes_n \Phi)(0, 1, x, y)$$
$$= \{(p \otimes \mathcal{F}_1[p_\Delta]) \otimes_n \mathfrak{R}_1 \otimes_n \Phi\}(0, 1, x, y) = p \otimes \mathcal{F}_1[p_\Delta \otimes_n' (\mathfrak{R}_1 \otimes_n \Phi)](0, 1, x, y).$$

Moreover,

$$(p \otimes M_4 \otimes_n' \Phi)(0, 1, x, y)$$
$$= \int_0^1 \mathrm{d}u \int p(0, u, x, v) \sum_{|\nu|=3} \frac{\mu_\nu(v)}{\nu!} D_v^\nu[(\tilde{\pi}_1 \otimes_n' \Phi)(u, 1, x, y)]\mathrm{d}v,$$

and, hence, the sum of the last three terms in (78) is equal to

$$(p \otimes \mathcal{F}_1[\tilde{\pi}_1 \otimes_n' \Phi + p_\Delta \otimes_n' (\mathfrak{R}_1 \otimes_n \Phi)])(0, 1, x, y).$$

For the proof of (70) it remains to show that

$$(p \otimes \mathcal{F}_1[\tilde{\pi}_1 \otimes_n' \Phi + p_\Delta \otimes_n' (\mathfrak{R}_1 \otimes_n' \Phi)])(0, 1, x, y)$$
$$\approx (p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p_\Delta]])(0, 1, x, y). \quad (79)$$

We shall show that

$$n^{-1}(p \otimes \mathcal{F}_1[(p - p_\Delta) \otimes_n' (\mathfrak{R}_1 \otimes_n \Phi)])(0, 1, x, y) \approx 0 \quad (80)$$

and

$$n^{-1}(p \otimes \mathcal{F}_1[p \otimes_n' (\mathfrak{R}_1 \otimes_n \Phi)])(0, 1, x, y) \approx n^{-1}(p \otimes \mathcal{F}_1[p \otimes (\mathfrak{R}_1 \otimes_n \Phi)])(0, 1, x, y). \quad (81)$$

Then (79) will follow from (80), (81) and (76). We now make use of the representation

$$p(u, j/n, x, y) - p_\Delta(u, j/n, x, y) = (p \otimes H - p \otimes_n' H)(u, j/n, x, y)$$
$$+ \{(p \otimes H - p \otimes_n' H) \otimes_n' \Phi_1\}(u, j/n, x, y), \quad (82)$$

where

$$(p \otimes H - p \otimes_n' H)\left(u, \frac{j}{n}, x, y\right) = \int_u^{j^*(u)/n} \mathrm{d}\tau \int p(u, \tau, x, z) H\left(\tau, \frac{j}{n}, z, y\right) \mathrm{d}z + R\left(u, \frac{j}{n}, x, y\right),$$

$$R\left(u, \frac{j}{n}, x, y\right) = \sum_{i=j^*(u)}^{j-1} \int_{i/n}^{(i+1)/n} (\tau - \frac{i}{n}) \int_0^1 \int [p(u, \tau, x, z)$$

$$\times \left(L - \tilde{L}\right)^2 \tilde{p}\left(\tau, \frac{j}{n}, z, y\right)\Big]\Big|_{\tau=\tau^*} \mathrm{d}z\, \mathrm{d}\delta\, \mathrm{d}\tau,$$

where $j^*(u) = [un] + 1$ (with the convention that $[x] = x - 1$ for $x \in \mathbb{N}$) and $\tau^* = \tau^*(i, \delta, \tau) = i/n + \delta(\tau - i/n)$. Representation (82) was obtained in the proof of Theorem 2.1 in Konakov and Mammen (2002). For the remainder term R the following estimate holds uniformly in $\delta \in [0, 1]$ and for $j \geqslant j^*(u) + 2$:

$$\left| R\left(u, \frac{j}{n}, x, y\right)\right| \leqslant \frac{C}{n} \sum_{i=j^*(u)}^{j-2} \frac{1}{n} \cdot \frac{1}{(j/n - i + 1/n)^{3/2}} \cdot \phi_\rho(y - x)$$

$$+ \int_{(j-1)/n}^{j/n} \frac{\mathrm{d}\tau}{\sqrt{j/n - \tau}} \cdot \phi_\rho(y - x) \leqslant \left\{ \frac{C}{n^{1/2-\varepsilon}} \int_u^{(j-1)/n} \frac{\mathrm{d}\tau}{(j - 1/n - \tau)^{1-\varepsilon}} + \frac{C}{\sqrt{n}} \right\} \phi_\rho(y - x)$$

$$\leqslant \frac{C}{n^{1/2-\varepsilon}} B(\varepsilon, 1) \phi_\rho(y - x), \tag{83}$$

where $\rho = \sqrt{j/n - u}$. For $j = j^*(u) + 1$ the estimate (83) follows directly from the definitions of p and p_Δ. Moreover,

$$\left| \int_u^{j^*(u)/n} \mathrm{d}\tau \int p(u, \tau, x, z) H\left(\tau, \frac{j}{n}, z, y\right) \mathrm{d}z \right| \leqslant C \sqrt{\frac{j^*(u)}{n} - u} \cdot \phi_\rho(y - x) \leqslant \frac{C}{\sqrt{n}} \cdot \phi_\rho(y - x) \tag{84}$$

and, hence, the estimate (83) holds for the first summand in (82). It is easy to obtain that the same estimate (83) remains true for the second summand in (82): it follows from the smoothing properties of the operation $\ldots \otimes_n' \Phi_1$. Hence, we obtain an estimate

$$\left| p\left(u, \frac{j}{n}, x, y\right) - p_\Delta\left(u, \frac{j}{n}, x, y\right)\right| \leqslant \frac{C}{n^{1/2-\varepsilon}} B(\varepsilon, 1) \phi_\rho(y - x). \tag{85}$$

We only sketch the proof of (80) and (81). From the definitions of $\mathfrak{R}_1$ and Φ_1 we have

$$\left| (\mathfrak{R}_1 \otimes_n \Phi_1)\left(\frac{j}{n}, 1, z, y\right)\right| \leqslant C n^\varepsilon \left(1 - \frac{j}{n}\right)^{\varepsilon - 1/2} B\left(\varepsilon, \frac{1}{2}\right) \cdot \phi_{(1-j/n)^{1/2}}(y - z), \tag{86}$$

and from (85) and (86),

$$\left| (p - p_\Delta) \otimes_n' \mathfrak{R}_1 \otimes_n \Phi_1(u, 1, v, y)\right| \leqslant \frac{C(\varepsilon)}{n^{1/2-2\varepsilon}} \cdot \phi_{(1-u)^{1/2}}(y - v). \tag{87}$$

Now for each summand in (80) we split the integral into two integrals

$$n^{-1}\int_0^1 \mathrm{d}u \int p(0, u, x, v)\frac{\mu_\nu(v)}{\nu!} D_v^\nu\left[\sum_{j=j^*(u)}^{n-1} \frac{1}{n}\int (p - p_\Delta)\left(u, \frac{j}{n}, v, z\right) \cdot (\mathfrak{R}_1 \otimes_n \Phi_1)\left(\frac{j}{n}, 1, z, y\right)\right] \mathrm{d}z$$

$$= n^{-1}\int_0^{1/2} \mathrm{d}u \ldots + n^{-1}\int_{1/2}^1 \mathrm{d}u = I + II. \tag{88}$$

By integration by parts we obtain from (87) that $II \approx 0$. To estimate I we consider two cases: (a) $j/n - u \geqslant \frac{1}{4}$; ($b$) $j/n - u < \frac{1}{4}$, $1 - j/n \geqslant \frac{1}{4}$. In case (a) we differentiate with respect to v in (88) and use (82). With the substitution $v + w' = w$, we have

$$\left| D_v^\nu\left[\int_u^{j^*(u)/n} \mathrm{d}\tau \int p(u, \tau, v, w) H\left(\tau, \frac{j}{n}, w, z\right) \mathrm{d}w\right]\right|$$

$$= \left| D_v^\nu\left[\int_u^{j^*(u)/n} \mathrm{d}\tau \int p(u, \tau, v, v + w') H\left(\tau, \frac{j}{n}, v + w', z\right) \mathrm{d}w'\right]\right|$$

$$\leqslant \frac{C}{n} \cdot \phi_{(j/n-u)^{1/2}}(z - v),$$

where we have used the fact that $j/n - \tau > \frac{1}{5}$ for $\tau \in [u, j^*(u)/n]$ for n large enough, and where we have used the inequality

$$|D_v^\nu p(u, \tau, v, v + w')| \leqslant \frac{C_1}{(\tau - u)^{p/2}} \cdot \exp\left[-C_2\frac{|w'|^2}{\tau - u}\right]. \tag{89}$$

This inequality is proved in Freedman (1964, p. 260). The other terms in (82) can be estimated analogously. Finally, in case (b) with similar substitutions we make use of (89) and of the following inequality (see Konakov and Mammen 2002):

$$|D_v^\nu H(u, \tau, v, v + w')| \leqslant \frac{C_1}{(\tau - u)^{(p+1)/2}} \cdot \exp\left[-C_2\frac{|w'|^2}{\tau - u}\right].$$

The proof of (81) is similar to the proof of Theorem 2.1 in Konakov and Mammen (2002), but with H in Konakov and Mammen (2002) replaced by $\mathfrak{R}_1$. We omit the details. This completes the proof of (70).

Asymptotic replacement of p_Δ by p. We start with a comparison of $n^{-1}(p \otimes \mathcal{F}_2[p])(0, 1, x, y)$ and $n^{-1}(p \otimes \mathcal{F}_2[p_\Delta])(0, 1, x, y)$. Note that by linearity the difference of these two terms is equal to $n^{-1}(p \otimes \mathcal{F}_2[p - p_\Delta])(0, 1, x, y)$. We will use the following simple estimates

$$\left|\int_0^{n^{-\delta}} \mathrm{d}u \int p(0, u, x, z) \cdot \chi_\nu(z) D_z^\nu p(u, 1, z, y)\mathrm{d}z\right| \leqslant Cn^{-\delta}\phi(y - x),$$

$$\left|\int_{1-n^{-\delta}}^1 \mathrm{d}u \int D_z^\nu[p(0, u, x, z) \cdot \chi_\nu(z)] \cdot p(u, 1, z, y)\mathrm{d}z\right| \leqslant Cn^{-\delta}\phi(y - x),$$

$$\left|\int_0^{n^{-\delta}} \mathrm{d}u \int p(0, u, x, z) \cdot \chi_\nu(z) D_z^\nu p_\Delta(u, 1, z, y) \mathrm{d}z\right| \leqslant C n^{-\delta} \phi(y - x),$$

$$\left|\int_{1-n^{-\delta}}^1 \mathrm{d}u \int D_z^\nu [p(0, u, x, z) \cdot \chi_\nu(z)] \cdot p_\Delta(u, 1, z, y) \mathrm{d}z\right| \leqslant C n^{-\delta} \phi(y - x).$$

From the first and third bounds we see that it suffices to consider $D_z^\nu(p_\Delta - p)(u, 1, z, y)$ for $u \in [n^{-\delta}, 1 - n^{-\delta}]$. We obtain

$$\int_{n^{-\delta}}^{1-n^{-\delta}} \mathrm{d}u \int p(0, u, x, z) \cdot \chi_\nu(z) \cdot D_z^\nu(p - p_\Delta)(u, 1, z, y) \mathrm{d}z$$

$$= \int_{n^{-\delta}}^{1/2} \mathrm{d}u \ldots + \int_{1/2}^{1-n^{-\delta}} \mathrm{d}u \ldots = I + II.$$

The relation $n^{-1} \cdot II \approx 0$ follows from (84) and from the following estimates:

$$\left|\int_{1/2}^{1-n^{-\delta}} \mathrm{d}u \int D_z^\nu [p(0, u, x, z) \cdot \chi_\nu(z)] \cdot \sqrt{\frac{j^*(u)}{n} - u} \cdot \phi_{(1-u)^{1/2}}(y - z) \mathrm{d}z\right|$$

$$\leqslant C\phi(y - x) \cdot \int_{1/2}^{1-n^{-\delta}} \sqrt{\frac{j^*(u)}{n} - u}\, \mathrm{d}u$$

$$\leqslant C\phi(y - x) \cdot \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} \sqrt{\frac{i+1}{n} - u}\, \mathrm{d}u \leqslant \frac{C}{\sqrt{n}} \phi(y - x), \tag{90}$$

$$\sum_{i=j^*(u)}^{n-1} \int_{i/n}^{(i+1)/n} (\tau - i/n) \int_0^1 \int L_v^t p(u, \tau, z, v) H(\tau, 1, v, y)|_{\tau=\tau^*} \mathrm{d}v\, \mathrm{d}\delta\, \mathrm{d}\tau$$

$$+ \sum_{i=j^*(u)}^{n-1} \int_{i/n}^{(i+1)/n} (\tau - i/n) \int_0^1 \int (\tilde{L}_v - L_v)^t p(u, \tau, z, v) \tilde{L}_v \tilde{p}(\tau, 1, v, y)]|_{\tau=\tau^*} \mathrm{d}v\, \mathrm{d}\delta\, \mathrm{d}\tau$$

$$= I' + II'. \tag{91}$$

Taking into account that $u \in [n^{-\delta}, 1 - n^{-\delta}]$, we obtain

$$|I'| \leqslant \frac{C}{n^{1-\delta}} (1 - u)^{\delta - 1/2} B\left(\delta, \frac{1}{2}\right) \cdot \phi_{\sqrt{1-u}}(y - z),$$

and an analogous estimate holds for II'. Thus, $n^{-1} \cdot II \approx 0$. For $u \in [n^{-\delta}, \frac{1}{2}]$, we have that

$$n^{-1} \cdot \int_{n^{-\delta}}^{1/2} \mathrm{d}u \int D_z^\nu [p(0, u, x, z) \cdot \chi_\nu(z)](p - p_\Delta)(u, 1, z, y) \mathrm{d}z \approx 0.$$

This can be shown by using

$$|D_z^\nu[p(0, u, x, z)\cdot\chi_\nu(z)]| \leqslant \frac{C}{u^2}\cdot\phi_{u^{1/2}}(z-x) \leqslant C\cdot n^{2\delta}\,\phi_{u^{1/2}}(z-x).$$

Thus, the only difference from the previous estimate of II is an additional factor $n^{2\delta}$, where $\delta > 0$ can be chosen arbitrary small. Using the estimates of I and II, we obtain

$$n^{-1}(p\otimes\mathcal{F}_2[p_\Delta])(0, 1, x, y) \approx n^{-1}(p\otimes\mathcal{F}_2[p])(0, 1, x, y). \tag{92}$$

We now prove that

$$\begin{aligned}&n^{-1}(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p]])(0, 1, x, y) - n^{-1}(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p_\Delta]])(0, 1, x, y)\\ &\quad = n^{-1}(p\otimes\mathcal{F}_1[p\otimes\mathcal{F}_1[p-p_\Delta]])(0, 1, x, y) \approx 0.\end{aligned} \tag{93}$$

We proceed as above. We consider a typical summand in (93):

$$\begin{aligned}&n^{-1}\int_0^1 \mathrm{d}u\int p(0, u, x, z)\mu_\nu(z)\\ &\quad\cdot D_z^\nu\left[\int_u^1 \mathrm{d}\tau\int p(u, \tau, z, v)\mu_\nu(v)D_v^\nu(p-p_\Delta)(\tau, 1, v, y)\mathrm{d}v\right]\mathrm{d}z.\end{aligned} \tag{94}$$

As in the proof of (92), it is enough to consider the integral over $u \in [n^{-\delta}, 1-n^{-\delta}]$. Now (94) is a sum of the integrals over the boundary regions and of the following integrals:

$$I_1 = n^{-1}\int_{n^{-\delta}}^{1/2}\mathrm{d}u\int\ldots D_z^\nu\int_u^{(1+u)/2}\mathrm{d}\tau\int\ldots,$$

$$I_2 = n^{-1}\int_{n^{-\delta}}^{1/2}\mathrm{d}u\int\ldots D_z^\nu\int_{(1+u)/2}^{1}\mathrm{d}\tau\int\ldots,$$

$$I_3 = n^{-1}\int_{1/2}^{1-n^{-\delta}}\mathrm{d}u\int\ldots D_z^\nu\int_u^{(1+u)/2}\mathrm{d}\tau\int\ldots,$$

$$I_4 = n^{-1}\int_{1/2}^{1-n^{-\delta}}\mathrm{d}u\int\ldots D_z^\nu\int_{(1+u)/2}^{1}\mathrm{d}\tau\int\ldots.$$

We show that $I_i \approx 0,\ i = 1, 2, 3, 4$. The proofs for all cases are similar. They use integration by parts and estimates for the derivatives of p or of $p - p_\Delta$. We consider only the case I_2. For this case $\tau - u \geqslant \frac{1}{4}$ and we obtain from (85) that

$$\begin{aligned}&n^{-1}\int_{n^{-\delta}}^{1/2}\mathrm{d}u\int p(0, u, x, z)\mu_\nu(z)\\ &\quad\cdot D_z^\nu\left[\int_{(1+u)/2}^{1}\mathrm{d}\tau\int p(u, \tau, z, v)\mu_\nu(v)D_v^\nu(p-p_\Delta)(\tau, 1, v, y)\mathrm{d}v\right]\mathrm{d}z\\ &\leqslant \frac{C(\varepsilon)}{n^{3/2-\varepsilon}}\phi(y-x) \approx 0,\end{aligned}$$

for $\varepsilon \in (0, \frac{1}{2})$. This shows (93).

We now consider the first term in (70):

$$n^{-1/2}(p \otimes \mathcal{F}_1[p_\Delta])(0, 1, x, y) = n^{-1/2}(p \otimes \mathcal{F}_1[p])(0, 1, x, y) \\ - n^{-1/2}(p \otimes \mathcal{F}_1[p - p_\Delta])(0, 1, x, y). \tag{95}$$

By (82) the last term in (95) is equal to

$$- n^{-1/2}(p \otimes \mathcal{F}_1[S_1])(0, 1, x, y) - n^{-1/2}(p \otimes \mathcal{F}_1[S_2])(0, 1, x, y) \\ - n^{-1/2}(p \otimes \mathcal{F}_1[S_3])(0, 1, x, y), \tag{96}$$

where

$$S_1(u, 1, z, y) = \int_u^{j^*(u)/n} d\tau \int p(u, \tau, z, v) H(\tau, 1, v, y) dv,$$

$$S_2(u, 1, z, y) = R(u, 1, z, y),$$

$$S_3(u, 1, z, y) = \{(p \otimes H - p \otimes'_n H) \otimes'_n \Phi_1\}(u, 1, z, y).$$

From (83),

$$\left| n^{-1/2} \int_{1-n^{-\delta}}^1 du \int D_z^\nu [p(0, u, x, z)\mu_\nu(z)] R(u, 1, z, y) dz \right| \leqslant \frac{C(\varepsilon)}{n^{1+(\delta-\varepsilon)}} \phi(y - x) \approx 0,$$

for $0 < \varepsilon < \delta$. Analogously,

$$\left| n^{-1/2} \int_0^{n^{-\delta}} du \int p(0, u, x, z)\mu_\nu(z) D_z^\nu R(u, 1, z, y) dz \right| \leqslant \frac{C(\varepsilon)}{n^{1+(\delta-\varepsilon)}} \phi(y - x) \approx 0.$$

For $u \in [n^{-\delta}, 1 - n^{-\delta}]$ we obtain

$$n^{-1/2} \int_{n^{-\delta}}^{1-n^{-\delta}} du \int p(0, u, x, z)\mu_\nu(z) D_z^\nu R(u, 1, z, y) dz$$

$$= n^{-1/2} \int_{n^{-\delta}}^{1/2} du \int p(0, u, x, z)\mu_\nu(z) D_z^\nu R(u, 1, z, y) dz$$

$$+ n^{-1/2} \int_{1/2}^{1-n^{-\delta}} du \int D_z^\nu [p(0, u, x, z)\mu_\nu(z)] R(u, 1, z, y) dz = I + II.$$

We have $I \approx 0$ and $II \approx 0$. This follows from simple estimates and from the following estimate for $n^{-1/2} R(u, 1, z, y)$ with $u \in [n^{-\delta}, 1 - n^{-\delta}]$:

$$|n^{-1/2}R(u,\,1,\,z,\,y)| \leqslant n^{-1/2}\left|\sum_{i=j^*(u)+1}^{n-2}\int_{i/n}^{(i+1)/n}\left(\tau-\frac{i}{n}\right)\right.$$

$$\left.\times\int_0^1\int[p(u,\,\tau,\,z,\,v)\big(L-\tilde{L}\big)^2\tilde{p}(\tau,\,1,\,v,\,y)]|_{\tau=\tau^*}\mathrm{d}v\,\mathrm{d}\delta\,\mathrm{d}\tau\right|$$

$$+\,n^{-1/2}\left|\int_{j^*(u)/n}^{(j^*(u)+1)/n}\left(\tau-\frac{i}{n}\right)\int_0^1\int[p(u,\,\tau,\,z,\,v)(L-\tilde{L})^2\tilde{p}(\tau,\,1,\,v,\,y)]|_{\tau=\tau^*}\mathrm{d}v\,\mathrm{d}\delta\,\mathrm{d}\tau\right|$$

$$+\,n^{-1/2}\left|\int_{1-1/n}^{1}\left(\tau-\frac{i}{n}\right)\int_0^1\int[p(u,\,\tau,\,z,\,v)(L-\tilde{L})^2\tilde{p}(\tau,\,1,\,v,\,y)]|_{\tau=\tau^*}\mathrm{d}v\,\mathrm{d}\delta\,\mathrm{d}\tau\right|$$

$$\leqslant \frac{C}{n^{3/2}}\sum_{i=j^*(u)+1}^{n-2}\frac{1}{n}\cdot\frac{1}{\left(1-\dfrac{i+1}{n}\right)^{3/2}}\phi_{(1-u)^{1/2}}(y-z)+n^{-1/2}\cdot\int_{j^*(u)/n}^{(j^*(u)+1)/n}\left(\tau-\frac{j^*(u)}{n}\right)$$

$$\times\frac{\mathrm{d}\tau}{(1-\tau)^{3/2}}+n^{-1/2}\int_{1-1/n}^{1}\left(\tau-\frac{n-1}{n}\right)\int_0^1\int[L^t p(u,\,\tau,\,z,\,v)\big(L-\tilde{L}\big)\tilde{p}(\tau,\,1,\,v,\,y)$$

$$+\,\tilde{L}^t(\tilde{L}-L)^t p(u,\,\tau,\,z,\,v)\tilde{p}(\tau,\,1,\,v,\,y)]\mathrm{d}v\,\mathrm{d}\delta\,\mathrm{d}\tau$$

$$\leqslant\left[\frac{C}{n^{3/2-\varepsilon}}\int_u^1\frac{\mathrm{d}t}{(1-t)^{1-\varepsilon}}+\frac{C}{n^{5/2-3\delta/2}}\right]\phi_{\sqrt{1-u}}(y-z).$$

Thus, we obtain $n^{-1/2}(p\otimes\mathcal{F}_1[S_2])(0,\,1,\,x,\,y)\approx 0$. The proof that $n^{-1/2}(p\otimes\mathcal{F}_1[S_1])(0,\,1,\,x,\,y)\approx 0$ is quite similar. First, we show that it is enough to consider $u\in[n^{-\delta},\,1-n^{-\delta}]$. Then the assertion follows from the following estimates

$$\left|n^{-1/2}\int_{n^{-\delta}}^{1/2}\mathrm{d}u\int p(0,\,u,\,x,\,z)\mu_\nu(z)D_z^\nu S_1(u,\,1,\,z,\,y)\mathrm{d}z\right|$$

$$=\left|n^{-1/2}\int_{n^{-\delta}}^{1/2}\mathrm{d}u\int p(0,\,u,\,x,\,z)\mu_\nu(z)\right.$$

$$\left.\times D_z^\nu\left[\int_u^{j^*(u)/n}\mathrm{d}\tau\int p(u,\,\tau,\,z,\,z+v')H(\tau,\,1,\,z+v',\,y)\mathrm{d}v'\right]\mathrm{d}z\right|$$

$$\leqslant Cn^{-1/2}\int_{n^{-\delta}}^{1/2}\left(\frac{j^*(u)}{n}-u\right)\mathrm{d}u\cdot\phi(y-x)\approx 0,$$

$$\left|n^{-1/2}\int_{1/2}^{1-n^{-\delta}}\mathrm{d}u\int D_z^{\nu}[p(0,\,u,\,x,\,z)\mu_\nu(z)]S_1(u,\,1,\,z,\,y)\mathrm{d}z\right|$$

$$\leqslant \frac{C}{n^{1/2-\delta}}\int_{1/2}^{1-n^{-\delta}}\left(\frac{j^*(u)}{n}-u\right)\mathrm{d}u\cdot\phi(y-x)\approx 0.$$

The same estimate holds true for the last summand in (96), that is,

$$n^{-1/2}(p\otimes\mathcal{F}_1[S_3])(0,\,1,\,x,\,y)\approx 0.$$

This follows from the smoothing properties of the operation $\ldots\otimes_n'\Phi_1$ and it can be shown by similar methods to those used above. This completes the proof of Theorem 4.1.

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