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Blocking measures for asymmetric exclusion processes via coupling

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We give sufficient conditions on the rates of two asymmetric exclusion processes such that the existence of an invariant blocking measure for the first implies the existence of such a measure for the second. The main tool is a coupling between the two processes under which the first dominates the second in an appropriate sense. In an appendix we construct a class of processes for which the existence of a blocking measure can be proven directly; these are candidates for comparison processes in applications of the main result.

Keywords: asymmetric exclusion processes; coupling; invariant blocking measure

1. Introduction

We consider the exclusion process η_t on $\{0, 1\}^{\mathbb{Z}}$ with generator L given by

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} \sum_{y\in\mathbb{Z}} p(x, y; \eta)[f(\eta^{x,y}) - f(\eta)]. \tag{1.1}$$

Here f is a continuous function on $\{0, 1\}^{\mathbb{Z}}$ (with the product topology) and, for $x, y, z \in \mathbb{Z}$,

$$\eta^{x,y}(z) = \begin{cases} \eta(x), & \text{if } z = y, \\ \eta(y), & \text{if } z = x, \\ \eta(z), & \text{otherwise.} \end{cases}$$

The jump rate of particles from x to y in configuration η, $p(x, y; \eta)$, is a continuous function of η which is zero unless $\eta(x) = 1 - \eta(y) = 1$.

Let

$$\mathcal{X}_n = \left\{ \eta \in \{0, 1\}^{\mathbb{Z}} : \sum_{x\leq n} \eta(x) = \sum_{x>n} (1 - \eta(x)) < \infty \right\}$$

and

$$\mathcal{X} = \bigcup_n \mathcal{X}_n.$$

The set $\mathcal{X}$ is countable; we will call elements of $\mathcal{X}$ *blocking configurations*, and call probability measures supported on $\mathcal{X}$ *blocking measures*. We are interested in finding sufficient conditions on the rates p for the existence of blocking measures which are invariant for the process η_t.

We will construct the process on blocking configurations directly. For the construction we will use two conditions on the rates, which we assume throughout the paper; these could be somewhat weakened at the expense of increasing the complexity of the exposition. (Liggett (1985) gives conditions on the rates which ensure the existence of the process started from an arbitrary initial condition.) First, we take the rates to be uniformly bounded; we can set the upper bound equal to one by means of a time-scale change, and thus assume that

$$0 \leqslant p(x, y; \eta) \leqslant 1. \tag{1.2}$$

Second, we assume that the total rate for exiting any configuration of $\mathcal{X}$ is finite:

$$\text{for any } \zeta \in \mathcal{X}, \qquad r(\zeta) \equiv \sum_x \sum_y p(x, y; \zeta) < \infty. \tag{1.3}$$

This condition follows from (1.2) if there is an upper bound on the range of jumps.

Various special cases are of interest. The rates are *simple* when they are independent of the configuration except for the exclusion condition, so that

$$p(x, y; \eta) = c(x, y)\eta(x)(1 - \eta(y)), \tag{1.4}$$

and are *translation-invariant* when

$$p(x, y; \eta) = p(0, y - x; \tau_{-x}\eta),$$

where τ_z is the operator of translation by z. When both of these conditions are satisfied the rates can be written in the form

$$p(x, y; \eta) = a(y - x)\eta(x)(1 - \eta(y)). \tag{1.5}$$

Liggett (1985) exhibits invariant blocking measures in the case of simple translation-invariant rates with jumps restricted to length 1: $a(z) = 0$ for $|z| > 1$ and $a(1) > a(-1)$. A trivial extension of his result is the following: if, for some $\alpha < 1$, the rates have the form (1.4) with

$$c(x, y) = \alpha^{x-y}c(y, x), \qquad \text{for all } x < y, \tag{1.6}$$

then the product measure μ with marginals

$$\mu(\eta(x) = 1) = \frac{1}{1 + \alpha^x} \tag{1.7}$$

is reversible for the process η_t. This is a special case of a more general construction which we describe in the Appendix.

For a more general set of rates, one might expect that blocking measures exist when the process has a sufficiently strong positive drift, for example in the simple translation-invariant case (that is, for rates satisfying (1.5)) when

$$\sum_{\{y\}} ya(y) > 0 \tag{1.8}$$

(positive mean drift for the underlying random walk). Proving that (1.8) or a similar condition implies the existence of blocking measures seems quite difficult; this is one of the open problems of Liggett (1985).[1] When the rates $p(x, y; \eta)$ depend on the configuration η at sites other than x and y, it is not even clear what necessary and/or sufficient condition to conjecture. We do not deal directly with conditions like (1.8), but give a different sort of sufficient condition, showing that when the rates of two processes are appropriately related, existence of a blocking measure for one implies existence for the other.

Note that if μ is any invariant blocking measure for η_t then $\mu(\mathcal{X}_n) \neq 0$ for some n; since each $\mathcal{X}_n$ is a closed set for the process, the conditional measure $\mu_n = \mu(\cdot|\mathcal{X}_n)$ is then also an invariant blocking measure. Thus, if we permit ourselves a translation of the entire system, there is no loss of generality in treating the existence of a blocking measure on $\mathcal{X}$ as equivalent to the existence of a blocking measure on $\mathcal{X}_0$. We remark that if the rates are simple and translation-invariant (see (1.5)) then $\mathcal{X}_n$ is irreducible whenever there is a positive rate for some forward and some backward jump, and the greatest common divisor of $\{x \neq 0\colon\ a(x) > 0\}$ is 1, so that under these conditions each μ_n is unique and extremal in the class of invariant blocking measures.

We now compare the process η_t with a second process $\bar{\eta}_t$ for which the generator $\bar{L}$ is constructed as in (1.1) but with rates $\bar{p}(x, y; \eta)$. Our main result, presented in Section 4, gives conditions on the rates p and $\bar{p}$ under which the existence of an invariant blocking measure for the process $\bar{\eta}_t$ implies the existence of such a measure for η_t. In the case in which the rates are simple and translation-invariant, it takes the following form:

Theorem 1.1. *Suppose that* $p(x, y; \eta) = a(y - x)\eta(x)(1 - \eta(y))$ *and that*

$$a(x) \geqslant \bar{a}(y), \qquad \text{for } 0 < x \leqslant y,$$

$$a(y) \leqslant \bar{a}(x), \qquad \text{for } y \leqslant x < 0.$$

Then if $\bar{\eta}_t$ *has an invariant blocking measure, so does* η_t.

For example, we may take the weights $\bar{p}$ to have the form (1.6), with $c(x, y) = a(y - x)$ for $x < y$, as in (1.5), so that the requisite blocking measure is given by (1.7).

We remark that establishing the existence of invariant blocking measures is a special case, and perhaps a first step towards the general case, of the problem of establishing the existence of *invariant shock measures*: measures on $\{0, 1\}^{\mathbb{Z}}$ which have distinct asymptotic limits to the right and left of the origin and which are time-invariant in some appropriate sense, usually for the process as seen from a suitable random viewpoint. Such measures are related to the shock solutions of the Burgers equation, which describes the process in the hydrodynamical limit. The left and right asymptotic measures will be time-invariant for the process in the usual sense, so that invariant shock measures appear in systems that have more than one translation-invariant state. Given two such asymptotic measures, the shock

[1] After this paper was submitted, Bramson and Mountford proved the existence of blocking measures for processes with simple translation-invariant rates when the range is finite and the drift is positive (M. Bramson, private communication).

measure describes one ultimate fate of the system when it starts with one of these on each side of the origin (another is the so-called *rarefaction fan*). The blocking measures are the simplest shock measures: conceptually, because they are invariant when seen from a fixed viewpoint, and technically, because they have support on a countable state space.

In the case of simple exclusion the extremal time- and translation-invariant measures are the one-parameter family of homogeneous product measures indexed by density. In nearest-neighbour asymmetric simple exclusion, existence of invariant shock measures has been established for the process as seen from a 'second class particle' (Ferrari *et al*. 1991; Ferrari 1992; Derrida *et al*. 1993; 1998). The approach of Ferrari *et al*. (1991) and Ferrari (1992) was closely based on the known blocking measures for this process, the product measures (1.7). Derrida *et al*. (1997; 1998) proposed other approaches to the problem of describing shock measures.

The paper is organized as follows. In Section 2 we construct η_t on $\mathcal{X}$ using Poisson processes (the Harris graphical construction); the construction is done in such a way as to facilitate an appropriate coupling of two such process. The key idea for the proof of our results is introduced in Section 3 – a certain partial order $\prec$ on the space $\mathcal{X}_0$ of blocking configurations with the property that, under the coupling, the conditions of Theorem 1.1 (or the more general conditions to be given later) imply that if the initial configurations η_0 and $\bar{\eta}_0$ satisfy $\eta_0 \prec \bar{\eta}_0$, then this ordering is preserved by the dynamics: $\eta_t \prec \bar{\eta}_t$ for all $t \geqslant 0$. In Section 4 we state and prove our general result, of which Theorem 1.1 is an immediate corollary. In Section 5 we present some applications, and in the Appendix we discuss the construction of a class of possible comparison processes $\bar{\eta}_t$.

2. Construction of the process

We now exhibit a special construction of the process in $\mathcal{X}$. The construction requires the rates $p(x, y; \zeta)$ to satisfy conditions (1.2) and (1.3).

The graphical construction of an exclusion process is usually based on families of independent Poisson processes that are associated with either lattice sites or particles. In the latter case, for example, the process for a particle at site x in configuration η has rate $q(x; \eta) = \sum_y p(x, y; \eta)$; at a Poisson event time the particle attempts to jump, choosing the target site y with probability $p(x, y; \eta)/q(x; \eta)$. In contrast, our construction associates distinct Poisson processes with each possible jump – specifically, with each pair (i, j), where i is a particle label and j is the label of an empty site (labels correspond to ordinary order). Two processes are associated with each (i, j), one controlling forward and one backward jumps; at the event time of the forward (backward) processes the ith particle attempts to jump to the jth site if that is a forward (backward) jump. All these processes have unit rate; to tune the actual jump rate, a uniform random variable (called a *mark*) is associated with each Poisson time event; the jump actually occurs if the mark satisfies a certain inequality. Once the jump rates are adjusted correctly the two approaches are clearly equivalent for describing a single exclusion process; the advantage of our approach is that it permits a useful coupling of two such processes.

We start by defining the labels of particles and empty sites. For a configuration $\eta \in \mathcal{X}$ we define ordered positions of the particles and empty sites by

$$x_0(\eta) = \min\{x : \eta(x) = 1\}, \tag{2.1}$$

$$x_k(\eta) = \min\{x > x_{k-1}(\eta) : \eta(x) = 1\}, \tag{2.2}$$

$$y_0(\eta) = \max\{x : \eta(x) = 0\}, \tag{2.3}$$

$$y_k(\eta) = \max\{x < y_{k-1}(\eta) : \eta(x) = 0\}. \tag{2.4}$$

For each pair (i, j) with $i, j \geqslant 0$, let

$$\Theta^{i,j} =: \{((T_n^{i,j}, U_n^{i,j}), (R_m^{i,j}, V_m^{i,j})) : n, m \geqslant 1\}$$

be a process with the following properties:

- Both $(T_n^{i,j} - T_{n-1}^{i,j})_{n\geqslant 1}$ and $(R_m^{i,j} - R_{m-1}^{i,j})_{m\geqslant 1}$, where by convention $T_0^{i,j} = R_0^{i,j} = 0$, are families of independent and exponentially distributed random variables of mean 1. In other words, $(T_n^{i,j})$ and $(R_m^{i,j})$ are Poisson processes of rate 1 for all i, j.
- Both $(U_n^{i,j})_{n\geqslant 1}$ and $(V_m^{i,j})_{m\geqslant 1}$ are families of independent random variables, uniformly distributed on $[0, 1]$.
- All four of these families of variables are mutually independent.

We also assume that $\{\Theta^{i,j} : i, j \geqslant 0\}$ is a family of mutually independent processes. The times $T_n^{i,j}$ and $R_m^{i,j}$ will be called Poisson events and the associated random variables $U_n^{i,j}$ and $V_m^{i,j}$ will be called *marks*.

We now construct the process η_t as a function of the marked Poisson processes and the initial configuration $\eta_0 \in \mathcal{X}$. Set $\tau_0 = 0$ and suppose inductively that we have defined times $\tau_0, \ldots, \tau_{n-1}$ and configurations $\eta_{\tau_0}, \ldots, \eta_{\tau_{n-1}}$. Define

$$\begin{aligned}\tau_n = \min\Big\{ &\inf_{i,j,k}\Big\{T_k^{i,j} > \tau_{n-1} : U_k^{i,j} < A_+(\eta_{\tau_{n-1}}, i, j)\Big\},\\ &\inf_{i,j,k}\Big\{R_k^{i,j} > \tau_{n-1} : V_k^{i,j} < A_-(\eta_{\tau_{n-1}}, i, j)\Big\}\Big\},\end{aligned} \tag{2.5}$$

where, for $i, j \geqslant 0$,

$$A_+(\eta, i, j) = p(x_i(\eta), y_j(\eta); \eta)\,\mathbf{1}\{y_j(\eta) > x_i(\eta)\}, \tag{2.6}$$

$$A_-(\eta, i, j) = p(x_i(\eta), y_j(\eta); \eta)\,\mathbf{1}\{y_j(\eta) < x_i(\eta)\}. \tag{2.7}$$

Here $\mathbf{1}S$ denotes the characteristic function of the set S. If (I_n, J_n) is the pair (i, j) such that $T_k^{i,j}$ or $R_k^{i,j}$ realizes the infimum τ_n for some k, set

$$X_n = x_{I_n}(\eta_{\tau_{n-1}}),$$

$$Y_n = y_{J_n}(\eta_{\tau_{n-1}}),$$

and define

$$\eta_{\tau_n} = (\eta_{\tau_{n-1}})^{X_n, Y_n}.$$

This completes the induction step. To finish the construction after all τ_n and η_{τ_n} are defined, set

$$\eta_t = \sum_{n \geqslant 0} \eta_{\tau_n} \mathbf{1}\{\tau_n \leqslant t < \tau_{n+1}\}, \qquad \text{for all } t \geqslant 0. \tag{2.8}$$

Note that it is a consequence of (2.1)–(2.4) that labels are not permanently attached to particles or holes. Because particles can jump over one another and thereby change their order (as can holes), particles and holes are effectively relabelled after each jump, so that, for all times t,

$$x_i(\eta_t) \leqslant x_{i+1}(\eta_t) \quad \text{and} \quad y_j(\eta_t) \geqslant y_{j+1}(\eta_t), \qquad i, j \geqslant 0.$$

The construction may be described in words as follows. We use independent times ($T_n^{i,j}$ and $R_m^{i,j}$, respectively) for jumps to the right and jumps to the left; this is not necessary for the construction here but ensures that the coupling we define later preserves a certain partial order on configurations. The instant τ_n is the first time after τ_{n-1} at which a jump is performed, and is the minimum of the first scheduled jump times to the right and to the left. The first scheduled jump time to the right is the first $T_k^{i,j}$ for which the corresponding uniform random variable $U_k^{i,j}$ is smaller than the threshold A_+, defined by (2.6) to ensure that the jump is indeed to the right and occurs at the correct rate (here we use the condition (1.2) that $p(x, y; \eta) \leqslant 1$). Similarly, the first scheduled jump time to the left is the first $R_k^{i,j}$ for which the corresponding uniform random variable $V_k^{i,j}$ is smaller than the threshold A_- defined by (2.7). The configuration at time τ_n is then the one obtained by interchanging the hole and the particle whose indices i, j correspond to the $R_k^{i,j}$ or $T_k^{i,j}$ that realizes the time τ_n.

To see that the above is well defined for initial configurations in $\mathcal{X}$ it suffices to see that, for any initial $\eta_0 \in \mathcal{X}$, τ_n is with probability 1 a strictly increasing sequence of (finite) times. The conditional distribution of $\tau_n - \tau_{n-1}$ given the past up to τ_{n-1} is

$$P(\tau_n - \tau_{n-1} > s | \eta_{\tau_{n-1}}) = \exp\left\{ -\sum_{x,y} p(x, y; \eta_{\tau_{n-1}}) \right\}, \tag{2.9}$$

by (2.5) – it is the minimum of independent random variables with exponential distribution and inverse mean $p(x, y; \eta_{\tau_{n-1}})$. Since $\eta_{\tau_{n-1}}$ is obtained by carrying out at most $n-1$ modifications to the initial configuration η_0, it belongs to $\mathcal{X}$. By condition (1.3), the conditional law (2.9) is that of a non-degenerate exponential random variable.

We now give a graphical interpretation of this construction, and of the coupling of the processes to be introduced later. For simplicity, assume $\mathcal{X} = \mathcal{X}_0$. With each configuration $\eta \in \mathcal{X}_0$ associate an interface $\Phi\eta$ corresponding to the integrated profile of η. Here $\Phi : \mathcal{X}_0 \to \mathbb{Z}_+^{\mathbb{Z}}$ is defined by either of two equivalent expressions

$$(\Phi\eta)(x) = -x + 2\sum_{y\leqslant x} \eta(y) \tag{2.10}$$

$$= x + 2\sum_{y>x} (1 - \eta(y)) \tag{2.11}$$

Note that $\Phi\eta$ increases by 1 when a particle is present at x or decreases by 1 when no particle is present at x, so that, in particular, $|\Phi\eta(x) - \Phi\eta(x+1)| = 1$. The graph $\{(x, (\Phi\eta)(x)) | x \in \mathbb{Z}\}$ is a subset of the lattice $\mathbb{Z}^2_{\text{even}} = \{(x, y) \in \mathbb{Z}^2 | x + y \text{ is even}\}$. The corresponding *interface*, which we will also refer to as $\Phi\eta$, is the path obtained by joining adjacent vertices of this graph with straight lines; each line segment with slope $+1$ (-1) of this interface is identified with a particle (hole). A typical interface is shown in Figure 1. The Heaviside configuration η^{H}, given by $\eta^{\mathrm{H}}(x) = \mathbf{1}\{x \geqslant 1\}$, gives rise to the interface $\Phi\eta^{\mathrm{H}}(x) = |x|$.

The interface picture yields a geometric interpretation of the construction of the process η_t. Index the squares of the lattice $\mathbb{Z}^2_{\text{even}}$ as $\{S_{i,j} | i, j \in \mathbb{Z}\}$ as shown in Figure 1 ($S_{i,j} = \{(x, y) | 2i < x + y < 2i + 2, 2j < y - x < 2j + 2\}$), and consider only those $S_{i,j}$ with $i, j \geqslant 0$. A line through $S_{i,j}$ of slope -1 $(+1)$ intersects the interface $\Phi\eta$ at the line segment corresponding to particle i (hole j). The interface lies above $S_{i,j}$ if $x_i(\eta) < y_j(\eta)$, that is, if the jump of particle i to hole j is to the right, and below $S_{i,j}$ if $x_i(\eta) > y_j(\eta)$, that is, if the jump is to the left. Now think of the marked processes $(T^{i,j}_n, U^{i,j}_n)$ and $(R^{i,j}_m, V^{i,j}_m)$

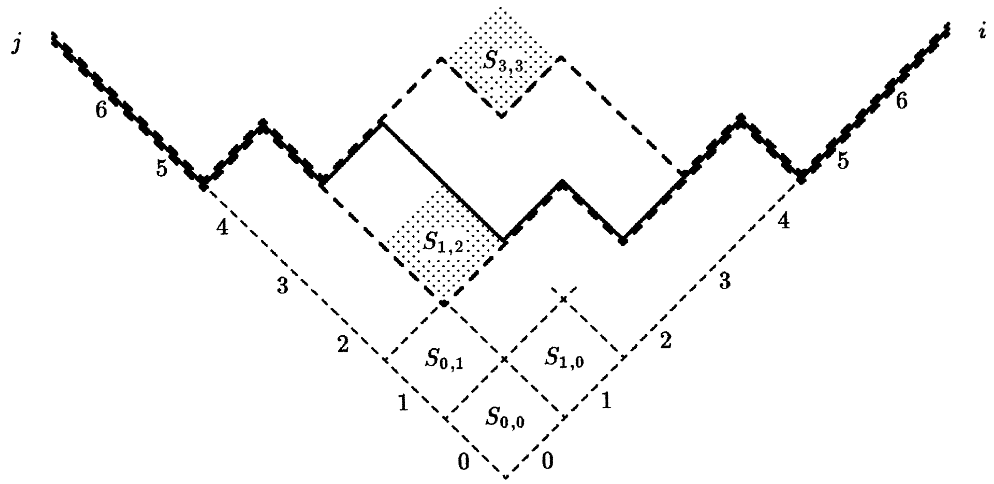

Figure 1. The heavy solid line is the interface for the configuration $\eta = \ldots 0001010010110111 \ldots$. The lower heavy dashed line is the interface after a jump of the particle at $x(1) = -2$ to the hole at $y(2) = 0$, triggered by the occurrence of a mark of the process $T^{1,2}$, that is, the interface for $\eta^{-2,0}$. The upper dashed line is the interface for $\eta^{3,-1}$, after a jump triggered by the occurrence of a mark of the process $T^{3,3}$.

as associated with $S_{i,j}$. When at the Poisson event $T_n^{i,j}$ the corresponding uniform variable $U_n^{i,j}$ is less than $p(x_i(\eta), y_j(\eta); \eta)$, then, if the interface $\Phi\eta$ lies above $S_{i,j}$, we update the interface by changing the line segment corresponding to particle i from slope $+1$ to slope -1, and similarly the line segment corresponding to hole j from slope -1 to slope $+1$; this has the effect of decreasing the interface height by two units in the interval $(x_i, y_j]$. Similarly, when at time $R_m^{i,j}$ the corresponding mark satisfies $V_m^{i,j} < p(x_i(\eta), y_j(\eta); \eta)$ and the interface lies below $S_{i,j}$, we increase by two units the height of the interface in the interval $(y_j, x_i]$. All of this is shown in Figure 1.

We now verify that this construction indeed produces the desired process.

Lemma 2.1. *The process* η_t *defined by* (2.8) *has generator* L *given by* (1.1).

Proof. Since the process is defined in $\mathcal{X}_0$, a countable state space, it suffices to show that if $\eta \in \mathcal{X}_0$ then for all i, j, with $x = x_i(\eta)$ and $y = y_j(\eta)$,

$$\lim_{h\to 0} h^{-1} P(\eta_{t+h} = \eta^{x,y} | \eta_t = \eta) = p(x, y; \eta), \tag{2.12}$$

and for all $\zeta \in \mathcal{X}_0$ with $\zeta \neq \eta$ and $\zeta \neq \eta^{x,y}$ for any $x = x_i(\eta)$, $y = y_i(\eta)$,

$$\lim_{h\to 0} h^{-1} P(\eta_{t+h} = \zeta | \eta_t = \eta) = 0. \tag{2.13}$$

To verify (2.12) we first write

$$\begin{aligned} &\mathbf{1}\{\eta_{t+h} = \eta^{x,y}, \eta_t = \eta\} \\ &= \mathbf{1}\{\eta_t = \eta\} \Bigg(\sum_k [\mathbf{1}\{y > x, T_k^{i,j} \in [t, t+h), U_k^{i,j} \leqslant p(x, y; \eta), A(\eta, h)\} \\ &\quad + \mathbf{1}\{y < x, R_k^{i,j} \in [t, t+h), V_k^{i,j} \leqslant p(x, y; \eta), A(\eta, h)\}] \Bigg) \\ &\quad + \mathbf{1}\{\eta_{t+h} = \eta^{x,y}, \eta_t = \eta, A^c(\eta, h)\}, \end{aligned} \tag{2.14}$$

where $A(\eta, h)$ is the event {if $\eta_t = \eta$ then at most one particle jumps in the time interval $(t, t+h]$}. Taking the expectation of all but the last term on the right-hand side of (2.14) conditioned on $\eta_t = \eta$, dividing by h, and then taking the limit $h \to 0$, we obtain the right-hand side of (2.12). Thus to complete the proof of (2.12) and prove (2.13) it suffices to show that

$$\lim_{h\to 0} h^{-1} P(A^c(\eta, h)) = 0. \tag{2.15}$$

We calculate $P(A^c(\eta, h))$ by conditioning on the time $t + t'$ at which the first jump in $(t, t+h]$ occurs and on the pair of sites x', y' involved. By an explicit consideration of the Θ processes, similar to the argument above, we find that

$$P(A^c(\eta,\, h)) = \sum_{x',y'} \int_0^h \mathrm{e}^{-t' r(\eta)} p(x',\, y';\, \eta)[1 - \mathrm{e}^{-(h-t')r(\eta^{x'y'})}]\mathrm{d}t'$$

$$\leqslant h \sum_{x'y'} p(x',\, y';\, \eta)[1 - \mathrm{e}^{-hr(\eta^{x'y'})}], \tag{2.16}$$

where $r(\eta)$, the total rate for leaving η, was defined in (1.3). Now (2.15) follows from (2.16), (1.3) and the Lebesgue dominated convergence theorem. □

We are really proving two somewhat different things here. One is that our condition (1.3) does in fact suffice to make our process well defined; this is standard and has nothing to do with the particular graphical construction we choose. The other is that our graphical construction works; this is immediate from (2.14).

We finally remark that the above construction works also if the process restricted to $\mathcal{X}$ has explosions, that is, if $T^* \equiv \lim_{n\to\infty}\tau_n$ satisfies $P(T^* < \infty) > 0$, but the resulting process is of course defined only for $t < T^*$.

3. An order relation on configurations

For configurations η and $\bar{\eta} \in \mathcal{X}_0$, we say that

$$\eta \prec \bar{\eta} \text{ if and only if for all } i,\, j \geqslant 0,\, x_i(\eta) \geqslant x_i(\bar{\eta}) \text{ and } y_j(\eta) \leqslant y_j(\bar{\eta}).$$

It is easy to see that this is a partial order which corresponds to the natural order on interfaces:

$$\eta \prec \bar{\eta} \text{ if and only if } (\Phi\eta)(\mathrm{x}) \leqslant (\Phi\bar{\eta})(\mathrm{x}) \text{ for all } x \in \mathbb{Z}.$$

Under this ordering, the Heaviside configuration η^{H} precedes every other configuration: $\eta^{\mathrm{H}} \prec \eta$ for any $\eta \in \mathcal{X}_0$. From (2.10) and (2.11) it follows that if $\eta \prec \bar{\eta}$ then, for all $z \in \mathbb{Z}$,

$$(\Phi\bar{\eta})(z) - (\Phi\eta)(z) = 2\sum_i \mathbf{1}\{x_i(\bar{\eta}) \leqslant z < x_i(\eta)\} \tag{3.1}$$

$$= 2\sum_j \mathbf{1}\{y_j(\eta) \leqslant z < y_j(\bar{\eta})\}, \tag{3.2}$$

and for all x, y such that $\eta(x) = 1$ and $\eta(y) = 0$ and all $z \in \mathbb{Z}$,

$$(\Phi\eta^{x,y})(z) = (\Phi\eta)(z) - 2\mathbf{1}\{x \leqslant z < y\} + 2\mathbf{1}\{y \leqslant z < x\}. \tag{3.3}$$

The following lemma says essentially that if we have two configurations which are ordered by $\prec$ then they will remain ordered after either (i) a jump in both configurations, in the same direction, of the ith particle to the jth hole, or (ii) certain jumps in only one of the configurations.

Lemma 3.1. *Assume $\eta \prec \bar{\eta}$, fix i and j, and let $x = x_i(\eta)$, $y = y_j(\eta)$, $\bar{x} = x_i(\bar{\eta})$ and $\bar{y} = y_j(\bar{\eta})$. Then jumps preserve ordering in the following cases:*

(i) *If $\bar{x} \leqslant x < y \leqslant \bar{y}$, then $\eta^{x,y} \prec \bar{\eta}$.*
(ii) *If $y \leqslant \bar{y} < \bar{x} \leqslant x$, then $\eta \prec \bar{\eta}^{\bar{x},\bar{y}}$.*
(iii) *If $\bar{x} \leqslant x < y \leqslant \bar{y}$, then $\eta^{x,y} \prec \bar{\eta}^{\bar{x},\bar{y}}$.*
(iv) *If $y \leqslant \bar{y} < \bar{x} \leqslant x$, then $\eta^{x,y} \prec \bar{\eta}^{\bar{x},\bar{y}}$.*
(v) *If $x > y$ and $\bar{x} < \bar{y}$, then $\eta \prec \bar{\eta}^{\bar{x},\bar{y}}$ and $\eta^{x,y} \prec \bar{\eta}$.*

Before giving a formal proof of this lemma, we describe its graphical interpretation. The interface $\Phi\eta$ lies below $\Phi\bar{\eta}$. In cases (i) and (iii) the square $S_{i,j}$ lies below both interfaces, so that for either interface a jump of the ith particle to the jth hole – or an (i, j) jump for short – lowers the interface; (i) and (iii) assert respectively that the order is preserved by either a jump in the lower interface only, or a jump for both interfaces. Similarly, in cases (ii) and (iv) $S_{i,j}$ lies above both interfaces, an (i, j) jump raises either interface, and the order is preserved by such a jump in either the upper interface alone or in both. Finally, in case (v) $S_{i,j}$ lies between the two interfaces, an (i, j) jump for the lower interface raises it and for the upper interface lowers it, and (v) asserts that such a jump for either interface alone preserves the order. These properties are easy to check in the graphical representation.

Proof. Statements (i) and (ii) follow immediately from (3.3). Under the hypothesis of (iii), $x < y$ and $\bar{x} < \bar{y}$. Hence, by (3.3),

$$(\Phi\eta^{x,y})(z) = (\Phi\eta)(z) - 2\mathbf{1}\{x \leqslant z < y\}; \tag{3.6}$$

an analogous identity holds for $\bar{\eta}$. Since $\eta \prec \bar{\eta}$ and $\bar{x} \leqslant x < y \leqslant \bar{y}$, by (3.1) and (3.2),

$$(\Phi\eta)(z) \leqslant (\Phi\bar{\eta})(z) - 2\mathbf{1}\{\bar{x} \leqslant z < x\} - 2\mathbf{1}\{y \leqslant z < \bar{y}\}. \tag{3.7}$$

Subtracting $2\mathbf{1}\{x < z \leqslant y\}$ from both sides of the above inequality, we obtain

$$(\Phi\eta)(z) - 2\mathbf{1}\{x \leqslant z < y\} \leqslant (\Phi\bar{\eta})(z) - 2\mathbf{1}\{\bar{x} \leqslant z < \bar{y}\},$$

which by (3.3) is the same as $(\Phi\eta^{x,y})(z) < (\Phi\bar{\eta}^{\bar{x},\bar{y}})(z)$. In this way we get $\eta^{x,y} \prec \bar{\eta}^{\bar{x},\bar{y}}$ and (iii) is proven. Statement (iv) is verified analogously.

By (3.1),

$$(\Phi\bar{\eta})(z) - (\Phi\eta)(z) \geqslant 2\mathbf{1}\{y \leqslant z < \bar{y}\},$$

$$(\Phi\bar{\eta})(z) - (\Phi\eta)(z) \geqslant 2\mathbf{1}\{\bar{x} \leqslant z < x\}.$$

Under the hypothesis of (v), this implies that

$$(\Phi\bar{\eta})(z) - (\Phi\eta)(z) \geqslant 2\mathbf{1}\{\min\{y, \bar{x}\} \leqslant z < \max\{\bar{y}, x\}\}.$$

Applying (3.3), we obtain (v). $\square$

4. Statement and proof of main result

We now consider two processes η_t and $\bar{\eta}_t$ with rates p and $\bar{p}$ respectively, as discussed in Section 1. Our main result is the following:

Theorem 4.1. *Suppose that whenever* $\eta \prec \bar{\eta}$ and $\eta(x) = \bar{\eta}(\bar{x}) = 1,\ \eta(y) = \bar{\eta}(\bar{y}) = 0,$

$$p(x, y; \eta) \geqslant \bar{p}(\bar{x}, \bar{y}; \bar{\eta}), \qquad \text{if } \bar{x} \leqslant x < y \leqslant \bar{y}, \tag{4.1}$$

$$p(x, y; \eta) \leqslant \bar{p}(\bar{x}, \bar{y}; \bar{\eta}), \qquad \text{if } y \leqslant \bar{y} < \bar{x} \leqslant x. \tag{4.2}$$

Then if $\bar{\eta}_t$ restricted to $\mathcal{X}$ has an invariant blocking measure, so does η_t.

Theorem 1.1 is an immediate corollary of Theorem 4.1.

We construct simultaneously the two processes η_t and $\bar{\eta}_t$ using the *same* marked Poisson processes $((T_n^{i,j}, U_n^{i,j}), (R_m^{i,j}, V_m^{i,j}))$. This joint construction is called *coupling* and is the key to the proof.

Lemma 4.2. *Assume that η_t and $\bar{\eta}_t$ are processes with rates p and $\bar{p}$ satisfying* (4.1)–(4.2) *and defined for* $0 \leqslant t < T^*$*, where T^* is a random time. Under the coupling, if $\eta_0 \prec \bar{\eta}_0$ are both configurations of $\mathcal{X}$, then for* $0 \leqslant t < T^*$, $\eta_t \prec \bar{\eta}_t$.

Proof. This is a mark-by-mark proof. Set $\theta_0 = 0$ and let $\theta_1 < \theta_2 < \ldots$ be the instants at which there is a jump for at least one of the processes $\eta_t, \bar{\eta}_t$. Assume inductively that $\eta_{\theta_{n-1}} \prec \bar{\eta}_{\theta_{n-1}}$, so that if (x_i, y_j) and $(\bar{x}_i, \bar{y}_j)$ are the sites and holes of $\eta_{\theta_{n-1}}$ and $\bar{\eta}_{\theta_{n-1}}$ respectively, at time θ_{n-1}, then

$$x_i \geqslant \bar{x}_i \quad \text{and} \quad y_j \leqslant \bar{y}_j, \qquad i, j \geqslant 0. \tag{4.3}$$

Let τ_n and $\bar{\tau}_n$ be the times defined as in (2.5) for the processes η_t and $\bar{\eta}_t$, so that

$$\theta_n = \min\{\min\{\tau_k > \theta_{n-1}\}, \min\{\bar{\tau}_k > \theta_{n-1}\}\}.$$

Let (I, J, K) be the indices which realize the infimum (2.5) defining the time θ_n, so that $\theta_n \in \{T_K^{I,J}, R_K^{I,J}\}$. Let $U \in \{U_K^{I,J}, V_K^{I,J}\}$ be the uniform random variable related with the indexes realizing the infimum, and let $\sigma = \pm$ indicate the direction of the jump at θ_n: $\sigma = +$ if $\theta_n = T_K^{I,J}$ and $U = U_K^{I,J}$, $\sigma = -$ if $\theta_n = R_K^{I,J}$ and $U = V_K^{I,J}$. Let

$$X = x_I, \qquad \overline{X} = \bar{x}_I, \qquad Y = y_J, \qquad \overline{Y} = \bar{y}_J;$$

$$\xi = \eta_{\theta_{n-1}}, \qquad \bar{\xi} = \bar{\eta}_{\theta_{n-1}};$$

$$B = A_\sigma(\xi, I, J), \qquad \overline{B} = A_\sigma(\bar{\xi}, I, J).$$

Since (4.3) implies that $\overline{X} \leqslant X$ and $Y \leqslant \overline{Y}$, there are three cases to consider

1. $\overline{X} \leqslant X < Y \leqslant \overline{Y}$. By hypothesis (4.1), $\overline{B} \leqslant B$. Hence there are two possibilities:
   (a) $U < \overline{B} \leqslant B$. In this case $\eta_{\theta_n} = \xi^{X,Y}$ and $\bar{\eta}_{\theta_n} = \bar{\xi}^{\overline{X},\overline{Y}}$. By Lemma 3.1(iii), $\eta_{\theta_n} \prec \bar{\eta}_{\theta_n}$.

   (b) $\bar{B} \leqslant U < B$. In this case $\eta_{\theta_n} = \xi^{X,Y}$ and $\bar{\eta}_{\theta_n} = \bar{\xi}$. By Lemma 3.1(i), $\eta_{\theta_n} \prec \bar{\eta}_{\theta_n}$.
2. $Y \leqslant \bar{Y} < \bar{X} \leqslant X$. By hypothesis (4.2), $B \leqslant \bar{B}$. Hence there are two possibilities:
   (a) $U < B \leqslant \bar{B}$. In this case $\eta_{\theta_n} = \xi^{X,Y}$ and $\bar{\eta}_{\theta_n} = \bar{\xi}^{\bar{X},\bar{Y}}$. By Lemma 3.1(iv), $\eta_{\theta_n} \prec \bar{\eta}_{\theta_n}$.
   (b) $B \leqslant U < \bar{B}$. In this case $\eta_{\theta_n} = \xi^{X,Y}$ and $\bar{\eta}_{\theta_n} = \bar{\xi}$. By Lemma 3.1(ii), $\eta_{\theta_n} \prec \bar{\eta}_{\theta_n}$.
3. $X > Y$ and $\bar{X} < \bar{Y}$. There are two possibilities:
   (a) $\sigma = +$ and $0 = B \leqslant U = U_K^{I,J} < \bar{B}$. In this case $\eta_{\theta_n} = \xi$ and $\bar{\eta}_{\theta_n} = \bar{\xi}^{\bar{X},\bar{Y}}$. By Lemma 3.1(v), $\eta_{\theta_n} \prec \bar{\eta}_{\theta_n}$.
   (b) $\sigma = -$ and $0 = \bar{B} \leqslant U = V_K^{I,J} < B$. In this case $\eta_{\theta_n} = \xi^{X,Y}$ and $\bar{\eta}_{\theta_n} = \bar{\xi}$. Again by Lemma 3.1(v), $\eta_{\theta_n} \prec \bar{\eta}_{\theta_n}$. □

Notice that if we had used the same Poisson process for both forward and backward jumps then in the situation of case 3 above jumps could have occurred simultaneously in η and $\bar{\eta}$, in opposite directions, which could destroy the ordering. For example, the Heaviside configuration $\eta = \eta^{\mathrm{H}}$ is below $\eta' = (\eta^{\mathrm{H}})^{0,1}$ (the configuration obtained from η^{H} when the first particle jumps one unit to the left). Now, a simultaneous jump of the first particle to the first hole of each of those configurations takes η to η' and η' to η, inverting the order.

We remark that Lemma 4.2 applies even if one of the processes η_t, $\bar{\eta}_t$ has explosions, that is, if $P(T^* < \infty) > 0$.

Proof of Theorem 4.1. As remarked in Section 1 it suffices to show that if $\bar{\eta}_t$ has an invariant measure in $\mathcal{X}_0$, then so does η_t. By restricting to a subset of $\mathcal{X}' \subset \mathcal{X}_0$ (if necessary) we may assume that $\bar{\eta}_t$ is ergodic with invariant measure $\bar{\mu}$ having support $\mathcal{X}'$. This excludes explosions for the process $\bar{\eta}_t$ starting with configurations in $\mathcal{X}'$.

Start the coupled process with any two configurations $\zeta \prec \bar{\zeta}$, with $\bar{\zeta} \in \mathcal{X}'$ and $\zeta \in \mathcal{X}_0$. We know that:

1. The process $\bar{\eta}_t$ is defined for all time, by the above remark.
2. Since $\bar{\eta}_t$ is a continuous-time ergodic Markov process in a countable state space, it converges in distribution to its unique invariant measure $\bar{\mu}$;
3. $\eta_t \prec \bar{\eta}_t$ for all t for which η_t is defined, by Lemma 4.2.

Further, we may show by an argument similar to the one in the proof of Lemma 4.2 that no explosions occur for η_t, so that η_t is defined for all time. Hence any weak Cesàro limit μ of the distribution of η_t is coupled with $\bar{\mu}$ in such a way that ν, the coupled measure with marginals μ and $\bar{\mu}$, satisfies

$$\nu((\eta, \bar{\eta}) : \eta \prec \bar{\eta}, \bar{\eta} \in \mathcal{X}') = 1.$$

In particular, this implies that $\mu(\mathcal{X}_0) = 1$, since $\mu(\mathcal{X}_0) = \nu(\mathcal{X}_0 \times \mathcal{X}')$ and $\mathcal{X}_0 \times \mathcal{X}' \supset \{(\eta, \bar{\eta}) : \eta \prec \bar{\eta}, \bar{\eta} \in \mathcal{X}'\}$. Since μ is a Cesàro limit, μ is invariant for η_t. This implies the theorem. □

5. Applications

To apply Theorem 4.1 one needs a suitable comparison process $\bar{\eta}$ which is known to have an invariant blocking measure. Obvious candidates are processes satisfying (1.6), for which the product measures (1.7) are invariant; in this section we draw some simple conclusions from this comparison. In the Appendix we discuss briefly the existence of other possible comparison processes: those which satisfy detailed balance with respect to a Gibbs measure obtained from a suitable potential (Hamiltonian).

Theorem 5.1. *Suppose that the exclusion process η_t has simple translation-invariant rates $p(x, y; \eta) = a(y - x)\eta(x)(1 - \eta(y))$ which, for some α with $0 \leqslant \alpha < 1$, satisfy*

$$a(-x) \leqslant \alpha^x \inf_{0<y\leqslant x} a(y)$$

for all $x > 0$. Then η_t has an invariant blocking measure.

Proof. The process with rates $\bar{p}(x, y; \eta) = \bar{a}(y - x)\eta(x)(1 - \eta(y))$, where, for $x > 0$,

$$\bar{a}(x) = \inf_{0<y\leqslant x} a(y) \quad \text{and} \quad \bar{a}(-x) = \alpha^x \bar{a}(x),$$

has an invariant measure of the form (1.7). Thus the process η_t has an invariant blocking measure by Theorem 1.1. □

As a second example, consider a process with symmetric 'disorder', in which translation-invariant, asymmetric, nearest-neighbour rates are perturbed by arbitrary, bounded, symmetric nearest-neighbour rates. Specifically, take $p(x, y; \eta) = c(x, y)\eta(x)(1 - \eta(y))$, where $c(x, y) = c_0(x, y) + c_1(x, y)$ with $c_0(x, y) = c_1(x, y) = 0$ if $|x - y| > 1$ and

$$c_0(x, x+1) = K, \qquad c_0(x+1, x) = 0, \qquad c_1(x, x+1) = c_1(x+1, x) = h(x),$$

with $K > 0$ and $h : \mathbb{Z} \to \mathbb{R}_+$ an arbitrary bounded function. It follows from Theorem 4.1 that this process has a blocking measure. A suitable comparison process has rates $\bar{p}(x, y; \eta) = \bar{c}(x, y)\eta(x)(1 - \eta(y))$ with $\bar{c}(x, x+1) = c(x, x+1)$, $\bar{c}(x+1, x) = \alpha c(x, x+1)$ and $\bar{c}(x, y) = 0$ if $|x - y| > 1$, where $\alpha = M/(M + K)$ with M an upper bound on $h(x)$; these rates satisfy (1.6) and hence have a blocking measure as given in (1.7). We single out this rather trivial example because in this case it is easy to see that the product measures with constant density are invariant measures, since if μ is such a measure then $L_0^* \mu = L_1^* \mu = 0$ and hence $(L_0^* + L_1^*)\mu = 0$, where L_i^* is the adjoint of the generator for the process with rates c_i.

Appendix

The remark that processes satisfying (1.6) have invariant product blocking measures of the form (1.7) can be generalized to processes which satisfy detailed balance with respect to a Gibbs measure obtained from a suitable potential (Hamiltonian). The latter is specified (see Liggett 1985) by a collection of real numbers $\{J_R\}$ indexed by finite subsets R of $\mathbb{Z}$ and satisfying $\sum_{R\ni x}|J_R| < \infty$ for each $x \in \mathbb{Z}$. We show that if these coupling constants are chosen appropriately, then blocking Gibbs measures for this potential arise as the limit of finite-volume measures.

Let $T_N = [-N+1, N] \cap \mathbb{Z}$ and $Y_N = \{0, 1\}^{T_N}$. For $\eta \in Y_N$, let $\eta^* \in \mathcal{X}$ be the configuration which agrees with η in T_N and with η^{H} outside T_N. The *energy* of the configuration η is

$$H_N(\eta) = \sum_{\{R|R\cap T_N \neq \varnothing\}} J_R \chi_R(\eta^*),$$

where $\mathcal{X}_R(\zeta) = \prod_{x\in R}(2\zeta(x) - 1)$; the variables $2\zeta(x) - 1$ are spins which take values ± 1. The corresponding finite-volume *Gibbs measure* ν_N on Y_N is defined by

$$\nu_N(\{\eta\}) = Z_N^{-1} \exp(-H_N(\eta))$$

for $\eta \in Y_N$, with $Z_N = \sum_{\zeta\in Y_N} \exp(-H_N(\zeta))$ a normalization constant; ν_N defines a measure on $\{0, 1\}^{\mathbb{Z}}$ by setting $\nu_N(A) = \nu_N(\{\eta \in Y_N | \eta^* \in A\})$. Now let us assume for simplicity that all n-body terms in the potential, for $n \geqslant 2$, are translation-invariant, that is, that $J_{R+k} = J_R$ for $k \in \mathbb{Z}$ and $|R| \geqslant 2$ (this assumption could easily be relaxed), and let $K = \sum_{R\ni x, |R|\geqslant 2}|J_R|$.

Theorem A.1. *Suppose that as x approaches* $\pm\infty$*, the one-particle potential* $J_{\{x\}}$ *approaches* $\mp\infty$ *sufficiently fast that*

$$\sum_{x\geqslant 1} \exp(2J_{\{x\}}) < \infty \quad \textit{and} \quad \sum_{x\leqslant 0} \exp(-2J_{\{x\}}) < \infty. \tag{A.1}$$

Then $\nu = \lim_{N\to\infty}\nu_N$ *exists and is a blocking measure. Moreover, if the rates* $p(x, y; \eta)$ *satisfy the detailed balance condition*

$$p(x, y; \eta)\exp\left(\sum_{\{R|x\in R \text{ or } y\in R\}} J_R \mathcal{X}_R(\eta)\right) = p(y, x; \eta^{x,y})\exp\left(\sum_{\{R|x\in R \text{ or } y\in R\}} J_R \mathcal{X}_R(\eta^{x,y})\right), \tag{A.2}$$

then ν *is reversible for the process with rates* p.

Proof. We wish to compare the measures ν_N and ν_M, where $N < M$. For $\eta \in Y_N$ we let $\eta' \in Y_M$ be the configuration which agrees with η in T_N and with η^{H} in $T_M \backslash T_N$, and for $\zeta \in Y_M$ we let $\hat{\zeta} \in Y_N$ be the restriction of ζ to T_N; thus $\hat{\zeta}' \equiv (\hat{\zeta})' \in T_M$. Now fix $\zeta \in Y_M$, let $S = \{x|\zeta(x) \neq \hat{\zeta}'(\mathrm{x})\}$, and set $S_+ = S \cap \{x \geqslant 1\}$, $S_- = S \cap \{x \leqslant 0\}$. Then

$$\begin{aligned} H_M(\zeta) &= H_M(\hat{\zeta}') - 2\sum_{x\in S_+} J_{\{x\}} + 2\sum_{x\in S_-} J_{\{x\}} + \sum_{\substack{R\cap T_M\neq\varnothing \\ |R|\geqslant 2}} J_R[\chi_R(\zeta^*) - \chi_R(\hat{\zeta}'^*)] \\ &\geqslant H_M(\hat{\zeta}') - 2\sum_{x\in S_+}(J_{\{x\}} + K) + 2\sum_{x\in S_-}(J_{\{x\}} - K). \end{aligned}$$

Thus if $\eta \in Y_N$,

$$e^{-H_M(\eta')} \leqslant \sum_{\hat{\zeta}=\eta} e^{-H_M(\zeta)} \leqslant e^{-H_M(\eta')} \prod_{x=N+1}^{M} (1+e^{2(J_{\{x\}}+K)}) \prod_{x=-N}^{M-1} (1+e^{2(-J_{\{x\}}+K)}).$$

Since the infinite products $\prod_{x\geqslant 1}(1+e^{2(J_{\{x\}}+K)})$ and $\prod_{x\leqslant 0}(1+e^{2(-J_{\{x\}}+K)})$ converge by (A.1), we have, for any $\epsilon > 0$,

$$e^{-H_M(\eta')} \leqslant \sum_{\hat{\zeta}=\eta} e^{-H_M(\zeta)} \leqslant e^{-H_M(\eta')}(1+\epsilon), \tag{A.3}$$

when N is sufficiently large, uniformly in M.

Now suppose that $A \subset \{0, 1\}^{\mathbb{Z}}$ is such that $\mathbf{1}A(\eta)$ depends on η only through the variables $\eta(x)$ for a finite number of sites – say, for $x \in T_L$. Since for $\eta \in Y_N$, $H_N(\eta) - H_M(\eta')$ is independent of η.

$$\nu_N(A) = \frac{\sum_{\eta\in Y_N,\eta^*\in A} e^{-H_N(\eta)}}{\sum_{\eta\in Y_N} e^{-H_N(\eta)}} = \frac{\sum_{\eta\in Y_N,\eta^*\in A} e^{-H_M(\eta')}}{\sum_{\eta\in Y_N} e^{-H_M(\eta')}},$$

and with (A.3) this implies that if $N \geqslant L$,

$$(1+\epsilon)^{-1}\nu_M(A) \leqslant \nu_N(A) \leqslant (1+\epsilon)\nu_M(A).$$

Hence $\lim_{N\to\infty}\nu_N(A)$ exists, so that ν exists. Similarly, if $B \subset \{0, 1\}^{\mathbb{Z}}$ is the event that $\eta(x) = \eta^{\mathrm{H}}(x)$ for $x \notin T_N$ then $\nu_M(B) = Z_M^{-1}\sum_{\eta\in Y_N}\exp(-H_M(\eta')) \geqslant (1+\epsilon)^{-1}$ by (A.3), so that ν is a blocking measure.

The measure ν is reversible for the process with rates p if, for any continuous f defined on $\{0, 1\}^{\mathbb{Z}}$ and any $x, y \in \mathbb{Z}$,

$$\int p(x, y; \eta)[f(\eta^{x,y}) - f(\eta)]\,d\nu = 0;$$

see the proof of the analogous result for stochastic Ising models in Liggett (1985). But this integral may be calculated to arbitrary accuracy by replacing ν with ν_N for suitably large N (here continuity of p in η is needed), and the fact that the integral with respect to ν_N vanishes is an immediate consequence of (A.2). $\square$

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