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Expansion of transition distributions of Lévy processes in small time

LUDGER RÜSCHENDORF* and JEANNETTE H.C. WOERNER**

Albert-Ludwigs-Universität Freiburg, Institut für Mathematische Stochastik, Eckerstr. 1, D-79104 Freiburg, Germany. E-mail: *ruschen@stochastik.uni-freiburg.de; **woerner@stochastik.uni-freiburg.de

We establish for small time t a series expansion of the transition density and the transition distribution function of Lévy processes in terms of the density and the spectral function of the Lévy measure, respectively. Furthermore, the integrals with respect to the distribution function weighted by $1/t$ are proved to converge to the integral with respect to the spectral function of the Lévy measure, when the integrated function does not increase too fast and $t \to 0$.

Keywords: Lévy processes; series expansion; transition density; transition distribution function

1. Introduction

In general, Lévy processes are determined by their characteristic functions. Typically, neither the transition density nor the transition distribution function of Lévy processes can be calculated explicitly. It is, however, known that the local behaviour of the transition distribution P_t of the Lévy process determines the Lévy measure $\mathrm{d}G(x)$, in the sense that, for every fixed $a > 0$, $(1/t)\mathrm{d}P_t(x)$ converges vaguely on $\{|x| > a\}$ to $\mathrm{d}G(x)$ as $t \to 0$. In this paper we give a detailed treatment of the behaviour of the transition distribution as $t \to 0$.

In the first part we establish a series expansion of the transition density as well as of the transition distribution function in small time. The proof involves splitting the process into a compound Poisson part and a small jump part, which leads to a sum representation where we can estimate each term separately. For the transition distribution function the important step is to find an estimate for the small jump part. The expansion holds true in general without any regularity conditions on the Lévy process.

In the case where transition densities and densities of the Lévy measure exist, we obtain an expansion of the density in small time under some regularity assumptions. The proof requires results due to Léandre (1987) and Ishikawa (1994) on first-order estimates of pure jump type Markov processes as $t \to 0$. Related estimates for more general Lévy measures are also given in Picard (1997). The special properties of Lévy processes allow us to simplify their arguments considerably and to obtain an expansion of the transition density in this case. Barndorff-Nielson (2000) independently obtained by different methods some related results on the local expansion of densities for subordinator processes. That paper also presents an interesting approach which leads to a complete infinite expansion.

From our results we obtain a local expansion of the transition distribution of a Lévy

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process in terms of the Lévy measure. Conversely, from the local behaviour of the transition distribution we can infer properties of the Lévy measure. Both directions of this relationship are of prime importance to questions of statistical inference in Lévy processes; these, however, are outside the scope of this paper.

In the second part of the paper the vague convergence as $t \to 0$ of the normalized distribution of the Lévy process to the Lévy measure is extended to the convergence of integrals of functions belonging to a certain class which do not increase too fast. The proof employs an extension of a result due to Kruglov (1970) on a characterization of the tail behaviour of infinitely divisible measures in terms of the finiteness of integrals. Further, we use the decomposition of the Lévy process into a compound Poisson part and a small jump part as well as an approximation technique in Sato (1999). In this case the influence of the big jumps is dominant, and we finally obtain approximations of the form $(1/t)\int f(x)\mathrm{d}P_t(x) \to \int f(x)\mathrm{d}G(x)$ as $t \to 0$. These results again have applications to the asymptotics of statistical estimators.

2. Expansion of the transition density

In this section we shall derive an approximation of the transition density in terms of the density of the Lévy measure for small time. Léandre (1987) and Ishikawa (1994) gave a first-order estimate of the transition density of a pure jump type Markov process, when the time tends to zero. We adopt their method for proving a series expansion of the transition density of Lévy processes, which involves rather fewer technicalities due to the special properties of Lévy processes.

Let X_t be a Lévy process with transition density $p_t(x)$ and characteristic function

$$\hat{p}_t(u) = \exp\left\{ t\left(\mathrm{i}\mu u - \frac{\sigma^2}{2}u^2 + \int \left(\mathrm{e}^{\mathrm{i}uz} - 1 - \frac{\mathrm{i}uz}{1+z^2} \right) g(z)\mathrm{d}z \right) \right\}, \tag{1}$$

where $g(z)$ denotes the density of the Lévy measure, such that $\int 1 \wedge x^2 g(x)\mathrm{d}x < \infty$.

Theorem 1. *Assume the following conditions:*

(i) *The Lévy process X_t possesses a C^∞ transition density $p_t(x)$.*
(ii) *The density of the Lévy measure $g(x)$ is C^∞ and, for $\varepsilon > 0$,*

$$\int_{|z|\geqslant\varepsilon} \frac{|g'(z)|^2}{g(z)} \mathrm{d}z < \infty, \tag{2}$$

and there exists $h \in C^\infty$ such that $h(z) \leqslant c|z|^2$, $h(z) > 0$ if $g(z) > 0$ and $z \neq 0$ for which

$$\int_{|z|\leqslant 1} \left| \frac{\mathrm{d}}{\mathrm{d}z} h(z) g(z) \right|^2 \frac{\mathrm{d}z}{g(z)} < \infty. \tag{3}$$

Then for $N \geqslant 1$ and $\eta > 0$ there exist $\varepsilon'(N) > 0$ and $t_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon'(N)$, $0 < t < t_0$ and $|x| > \eta > 0$,

$$p_t(x) = \exp\left(-t\int g_\varepsilon(y)\mathrm{d}y\right)\sum_{i=0}^{N-1}\frac{t^i}{i!}g_\varepsilon(x)^{*i} + O_{\varepsilon,\eta}(t^N), \tag{4}$$

where $g_\varepsilon(x) = 1_{|x|\geqslant\varepsilon}g(x)$ is the truncated Lévy density.

Remark. Condition (i) may be ensured by conditions on $g(x)$. For example, by a result of Orey (1968) an infinitely divisible distribution μ on $\mathbb{R}$ has a density of class C^∞ and all derivatives of the density tend to 0 as $|x| \to \infty$ if the Lévy measure ν satisfies

$$\liminf_{r\to 0}\frac{\int_{[-r,r]}x^2\nu(\mathrm{d}x)}{r^{2-\alpha}} > 0, \tag{5}$$

for some $0 < \alpha < 2$.

Condition (ii) is used to establish an upper bound for the transition density by Léandre (1987).

Proof. For the proof we split the process X_t into one part (X_t^ε) with jumps smaller than ε, where the transition density is denoted by $p_t^\varepsilon(x)$, and another part $(X_t^{c,\varepsilon})$ which is a compound Poisson process whose transition density for $x \neq 0$ is denoted by $p_t^{\varepsilon,c}(x)$ and where the density of the corresponding Lévy measure is given by $g_\varepsilon(x) = 1_{|x|\geqslant\varepsilon}g(x)$. This decomposition yields, for all $\varepsilon > 0$,

$$\begin{aligned} p_t(x) &= p_t^\varepsilon * p_t^{\varepsilon,c}(x) + \exp\left(-t\int g_\varepsilon(y)\mathrm{d}y\right)p_t^\varepsilon(x) \\ &= \exp\left(-t\int g_\varepsilon(y)\mathrm{d}y\right)\left(p_t^\varepsilon(x) + \sum_{i=1}^{\infty}\frac{t^i}{i!}p_t^\varepsilon * g_\varepsilon^{*i}(x)\right) \\ &= \exp\left(-t\int g_\varepsilon(y)\mathrm{d}y\right)\left(p_t^\varepsilon(x) + \sum_{i=1}^{N-1}\frac{t^i}{i!}p_t^\varepsilon * g_\varepsilon^{*i}(x)\right) + \overline{p}_t^\varepsilon(N, x). \end{aligned} \tag{6}$$

For the estimate of the first term of the sum we use an estimate due to Léandre (1987) which states that, for every $p > 1$ and any $\eta > 0$, there exists $\varepsilon'(p) > 0$ such that for all $t \leqslant 1$ and every $\varepsilon \in (0, \varepsilon'(p))$,

$$\sup_{|x|\geqslant\eta}\exp\left(-t\int g_\varepsilon(y)\mathrm{d}y\right)p_t^\varepsilon(x) \leqslant C(\varepsilon, \eta)t^p. \tag{7}$$

For the estimate of the remaining term we apply a lemma of Ishikawa (1994) which yields that, for every $\varepsilon > 0$ and $t > 0$,

$$\sup_{x\in\mathbb{R}}\overline{p}_t^\varepsilon(N, x) \leqslant C(\varepsilon)t^N. \tag{8}$$

Hence, in particular as $t \to 0$, for every $\varepsilon > 0$,

$$\overline{p}_t^\varepsilon(N, x) = O_\varepsilon(t^N). \tag{9}$$

By the following lemma we finally obtain an upper and lower bound for each summand in the convolution (6). □

Lemma 1. *Let $\mathcal{K}$ be the class of non-negative, bounded functions. Then, given $\delta > 0$ and $\eta > 0$, there exists $t_0 > 0$ such that*

$$f(y) - \delta \leqslant \int f(y-z) p_t^\varepsilon(z) \mathrm{d}z \leqslant f(y) + \delta + C(\varepsilon, \eta) t^p, \tag{10}$$

for all $f \in \mathcal{K}$, $|y| > \eta$, $y \in C_f$, $t \in (0, t_0)$ and $\varepsilon \in (0, \varepsilon'(p))$, where $\varepsilon'(p)$ is given in (7).

Proof. The main part of the proof is similar to the proof in Ishikawa (1994). Given $\delta > 0$, choose $\chi > 0$ such that $|f(y) - f(z)| < \delta/4$ for $|y - z| < \chi$. We may assume $\chi < \eta$. Then,

$$\begin{aligned}
\int f(y-z) p_t^\varepsilon(z) \mathrm{d}z &\geqslant \int_{|z| \leqslant \chi} f(y-z) p_t^\varepsilon(z) \mathrm{d}z \\
&= \int_{|z| \leqslant \chi} (f(y-z) - f(y)) p_t^\varepsilon(z) \mathrm{d}z + f(y) \int_{|z| \leqslant \chi} p_t^\varepsilon(z) \mathrm{d}z \\
&\geqslant (f(y) - \delta/4) \int_{|z| \leqslant \chi} p_t^\varepsilon(z) \mathrm{d}z \\
&\geqslant f(y) - \delta
\end{aligned}$$

for $t \in (0, t_1)$ for some appropriate $t_1 > 0$, since

$$\lim_{t \to 0} \int_{|z| \leqslant \chi} p_t^\varepsilon(z) \mathrm{d}z = \lim_{t \to 0} (P_t^\varepsilon(\chi) - P_t^\varepsilon(-\chi)) = P_0^\varepsilon(\chi) - P_0^\varepsilon(-\chi) = 1.$$

For inequality (10) in the opposite direction we obtain, for $t \in (0, t_2)$, by choosing an appropriate t_2,

$$\begin{aligned}
\int_{|z| \leqslant \eta} f(y-z) p_t^\varepsilon(z) \mathrm{d}z &= \int_{|z| \leqslant \chi} (f(y-z) - f(y)) p_t^\varepsilon(z) \mathrm{d}z + f(y) \int_{|z| \leqslant \eta} p_t^\varepsilon(z) \mathrm{d}z \\
&\quad + \int_{\chi < |z| \leqslant \eta} (f(y-z) - f(y)) p_t^\varepsilon(z) \mathrm{d}z \\
&\leqslant (f(y) + \delta/4) \int_{|z| \leqslant \eta} p_t^\varepsilon(z) \mathrm{d}z + \sup_x f(x) \int_{\chi < |z| \leqslant \eta} p_t^\varepsilon(z) \mathrm{d}z \\
&\leqslant f(y) + \delta
\end{aligned}$$

and, for $\varepsilon \in (0, \varepsilon'(p))$, by Léandre's estimate (7),

$$\int_{|z|>\eta} f(y-z)p_t^\varepsilon(z)\mathrm{d}z \leqslant C'(\varepsilon, \eta)t^p \int f(y-z)\mathrm{d}z.$$

This yields $\int f(y-z)p_t^\varepsilon(z)\mathrm{d}z \leqslant f(y) + \delta + C'(\varepsilon, \eta)t^p \int f(y-z)\mathrm{d}z$. Since $\delta > 0$ is arbitrary we obtain, for $1 \leqslant n \leqslant N-1$,

$$\exp\left(-t\int g_\varepsilon(y)\mathrm{d}y\right)\frac{t^n}{n!}p_t^\varepsilon * g_\varepsilon^{n*}(x) = \exp\left(-t\int g_\varepsilon(y)\mathrm{d}y\right)\frac{t^n}{n!}g_\varepsilon^{n*}(x) + O_{\varepsilon,\eta}\left(t^N\right)$$

for $t < t_0 = \min\{t_1, t_2\}$ and $\varepsilon \in (0, \varepsilon'(N))$. □

Lemma 1 and the steps described before it yield, for any $|x| > \eta > 0$, $\varepsilon \in (0, \varepsilon'(N))$ and $t \in (0, t_0)$,

$$p_t(x) = \exp\left(-t\int g_\varepsilon(x)\mathrm{d}x\right)\sum_{i=0}^{N-1}\frac{t^i}{i!}g_\varepsilon^{*i}(x) + O_{\varepsilon,\eta}\left(t^N\right). \tag{11}$$

Corollary 1. *Under the conditions of Theorem 1 we obtain, for fixed* $x \neq 0$,

$$\lim_{t\to 0}\frac{1}{t}p_t(x) = g(x). \tag{12}$$

Example 1 *Gamma process*.
For the gamma process the transition density is given by

$$p_t(x, \alpha, \beta) = \frac{\beta^{\alpha t}x^{\alpha t-1}\mathrm{e}^{-\beta x}}{\Gamma(\alpha t)} = \frac{\alpha t\beta^{\alpha t}x^{\alpha t-1}\mathrm{e}^{-\beta x}}{\Gamma(\alpha t+1)}$$

and

$$\lim_{t\to 0}\frac{1}{t}p_t(x, \alpha, \beta) = \frac{\alpha\mathrm{e}^{-\beta x}}{x},$$

which is indeed the density of the Lévy measure.

Example 2 *Normal inverse Gaussian process*.
For the normal inverse Gaussian process the transition density is given by

$$p_t(x, \alpha, \beta, \delta, \mu)$$

$$= \frac{\alpha\exp\left(t\delta\sqrt{\alpha^2-\beta^2} - t\beta\mu\right)}{\pi}\frac{t\delta}{\sqrt{t^2\delta^2+(x-t\mu)^2}}K_1\left(\alpha\sqrt{t^2\delta^2+(x-t\mu)^2}\right)\mathrm{e}^{\beta x}.$$

For $x \neq 0$ this leads to

$$\lim_{t\to 0} \frac{1}{t} p_t(x, \alpha, \beta, \delta, \mu) = \frac{\delta\alpha K_1(\alpha|x|)\mathrm{e}^{\beta x}}{\pi|x|}, \tag{13}$$

which is the density of the Lévy measure.

3. Approximate distribution function

In this section we give an expansion of the transition distribution function $P_t(x)$, without regularity assumptions on the existence of the densities of the Lévy measure or the transition distribution function. Let $G(x)$ denote the spectral function of the Lévy measure, i.e. $G(x) = \nu([x, \infty))$ for $x > 0$ and $G(x) = \nu((-\infty, x])$ for $x < 0$.

The proof of the expansion is similar to that in the case with transition densities. We have to find analogues to the estimates of the different terms in that proof.

Lemma 2. *Let A be the smallest non-negative number such that $\{x : |x| \leqslant A\}$ contains the support of the Lévy measure* $\mathrm{d}G(x)$*, which should not be identically zero. Then, for all $x > 0$ and $a < 1/A$, there exists a constant $K > 0$ such that, as $t \to 0$,*

$$1 - P_t(x) \leqslant K \exp\{ax - ax\log x\}t^{xa},$$

$$P_t(-x) \leqslant K \exp\{ax - ax\log x\}t^{xa}.$$

Proof. We use essentially ideas from the proof of Sato (1973, Lemma 1) and of Zolotarev (1965, Theorem 1).

For Lévy measures with bounded support we may write

$$\hat{P}_t(u) = \exp\left\{ t\left(\mathrm{i}\mu u + \frac{\sigma^2}{2}(\mathrm{i}u)^2 + \int (\mathrm{e}^{\mathrm{i}uz} - 1 - \mathrm{i}uz)\mathrm{d}G(z) \right) \right\}$$
$$= \exp\{t\psi(\mathrm{i}u)\},$$

and

$$\psi(x) = \mu x + \frac{\sigma^2}{2x^2} + \int (\mathrm{e}^{xz} - 1 - xz)\mathrm{d}G(z)$$

exists for any real number x. We have

$$\psi'(x) = \mu + \sigma^2 x + \int (\mathrm{e}^{xz} - 1)z\,\mathrm{d}G(z),$$

$$\psi''(x) = \sigma^2 + \int z^2\mathrm{e}^{xz}\,\mathrm{d}G(z).$$

Since $\psi''(x) > 0$ the function $\psi'(x)$ is monotone increasing and hence there exists a monotone increasing inverse function τ for $0 \leqslant x < \infty$, which is given by

$$x = \psi'(\tau(x)) = \mu + \sigma^2 \tau(x) + \int (\mathrm{e}^{\tau(x)y} - 1) y \,\mathrm{d}G(y), \tag{14}$$

for $0 \leqslant x < B \leqslant \infty$. We consider $\tau(x)$ as defined on $0 \leqslant x < \infty$, letting $\tau(x) = \infty$ for $x \geqslant B$ if $B < \infty$.

We now wish to find an upper bound for $1 - P_t(x)$. From Chebyshev's inequality we obtain

$$1 - P_t(x) = P(X_t \geqslant x) \leqslant \frac{\mathrm{E}\exp\{sX_t\}}{\mathrm{e}^{sx}} = \exp\{t\psi(s) - sx\}.$$

Minimizing the right-hand side over s, this implies

$$\inf_s \exp\{t\psi(s) - sx\} = \exp\left\{ -t \int_0^{x/t} \tau(z)\mathrm{d}z \right\}.$$

To show this, note that

$$\frac{\partial}{\partial s} \exp\{t\psi(s) - sx\} = (t\psi'(s) - x)\exp\{t\psi(s) - sx\} = 0$$

and, therefore, $\psi'(s) = x/t$ and $s = \tau(\psi'(s)) = \tau(x/t)$. Hence

$$\inf_s \exp\{t\psi(s) - sx\} = \exp\left\{ t\left(\psi\left(\tau\left(\frac{x}{t}\right)\right) - \tau\left(\frac{x}{t}\right)\psi'\left(\tau\left(\frac{x}{t}\right)\right)\right)\right\}.$$

This, together with

$$\begin{aligned}
\psi\left(\tau\left(\frac{x}{t}\right)\right) - \tau\left(\frac{x}{t}\right)\psi'\left(\tau\left(\frac{x}{t}\right)\right) &= \int_0^{\tau(x/t)} \psi'(u)\mathrm{d}u - \tau\left(\frac{x}{t}\right)\psi'\left(\tau\left(\frac{x}{t}\right)\right) \\
&= \int_{\psi'(0)}^{x/t} \psi'(\tau(w))\mathrm{d}\tau(w) - \tau\left(\frac{x}{t}\right)\psi'\left(\tau\left(\frac{x}{t}\right)\right) \\
&= \int_{\psi'(0)}^{x/t} w\,\mathrm{d}\tau(w) - \tau\left(\frac{x}{t}\right)\frac{x}{t} = -\int_{\psi'(0)}^{x/t} \tau(z)\mathrm{d}z,
\end{aligned}$$

leads to the desired estimate.

The next step is to find an estimate for τ. Since by Taylor expansion $(\mathrm{e}^{\tau(x)y} - 1)y \leqslant \mathrm{e}^{\tau(x)A}\tau(x)y^2$ for $|y| \leqslant A$, we obtain from (14)

$$x \leqslant \mu + \sigma^2\tau(x) + \mathrm{e}^{\tau(x)A}\tau(x)\int y^2 \,\mathrm{d}G(y).$$

Therefore, letting $a < 1/A$, we obtain $\log z \leqslant \tau(z)/a$ for large enough z, say $z > x_0$. Hence, for t small enough, i.e. such that $x/t > x_0$,

$$1 - P_t(x) \leqslant \exp\left\{ -ta \int_{x_0}^{x/t} \log z \, \mathrm{d}z \right\}$$
$$= \exp\{-ta[-z + z \log z]_{x_0}^{x/t}\}$$
$$= \exp\{-tax_0 + tax_0 \log x_0 + ax - ax \log x\} t^{ax}. \tag{15}$$

Similar calculations yield the corresponding result for $P_t(-x)$. □

We can now give the analogue to Léandre's estimate (7) of the part with small jumps.

Corollary 2. *Let ε be the smallest non-negative number such that $\{x : |x| \leqslant \varepsilon\}$ contains the support of the Lévy measure $\mathrm{d}G(x)$, which should not be identically zero. Then, for all $\eta > 0$ and $N \geqslant 1$, there exists $\varepsilon'(N) > 0$ such that, for any $\varepsilon \in (0, \varepsilon'(N))$,*

$$\sup_{x>\eta}(1 - P_t^\varepsilon(x)) = O_{\varepsilon,\eta}(t^N),$$
$$\sup_{-x<-\eta} P_t^\varepsilon(-x) = O_{\varepsilon,\eta}(t^N) \tag{16}$$

as $t \to 0$.

Proof. Choose $\varepsilon'(N)$ such that $\eta/\varepsilon'(N) > N$. Then there exists $a < 1/\varepsilon'(N)$ with $a\eta \geqslant N$. Hence, we have $a < 1/\varepsilon$ for all $\varepsilon \in (0, \varepsilon'(N))$, and $ax \geqslant N$ for all $x > \eta$. Furthermore, $\sup_{x>\eta} \exp\{ax - ax \log x\} = c(\varepsilon, \eta)$. □

The next lemma is analogous to Lemma 1 and gives estimates for the convolution terms $1 \leqslant n \leqslant N - 1$.

Lemma 3. *Let $\mathcal{K}$ be the class of non-negative, bounded functions. Fix $N \geqslant 1$ and $\eta > 0$; then there exist $\varepsilon'(N)$ and $t_0 > 0$ such that, for given $\delta > 0$, $\varepsilon \in (0, \varepsilon'(N))$ and $t \in (0, t_0)$, we obtain*

$$f(y) - \delta \leqslant \int f(y - z) \mathrm{d}P_t^\varepsilon(z) \leqslant f(y) + \delta + C(\varepsilon, \eta) t^N, \tag{17}$$

for all $f \in \mathcal{K}$ and $|y| > \eta$, $y \in C_f$.

Proof. The proof is essentially the same as that of Lemma 1. At the final estimate we apply Corollary 2 to obtain

$$\int_{|z|>\eta} f(y-z)\mathrm{d}P_t^\varepsilon(z) = \int_\eta^\infty f(y-z)\mathrm{d}P_t^\varepsilon(z) + \int_{-\infty}^{-\eta} f(y-z)\mathrm{d}P_t^\varepsilon(z)$$

$$\leqslant \sup_x f(x)(1-P_t^\varepsilon(\eta)+P_t^\varepsilon(-\eta))$$

$$\leqslant \sup_x f(x)C(\varepsilon, \eta)t^N.$$

□

Next we need an estimate for the remainders.

Lemma 4. *Let* $\overline{P}_t(\varepsilon, N, x) = \sum_{i=N}^\infty (t^i/i!)P_t^\varepsilon * G_\varepsilon^{*i}(x)$. *For every* $\varepsilon > 0$ *and* $t > 0$, *we have*

$$\sup_x \overline{P}_t(\varepsilon, N, x) \leqslant C(\varepsilon)t^N. \tag{18}$$

Proof. The proof is similar to that of Ishikawa (1994, Lemma 4.1). □

Theorem 2. *Let* P_t *be the distribution function of* X_t *and* G *the spectral function of the associated Lévy measure, i.e.* $G(x) = \nu([x, \infty))$ *for* $x > 0$ *and* $G(x) = \nu((-\infty, x])$ *for* $x < 0$. *Fix* $N \geqslant 1$; *then there exist* $\varepsilon'(N) > 0$ *and* $t_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon'(N))$ *and* $t \in (0, t_0)$, *we obtain:*

(i) *for* $x > \eta > 0$,

$$1 - P_t(x) = \sum_{i=1}^{N-1} \frac{t^i}{i!} G_\varepsilon^{*i}(x) + O_{\varepsilon,\eta}\left(t^N\right); \tag{19}$$

(ii) *for* $x < -\eta < 0$,

$$P_t(x) = \sum_{i=1}^{N-1} \frac{t^i}{i!} G_\varepsilon^{*i}(x) + O_{\varepsilon,\eta}\left(t^N\right), \tag{20}$$

where $G_\varepsilon(x)$ *denotes the truncated spectral function* $1_{|x|\geqslant\varepsilon}G(x)$.

Proof. We split the process X_t into two components, one ($X_t^{c,\varepsilon}$) a compound Poisson process with Lévy measure $1_{|x|\geqslant\varepsilon}\,\mathrm{d}G(x) = \mathrm{d}G_\varepsilon(x)$, and the other ($X_t^\varepsilon$) with Lévy measure $1_{|x|<\varepsilon}\,\mathrm{d}G(x)$ which is a process of the type as in Lemma 2. Hence

$$
\begin{aligned}
P^{X_t^\varepsilon} &= P^{X_t^\varepsilon} * P^{X_t^{\varepsilon,c}} \\
&= \exp(-t|G_\varepsilon|)\left(\sum_{i=0}^{\infty}\frac{t^i}{i!}P^{X_t^\varepsilon} * \nu_\varepsilon^{*i}\right),
\end{aligned}
$$

$$
\begin{aligned}
1 - P_t(x) &= P^{X_t}([x, \infty)) \\
&= \exp(-t|G_\varepsilon|)\left(1 - P_t^\varepsilon(x) + \sum_{i=1}^{\infty}\frac{t^i}{i!}P^{X_t^\varepsilon} * \nu_\varepsilon^{*i}([x, \infty))\right) \\
&= \exp(-t|G_\varepsilon|)\left(1 - P_t^\varepsilon(x) + \sum_{i=1}^{N-1}\frac{t^i}{i!}P_t^\varepsilon * G_\varepsilon^{*i}(x) + \overline{P}_t(\varepsilon, N, x)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
P_t(x) &= P^{X_t}((-\infty, x]) \\
&= \exp(-t|G_\varepsilon|)\left(P_t^\varepsilon(x) + \sum_{i=1}^{\infty}\frac{t^i}{i!}P^{X_t^\varepsilon} * \nu_\varepsilon^{*i}((-\infty, x])\right) \\
&= \exp(-t|G_\varepsilon|)\left(P_t^\varepsilon(x) + \sum_{i=1}^{N-1}\frac{t^i}{i!}P_t^\varepsilon * G_\varepsilon^{*i}(x) + \overline{P}_t(\varepsilon, N, x)\right).
\end{aligned}
$$

Now the proof is similar to that of Theorem 1, using Corollary 2, Lemma 3 and Lemma 4. □

As a corollary we obtain that the Lévy measure can be obtained from the distribution function of the transition probability measure.

Corollary 3. *Let P_t be the distribution function of X_t and G the associated spectral function. Then we obtain:*

(i) *for each fixed $x > 0$,*

$$\lim_{t\to 0}\frac{1}{t}(1 - P_t(x)) = G(x); \tag{21}$$

(ii) *for each fixed $x < 0$,*

$$\lim_{t\to 0}\frac{1}{t}P_t(x) = G(x). \tag{22}$$

If P_t and G possess densities p_t and g, then for $x \neq 0$

$$\lim_{t\to 0}\frac{1}{t}p_t(x) = \frac{\partial}{\partial t}p_t(x)|_{t=0} = g(x), \tag{23}$$

where we additionally assume that $P_t(x)$ is continuous in a neighbourhood of $(t=0, x)$ and furthermore that $(\partial/\partial t)P_t(x)$, $(\partial/\partial x)P_t(x)$, $(\partial/\partial t)(\partial/\partial x)P_t(x)$ exist and are continuous in $(t=0, x)$.

Remark. Since G is a spectral function, density in this case means $g \geqslant 0$ and $G'(x) = g(x)$ for $x < 0$ and $G'(x) = -g(x)$ for $x > 0$.

This theorem sharpens the well-known property that, for every fixed $a > 0$, $(1/t)P(X_t \in \mathrm{d}x)$ converges vaguely on $\{|x| > a\}$ to $\mathrm{d}G(x)$ as $t \to 0$ (cf. Bertoin 1996, p. 39).

Proof. The estimates for the distribution functions are an obvious consequence of Theorem 2, by choosing $\varepsilon < |x|$.

Note that $P_0(x)$ equals 0 as $x < 0$ and 1 as $x > 0$, and define $p_0(x) = 0$ for $x \neq 0$. Hence, for $x > 0$, and interchanging the differentiations,

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t} p_t(x) &= \frac{\partial}{\partial t} p_t(x)|_{t=0} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}(P_t(x))|_{t=0} \\
&= -\frac{\partial}{\partial t}\frac{\partial}{\partial x}(1 - P_t(x))|_{t=0} = -\frac{\partial}{\partial x}\frac{\partial}{\partial t}(1 - P_t(x))|_{t=0} \\
&= -\frac{\partial}{\partial x} G(x) = g(x).
\end{aligned}$$

Similar calculations for $x < 0$ yield the desired result. □

4. Asymptotics for integrals with respect to $\mathrm{d}P_t$

Since the results in Sections 2 and 3 are only valid for fixed $x \neq 0$, they do not provide enough information about the asymptotic behaviour of integrals with respect to $\mathrm{d}P_t$. In this section we shall derive such an asymptotic result for functions which basically do not increase too fast. The idea is again to split the integral in two parts, one with the small jumps and one with big jumps.

For the purposes of this section, P denotes an infinitely divisible probability measure with Lévy measure $\mathrm{d}G$. The tail behaviour of the distribution for Lévy measures with bounded support is estimated by the following result due to Sato (1973) which is used throughout this section.

Theorem 3 (Sato 1973). *Let A be the smallest non-negative number (possibly $+\infty$) such that the set $\{x : |x| \leqslant A\}$ contains the support of $G(x)$. Then, for every positive $\alpha < 1/A$,*

$$\int_{|x|>r} \mathrm{d}P(x) = o(r^{-\alpha r}) \tag{24}$$

as $r \to \infty$ and P has finite $\mathrm{e}^{\alpha|x|\log|x|}$-moment; for every $\alpha > 1/A$,

$$\int_{|x|>r} \mathrm{d}P(x)/r^{-\alpha r} \to \infty \tag{25}$$

as $r \to \infty$ *and* P *has infinite* $\mathrm{e}^{\alpha|x|\log|x|}$*-moments.*

Denote by $\mathcal{S}$ the class of functions f that can be decomposed into a submultiplicative and subadditive component:

$$\mathcal{S} = \{f(x)|\exists H,\ K \text{ such that } \forall x,\ y \in \mathbb{R}: \tag{26}$$

$$f(x+y) = h(x+y)k(x+y) \leqslant HK(h(x)+h(y))k(x)k(y)\}$$

Functions belonging to $\mathcal{S}$ cannot increase faster than $x^k \mathrm{e}^{cx}$ (Kruglov 1970). The tail behaviour of the transition distribution function is now formulated in terms of integrability conditions with respect to the Lévy measure.

Theorem 4. *For all functions* ℓ *such that* $|\ell| \leqslant f$, $f \in \mathcal{S}$ *with* $\int_{|x|\geqslant 1} f(x)\mathrm{d}G(x) < \infty$ *and* $\int_{|x|\geqslant 1}|k(x)|\mathrm{d}G(x) < \infty$*, we obtain* $\int \ell(x)\mathrm{d}P(x) < \infty$.

Proof. Theorem 4 is an extension of a theorem in Kruglov (1970) formulated for functions which are either submultiplicative or subadditive. Kruglov's proof can be extended to the more general class. In the first step P is decomposed as $P(x) = F_1 * F_2(x)$, where F_1 is a compound Poisson distribution and F_2 corresponds to the small jumps. We use the refined estimate in Theorem 3 for the decay of F_2 with Lévy measure with bounded support. Kruglov's induction proof for the decay of F_1 can be extended to the more general class $\mathcal{S}$ to yield the result. □

Lemma 5. *Weak convergence of* $P^{X_t^\varepsilon}(\mathrm{d}x)$ *to* $\varepsilon_{\{0\}}(\mathrm{d}x)$ *as* $t \to 0$ *may be extended to functions* $f \in C^1$ *such that* $|f| \leqslant g_1$ *and* $|f'| \leqslant g_2$*, for* $g_1,\ g_2 \in \mathcal{S}$*, i.e.*

$$\lim_{t\to 0} \int f(x)\mathrm{d}P_t^\varepsilon(x) = f(0). \tag{27}$$

Remark. Actually the proof only requires that f and $|f'|$ do not increase faster than $x^k \mathrm{e}^{cx}$ for some k and c.

Proof. Integration by parts leads to

$$\int_0^b f'(x)\int_x^\infty \mathrm{d}P_t^\varepsilon(y)\mathrm{d}x = \left[f(x)\int_x^\infty \mathrm{d}P_t^\varepsilon(y)\right]_0^b + \int_0^b f(x)\mathrm{d}P_t^\varepsilon(x),$$

which yields

$$\int_0^\infty f'(x)\int_x^\infty \mathrm{d}P_t^\varepsilon(y)\mathrm{d}x = -f(0)\int_0^\infty \mathrm{d}P_t^\varepsilon(y) + \int_0^\infty f(x)\mathrm{d}P_t^\varepsilon(x)$$

since $f(b)\int_b^\infty \mathrm{d}P_t^\varepsilon(x) \to 0$ for $b \to \infty$ as $|f| \leqslant g_1$ with $g_1 \in \mathcal{S}$ by Lemma 2. Analogous calculations for the negative part yield

$$\int_{-\infty}^{0} f'(x) \int_{-\infty}^{x} \mathrm{d}P_t^{\varepsilon}(y)\mathrm{d}x = f(0)\int_{-\infty}^{0} \mathrm{d}P_t^{\varepsilon}(y) - \int_{-\infty}^{0} f(x)\mathrm{d}P_t^{\varepsilon}(x).$$

Hence,

$$\int f(x)\mathrm{d}P_t^{\varepsilon}(x) = f(0) + \int_0^{\infty} f'(x)\int_x^{\infty} \mathrm{d}P_t^{\varepsilon}(y)\mathrm{d}x - \int_{-\infty}^{0} f'(x)\int_{-\infty}^{x} \mathrm{d}P_t^{\varepsilon}(y)\mathrm{d}x.$$

Looking at the second term, we obtain, by using Lemma 2,

$$\begin{aligned}
&\left|\int_0^{\infty} f'(x)\int_x^{\infty} \mathrm{d}P_t^{\varepsilon}(y)\mathrm{d}x\right| \\
&\qquad \leqslant \int_0^{\infty} |f'(x)|(1 - P_t^{\varepsilon}(x))\mathrm{d}x \\
&\qquad \leqslant \int_0^{\infty} |f'(x)|\exp\{-tax_0 + tax_0 \log x_0 + ax - ax \log x\} t^{ax}\,\mathrm{d}x \\
&\qquad \leqslant \exp\{-tax_0 + tax_0 \log x_0\}\int_0^{\infty} |f'(x)|\exp\{ax - ax\log x\}\mathrm{d}x < \infty
\end{aligned}$$

as $t \leqslant 1$ and $x/t \geqslant x_0$. Hence, by dominated convergence,

$$\lim_{t\to 0}\int_0^{\infty} f'(x)\int_x^{\infty} \mathrm{d}P_t^{\varepsilon}(y)\mathrm{d}x = 0.$$

Analogous calculations for the third term yield

$$\lim_{t\to 0}\int_{-\infty}^{0} f'(x)\int_{-\infty}^{x} \mathrm{d}P_t^{\varepsilon}(y)\mathrm{d}x = 0.$$

Therefore, we conclude that

$$\lim_{t\to 0}\int f(x)\mathrm{d}P_t^{\varepsilon}(x) = f(0). \qquad \square$$

Lemma 6. *If* $\int_{|x|\geqslant 1} f(x)\mathrm{d}G(x) < \infty$, $f(x)(|x|^2\Lambda 1)^{-1}|_{x=0} = 0$, *and* $f(x)(|x|^2 \wedge 1)^{-1} \in C_b$, *then*

$$\lim_{t\to 0}\frac{1}{t}\int f(x)\mathrm{d}P_t(x) = \int f(x)\mathrm{d}G(x). \tag{28}$$

Proof. The proof is analogous to those of Theorems 8.1, 8.7 and 8.9 in Sato (1999). Let $\{t_n\}$ be any sequence such that $t_n\downarrow 0$, and $\mu = P_1$ a given infinitely divisible probability measure. Define μ_n by

$$\hat{\mu}_n(z) = \exp[t_n^{-1}(\hat{\mu}(z)^{t_n} - 1)]$$

$$= \exp\left[t_n^{-1}\int_{\mathbb{R}\backslash\{0\}}(\mathrm{e}^{\mathrm{i}xz} - 1)\mathrm{d}P_{t_n}(x)\right].$$

Hence μ_n is compound Poisson and, for each z as $n \to \infty$,

$$\hat{\mu}_n(z) = \exp[t_n^{-1}(\exp(t_n \log \hat{\mu}(z)) - 1)]$$

$$= \exp[t_n^{-1}(t_n \log \hat{\mu}(\mathrm{z}) + \mathrm{O}(t_n^2))].$$

This implies $\hat{\mu}_n(z) \to \hat{\mu}(z)$ and thus $\mu_n \xrightarrow{\mathcal{D}} \mu$.

Now define $\rho_n(\mathrm{d}x) = (|x|^2 \wedge 1)\nu_n(\mathrm{d}x)$, where $\nu_n(\mathrm{d}x)$ denotes the Lévy measure of $\mu_n(\mathrm{d}x)$. Then, from Sato (1999, Theorem 8.7), we obtain that $\{\rho_n\}$ is tight, i.e. $\sup_n \rho_n(\mathbb{R}^d) < \infty$ and $\lim_{\ell\to\infty} \sup_n \int_{|x|>\ell}\rho_n(\mathrm{d}x) = 0$. So there exists a subsequence $\{\rho_{n_k}\}$ such that $\rho_{n_k} \xrightarrow{\mathcal{D}} \rho$, where ρ is a finite measure. Define $\nu(\mathrm{d}x) = (|x|^2 \wedge 1)^{-1}\rho(\mathrm{d}x)$ on $\{|x| > 0\}$ and $\nu(\{0\}) = 0$. As $f(x)(|x|^2 \wedge 1)^{-1} \in C_b$ we obtain

$$\lim_{k\to\infty}\int f(x)\nu_{n_k}(\mathrm{d}x) = \lim_{k\to\infty}\int f(x)(|x|^2 \wedge 1)^{-1}\rho_{n_k}(\mathrm{d}x)$$

$$= \int f(x)(|x|^2 \wedge 1)^{-1}\rho(\mathrm{d}x)$$

$$= \int f(x)\nu(\mathrm{d}x).$$

This implies the result since $\nu(\mathrm{d}x)$ is unique by Sato (1999, p. 43) and noting that $\nu_n(\mathrm{d}x) = t_n^{-1}\,\mathrm{d}P_{t_n}(x)$. □

Theorem 5. *Let f be any function in C^1. Suppose that the following assumptions can be made:*

(i) $|f| \leqslant g_1$ *and* $|f'| \leqslant g_2$ *for* $g_1, g_2 \in \mathcal{S}$;
(ii) $1_{|x|\leqslant 1}f(x)(|x|^2 \wedge 1)^{-1} \in C_b$ *and* $f(x)(|x|^2\Lambda 1)^{-1}|_{x=0} = 0$;
(iii) $|\int f(x+y)\mathrm{d}G_\varepsilon(y)| \leqslant g_3$, $|(\partial/\partial x)\int f(x+y)\mathrm{d}G_\varepsilon(y)| \leqslant g_4$ *for* $g_3, g_4 \in \mathcal{S}$. *Then we have that*

$$\lim_{t\to 0}\frac{1}{t}\int f(x)\mathrm{d}P_t(x) = \int f(x)\mathrm{d}G(x). \tag{29}$$

Remark. For the case where $\int(|x| \wedge 1)\mathrm{d}G(x) < \infty$ it is sufficient to postulate $1_{|x|\leqslant 1}f(x)(|x| \wedge 1)^{-1} \in C_b$, and when $\int \mathrm{d}G(x) < \infty$ to assume that $1_{|x|\leqslant 1}f \in C_b$.

Proof. We again use the technique of splitting the process into a compound Poisson process and a process with small jumps,

$$P_t(x) = \exp\left\{-t\int \mathrm{d}G_\varepsilon(x)\right\}\left(P_t^\varepsilon(x) + \sum_{i=1}^{\infty}\frac{t^i}{i!}G_\varepsilon^{i*} * P_t^\varepsilon(x)\right),$$

where $G_\varepsilon = 1_{|x|>\varepsilon}G$ and P_t^ε denotes the distribution function belonging to the jumps less than or equal to ε, i.e. with Lévy measure $1_{|x|\leqslant\varepsilon}\,\mathrm{d}G(x)$. For the limit we then only have to consider the first two terms of the sum. The second leads to

$$\begin{aligned}\lim_{t\to 0}\int f(x)\mathrm{d}G_\varepsilon * P_t^\varepsilon(x) &= \lim_{t\to 0}\int\int f(x+y)\mathrm{d}G_\varepsilon(y)\mathrm{d}P_t^\varepsilon(x)\\ &= \int f(x)\mathrm{d}G_\varepsilon(x)\\ &= \int_{|x|>\varepsilon} f(x)\mathrm{d}G(x),\end{aligned}$$

which follows immediately from Lemma 5 and assumption (iii).

For the first summand we split the integral into two parts, one pertaining to the integral over $|x| \leqslant \varepsilon$ and one to the integral over $|x| > \varepsilon$. From the preceding lemma we already know that

$$\lim_{t\to 0}\frac{1}{t}\int_{|x|\leqslant\varepsilon} f(x)\mathrm{d}P_t(x) = \int_{|x|\leqslant\varepsilon} f(x)\mathrm{d}G(x).$$

For the other part of the first summand we use the same technique as in Lemma 5. By assumption (i) we have

$$\int f(x)\mathrm{d}P_t^\varepsilon(x) = f(0) + \int_0^\infty f'(x)\int_x^\infty \mathrm{d}P_t^\varepsilon(y)\mathrm{d}x - \int_{-\infty}^0 f'(x)\int_{-\infty}^x \mathrm{d}P_t^\varepsilon(y)\mathrm{d}x,$$

hence

$$\int_{|x|>\varepsilon} f(x)\mathrm{d}P_t^\varepsilon(x) = \int_{x>\varepsilon} f'(x)\int_x^\infty \mathrm{d}P_t^\varepsilon(y)\mathrm{d}x - \int_{x<-\varepsilon} f'(x)\int_{-\infty}^x \mathrm{d}P_t^\varepsilon(y)\mathrm{d}x.$$

For the first term on the right-hand side we then obtain, by Lemma 2,

$$\begin{aligned}&\left|\frac{1}{t}\int_{x>\varepsilon} f'(x)\int_x^\infty \mathrm{d}P_t^\varepsilon(y)\mathrm{d}x\right|\\ &\qquad\leqslant \frac{1}{t}\int_{x>\varepsilon}|f'(x)|(1-P_t^\varepsilon(x))\mathrm{d}x\\ &\qquad\leqslant \int_{x>\varepsilon}|f'(x)|\exp\{-tax_0 + tax_0\log x_0 + ax - ax\log x\}t^{ax-1}\,\mathrm{d}x\\ &\qquad\leqslant \exp\{-tax_0 + tax_0\log x_0\}\int_{x>\varepsilon}|f'(x)|\exp\{ax - ax\log x\}\mathrm{d}x < \infty\end{aligned}$$

for $t \leqslant 1$, $x/t \geqslant x_0$ and $a < 1/\varepsilon$. Hence by dominated convergence we may interchange limit and integration and obtain

$$\lim_{t \to 0} \frac{1}{t} \int_{x > \varepsilon} f'(x) \int_x^\infty \mathrm{d}P_t^\varepsilon(y) \mathrm{d}x = 0.$$

Analogous calculations for the second term yield

$$\lim_{t \to 0} \frac{1}{t} \int_{x < -\varepsilon} f'(x) \int_{-\infty}^x \mathrm{d}P_t^\varepsilon(y) \mathrm{d}x = 0,$$

and hence

$$\lim_{t \to 0} \frac{1}{t} \int_{|x| > \varepsilon} f(x) \mathrm{d}P_t^\varepsilon(x) = 0.$$

Piecing together the three estimates, we obtain the desired result. □

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