The categorified Diassociative cooperad

Frédéric Chapoton

Abstract

Using representations of quivers of type $\mathbb{A}$, we define an anticyclic cooperad in the category of triangulated categories, which is a categorification of the linear dual of the Diassociative operad.

1 Introduction

The Diassociative operad has been introduced by Loday [Lod95, Lod97, Lod01]. It can be described as a collection of free abelian groups $\mathrm{Dias}(n)$ of rank n and maps $\circ_i$ from $\mathrm{Dias}(m) \otimes \mathrm{Dias}(n)$ to $\mathrm{Dias}(m+n-1)$ satisfying some kind of associativity. The composition maps $\circ_i$ have a simple combinatorial description, using grafting of planar trees with a distinguished path from the root to a leaf.

It has been shown in [Cha05] that one can endow this operad with a refined structure of anticyclic operad. This means that there exists a map of order $n+1$ on $\mathrm{Dias}(n)$, with some compatibility with the $\circ_i$ maps.

The aim of this article is to prove that this whole structure (or rather its linear dual, which is an anticyclic cooperad) is the shadow of a natural representation-theoretic object, related to the Dynkin diagrams of type $\mathbb{A}$.

We will not assume any knowledge of operads, but the interested reader can consult [GK94, Lod01, Smi01, MSS02] for basics of this theory and [Mar99, Cha05] for the notion of anticyclic operad.

We first define a cooperad $\mathcal{A}$ in the category of abelian categories. This amounts to a collection of abelian categories $\mathcal{A}_n$ for $n \geq 1$ and some functors ∇ from these categories to products of two of these categories. The categories $\mathcal{A}_n$ involved are just the categories of modules over the $\overrightarrow{\mathbb{A}}_n$ quivers. These are very classical objects in representation theory.

Received by the editors October 2009.
Communicated by M. Van den Bergh.

The ∇ functors are defined as tensor product by some specific multiplicity-free bimodules. The axioms of cooperads are checked by using a combinatorial description of the tensor product of such bimodules.

At the level of the Grothendieck groups, one then checks that the induced cooperad is the linear dual of the Diassociative operad. The classes of simple modules correspond to the usual basis of $\mathrm{Dias}(n)$ and the ∇ functors give the linear dual of the $\circ_i$ maps.

As the ∇ functors are given by the tensor product with projective bimodules, they are exact. Going to the derived categories $D\mathcal{A}_n$, we prove that there is some compatibility between the ∇ functors and the Auslander-Reiten translations. At the level of Grothendieck groups, this amounts to the structure of anticyclic cooperad on the Diassociative cooperad.

2 General facts

2.1 Quivers of type $\mathbb{A}$

For each integer $n \geq 1$, let $\overrightarrow{\mathbb{A}}_n$ be the quiver $1 \to 2 \to \cdots \to n$. This is a quiver on the graph of type $\mathbb{A}_n$ in the classification of Dynkin diagrams.

Let k be a fixed ground field. Let $\mathcal{A}_n$ be the category of finite dimensional right-modules over the path algebra of $\overrightarrow{\mathbb{A}}_n$ over k. This is an abelian category, with a finite number of isomorphism classes of indecomposable objects.

Let $D\mathcal{A}_n$ be the bounded derived category of the category $\mathcal{A}_n$. This is a triangulated category, with a shift functor that will be denoted by S. Indecomposable objects in $D\mathcal{A}_n$ are just shifts of the images of the indecomposable objects in $\mathcal{A}_n$.

There exists a canonical self-equivalence of $D\mathcal{A}_n$, called the Auslander-Reiten translation and denoted by τ_n.

The Nakayama functor ν_n is the composite $\tau_n S = S\tau_n$. This functor maps, for each vertex i of $\overrightarrow{\mathbb{A}}_n$, the projective module P_i to the injective module I_i.

2.2 Products of quivers

Let $\overrightarrow{\mathbb{A}}_{m_1,m_2,\ldots,m_k}$ be the product quiver $\overrightarrow{\mathbb{A}}_{m_1} \times \overrightarrow{\mathbb{A}}_{m_2} \times \cdots \times \overrightarrow{\mathbb{A}}_{m_k}$. We consider this as a quiver with relations by imposing all possible commutation relations.

A module over this quiver amounts to a module over the tensor product of the path algebras of the quivers $\overrightarrow{\mathbb{A}}_{m_i}$. Therefore, one can forget the action of some of the factors to define restricted modules.

As there is a canonical isomorphism of quivers from $\overrightarrow{\mathbb{A}}_{m,n}$ to $\overrightarrow{\mathbb{A}}_{n,m}$, there are canonical equivalences between the corresponding module and derived categories. Let us denote by X these flip functors.

More generally, any permutation of the factors in a multiple product of quivers $\overrightarrow{\mathbb{A}}_n$ give rise to a corresponding equivalence.

2.3 Standard modules

Let M be a module over a quiver $\overrightarrow{\mathbb{A}}_{m_1,m_2,\ldots,m_k}$. Let M_s be the vector space associated with a vertex s. One says that M is **multiplicity-free** if the dimension of M_s is at most 1 for every vertex s. Let then $\mathsf{S}(M)$ be the support of M, which is the set of vertices s such that $\dim M_s = 1$.

Let M be a multiplicity-free module. One says that M is **standard** if, for any two adjacent vertices s, s' in $\mathsf{S}(M)$, the map between M_s and $M_{s'}$ is an isomorphism.

To describe a standard module M up to isomorphism, it is clearly enough to give its support $\mathsf{S}(M)$. The support of a standard module cannot be arbitrary, because of the commuting conditions that must be satisfied. One can then build back the module using copies of the field k and identity maps between them.

2.4 Tensor product of projective standard modules

There is a simple combinatorial description of the tensor product of two projective standard modules.

Let us consider only the special case that we will need. Let $M_{a\,;\,b,c}$ be a $\overrightarrow{\mathbb{A}}_a^{op} \times \overrightarrow{\mathbb{A}}_b \times \overrightarrow{\mathbb{A}}_c$-module and let $M_{c\,;\,d,e}$ be a $\overrightarrow{\mathbb{A}}_c^{op} \times \overrightarrow{\mathbb{A}}_d \times \overrightarrow{\mathbb{A}}_e$-module. Assume that $M_{a\,;\,b,c}$ is $\overrightarrow{\mathbb{A}}_c$-projective and that $M_{c\,;\,d,e}$ is $\overrightarrow{\mathbb{A}}_c^{op}$-projective.

Then one can define the tensor product of $M_{a\,;\,b,c}$ and $M_{c\,;\,d,e}$ over (the path algebra of) $\overrightarrow{\mathbb{A}}_c$. This is a $\overrightarrow{\mathbb{A}}_a^{op} \times \overrightarrow{\mathbb{A}}_b \times \overrightarrow{\mathbb{A}}_d \times \overrightarrow{\mathbb{A}}_e$-module denoted by $M_{a\,;\,b,c} \otimes_{\overrightarrow{\mathbb{A}}_c} M_{c\,;\,d,e}$.

Assume that $M_{a\,;\,b,c}$ and $M_{c\,;\,d,e}$ are standard modules with support $\mathsf{S}_{a\,;\,b,c}$ and $\mathsf{S}_{c\,;\,d,e}$. Let us define a set $\mathsf{S}_{a\,;\,b,c} \times_c \mathsf{S}_{c\,;\,d,e}$ as follows:

$$\mathsf{S}_{a\,;\,b,c} \times_c \mathsf{S}_{c\,;\,d,e} = \{(\alpha,\beta,\delta,\epsilon) \mid \exists \gamma\, (\alpha,\beta,\gamma) \in \mathsf{S}_{a\,;\,b,c} \text{ and } (\gamma,\delta,\epsilon) \in \mathsf{S}_{c\,;\,d,e}\}.$$

Proposition 2.1. *The tensor product $M_{a\,;\,b,c} \otimes_{\overrightarrow{\mathbb{A}}_c} M_{c\,;\,d,e}$ is isomorphic to the standard module with support $\mathsf{S}_{a\,;\,b,c} \times_c \mathsf{S}_{c\,;\,d,e}$.*

Proof. The tensor product over the field k has a basis indexed by tuples $(\alpha,\beta,\gamma,\gamma',\delta,\epsilon)$ with $(\alpha,\beta,\gamma) \in \mathsf{S}_{a\,;\,b,c}$ and $(\gamma',\delta,\epsilon) \in \mathsf{S}_{c\,;\,d,e}$. Then one has to take the quotient by the action of the idempotents and arrows of the quiver $\overrightarrow{\mathbb{A}}_c$. Abusing notation, we will identify tuples with the corresponding vectors.

By the action of the idempotents in the path algebra, one can see that all vectors $(\alpha,\beta,\gamma,\gamma',\delta,\epsilon)$ with $\gamma \neq \gamma'$ vanish in the tensor product.

There remains to quotient by the action of the arrows. This means that one has to identify $(\alpha,\beta,\gamma,\gamma,\delta,\epsilon)$ and $(\alpha,\beta,\gamma+1,\gamma+1,\delta,\epsilon)$, provided that one has $(\alpha,\beta,\gamma) \in \mathsf{S}_{a\,;\,b,c}$ and $(\gamma+1,\delta,\epsilon) \in \mathsf{S}_{c\,;\,d,e}$.

By the hypothesis made (considered modules are projective), in this situation, one also has $(\alpha,\beta,\gamma+1) \in \mathsf{S}_{a\,;\,b,c}$ and $(\gamma,\delta,\epsilon) \in \mathsf{S}_{c\,;\,d,e}$. Hence both $(\alpha,\beta,\gamma,\gamma,\delta,\epsilon)$ and $(\alpha,\beta,\gamma+1,\gamma+1,\delta,\epsilon)$ are non-zero vectors.

Therefore, the tensor product has a basis indexed by tuples $(\alpha,\beta,\delta,\epsilon)$ such that there exists γ with $(\alpha,\beta,\gamma) \in \mathsf{S}_{a\,;\,b,c}$ and $(\gamma,\delta,\epsilon) \in \mathsf{S}_{c\,;\,d,e}$.

One can also see by construction that indeed the module is standard. ∎

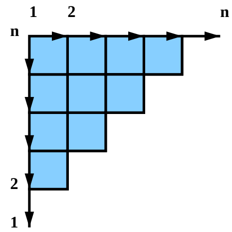

Figure 1: The bimodule N_6 corresponding to Nakayama functor ν_6

2.5 Fiber-reversal and action of τ

Let N_n be the standard $\overrightarrow{\mathbb{A}}_n^{op} \times \overrightarrow{\mathbb{A}}_n$ module with support

$$\{(i,j) \in [1,n] \times [1,n] \mid i \geq j\}. \tag{1}$$

Note that N_n is injective as a $\overrightarrow{\mathbb{A}}_n^{op}$-module and as a $\overrightarrow{\mathbb{A}}_n$-module.

Lemma 2.2. *The Nakayama functor ν_n on the category $D\mathcal{A}_n$ is the derived tensor product $? \otimes^L_{\overrightarrow{\mathbb{A}}_n} N_n$.*

Proof. This follows from the fact that the image by ν of the projective module P_i is the injective module I_i, by the standard way of representing functors by bimodules. ■

Let us now introduce some operations on support sets.

Let S be a subset in the product $[1,m_1] \times \cdots \times [1,m_k]$. Fix an index i. Assume that S is projective in the direction i, *i.e.* that

$$\text{if } (j_1,\ldots,j_i,\ldots,j_k) \in \mathsf{S} \text{ then } (j_1,\ldots,\ell,\ldots,j_k) \in \mathsf{S} \text{ for all } \ell \geq j_i. \tag{2}$$

The **fiber-reversal** of S in the direction i is

$$\{(j_1,\ldots,j_i,\ldots,j_k) \in [1,m_1] \times \cdots \times [1,m_k] \mid (j_1,\ldots,j_i-1,\ldots,j_k) \notin \mathsf{S}\}. \tag{3}$$

Note that the fiber-reversal of S in direction i is never disjoint from S, and really depends on the index i.

One can give a similar definition of the fiber-reversal if the set S is injective in the direction i, *i.e.* if the following condition holds:

$$\text{if } (j_1,\ldots,j_i,\ldots,j_k) \in \mathsf{S} \text{ then } (j_1,\ldots,\ell,\ldots,j_k) \in \mathsf{S} \text{ for all } \ell \leq j_i. \tag{4}$$

Let us now describe the (derived) tensor product with N_n. We consider only the special case that we will need.

Let $M_{n;c;d}$ be a $\overrightarrow{\mathbb{A}}_n^{op} \times \overrightarrow{\mathbb{A}}_c \times \overrightarrow{\mathbb{A}}_d$ standard module with support $\mathsf{S}_{n;c;d}$. Assume that $M_{n;c,d}$ is projective as a $\overrightarrow{\mathbb{A}}_n^{op}$-module.

Proposition 2.3. *The derived tensor product of $N_n \otimes^L_{\overrightarrow{\mathbb{A}}_n} M_{n;c,d}$ is isomorphic to the standard module with support the fiber-reversal of $\mathsf{S}_{n;c;d}$ in the direction of length n.*

Proof. The tensor product $N_n \otimes_k M_{n;c,d}$ has a basis indexed by tuples $(\alpha, \beta, \beta', \gamma, \delta)$ with $\alpha \geq \beta$ and $(\beta', \gamma, \delta) \in \mathsf{S}_{n;c;d}$. Abusing notation, we will identify tuples with the corresponding vectors.

Using the idempotents in the path algebra, the tensor product over $\overrightarrow{\mathbb{A}}_n$ is spanned by tuples $(\alpha, \beta, \beta, \gamma, \delta)$. Then one has to identify $(\alpha, \beta, \beta, \gamma, \delta)$ and $(\alpha, \beta + 1, \beta + 1, \gamma, \delta)$ as soon as $\alpha \geq \beta$ and $(\beta + 1, \gamma, \delta) \in \mathsf{S}_{n;c,d}$.

Using the hypothesis that $M_{n;c,d}$ is projective, one has in this situation that $(\beta, \gamma, \delta) \in \mathsf{S}_{n;c,d}$.

The only case where one of the two vectors $(\alpha, \beta, \beta, \gamma, \delta)$ and $(\alpha, \beta + 1, \beta + 1, \gamma, \delta)$ is zero and the other is not zero happens if $\beta + 1 > \alpha$, *i.e.* $\alpha = \beta$.

It follows that the vector $(\alpha, \beta, \beta, \gamma, \delta)$ is mapped to zero in the tensor product over $\overrightarrow{\mathbb{A}}_n$ if and only if $(\alpha + 1, \gamma, \delta) \in \mathsf{S}(M)$ and are otherwise just identified. This is exactly the definition of the fiber-reversal of $\mathsf{S}_{n;c,d}$ in the first direction.

One can easily check that the tensor product is standard. ■

3 The ∇ functors on module categories

In this section, we define a cooperad structure on the collection of module categories $(\mathcal{A}_n)_{n \geq 1}$. This means that several functors ∇ are introduced and some kind of associativity properties are proved.

Let $n \geq 1$ be an integer and let m, i be integers such that $1 \leq i \leq m$.

Consider the set $\mathsf{S}^n_{m;i}$ of integer triples (γ, μ, ν) in $[1, m + n - 1] \times [1, m] \times [1, n]$ such that

$$\begin{aligned} &(\mu \leq i - 1 \text{ and } \gamma \leq \mu) \\ \text{or } &(\mu = i \text{ and } \gamma \leq i + \nu - 1) \\ \text{or } &(i + 1 \leq \mu \text{ and } \gamma \leq \mu + n - 1). \end{aligned}$$

This is illustrated in Figure 2 with $m = 6, n = 4$ and $i = 3$.

For later use, here is an equivalent description of $\mathsf{S}^n_{m;i}$:

$$(\gamma \leq i \text{ and } \gamma \leq \mu) \tag{5}$$
$$\text{or } (i + 1 \leq \gamma \leq i + n - 1 \text{ and } \gamma \leq i + \nu - 1) \text{ and } i \leq \mu \tag{6}$$
$$\text{or } (i + 1 \leq \gamma \leq i + n - 1 \text{ and } i + \nu \leq \gamma) \text{ and } i + 1 \leq \mu \tag{7}$$
$$\text{or } (i + n \leq \gamma \text{ and } \gamma - n + 1 \leq \mu). \tag{8}$$

One can easily check that the set $\mathsf{S}^n_{m;i}$ has the following property : if $(\gamma, \mu, \nu) \in \mathsf{S}^n_{m;i}$ and if $(\gamma', \mu', \nu') \in [1, m + n - 1] \times [1, m] \times [1, n]$ is such that $\gamma' \leq \gamma$, $\mu \leq \mu'$ and $\nu \leq \nu'$, then $(\gamma', \mu', \nu', \gamma') \in \mathsf{S}^n_{m;i}$.

This implies that one can define a representation $M^n_{m;i}$ of the quiver $\overrightarrow{\mathbb{A}}^{op}_{m+n-1} \times \overrightarrow{\mathbb{A}}_m \times \overrightarrow{\mathbb{A}}_n$ as the standard module with support $\mathsf{S}^n_{m;i}$. Note that $M^n_{m;i}$ is projective with respect to $\overrightarrow{\mathbb{A}}_m$, $\overrightarrow{\mathbb{A}}_n$ and $\overrightarrow{\mathbb{A}}^{op}_{m+n-1}$.

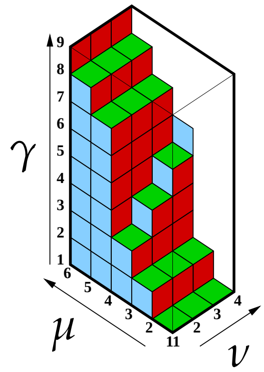

Figure 2: The module $M^4_{6;3}$ and its symbolic description

Let then $\nabla^n_{m;i}$ be the functor from $\mathcal{A}_{m+n-1}$ to $\mathcal{A}_{m,n}$ defined by the tensor product over $\overrightarrow{\mathbb{A}}_{m+n-1}$ by $M^n_{m;i}$:

$$\nabla^n_{m;i} = ? \otimes_{\overrightarrow{\mathbb{A}}_{m+n-1}} M^n_{m;i}. \tag{9}$$

Note that $\nabla^n_{1;1}$ is the identity functor of $\mathcal{A}_n$ and that $\nabla^1_{m;i}$ is the identity functor of $\mathcal{A}_m$ for all i.

3.1 Relation to the Dias cooperad

Let n be an integer and let $1 \leq j \leq n$. Let S^n_j be the simple $\overrightarrow{\mathbb{A}}_n$-module supported at vertex j. Let P^n_j be the projective $\overrightarrow{\mathbb{A}}_n$-module associated with vertex j. The class of a module M in the Grothendieck group $K^0(\mathcal{A}_n)$ of $\mathcal{A}_n$ will be denoted by $[M]$. The elements $[S^n_j]$ for $1 \leq j \leq n$ form a basis of $K^0(\mathcal{A}_n)$.

Let us now compute the class $[\nabla^n_{m;i}(S^{m+n-1}_j)]$.

From the explicit description of the module $M^n_{m;i}$, one has

$$[\nabla^n_{m;i}(P^{m+n-1}_j)] = \begin{cases} [P^{m,n}_{j,1}] & \text{if } 1 \leq j \leq i, \\ [P^{m,n}_{i+1,1}] + [P^{m,n}_{i,j-i+1}] - [P^{m,n}_{i+1,j-i+1}] & \text{if } i+1 \leq j \leq i+n-1, \\ [P^{m,n}_{j-n+1,1}] & \text{if } i+n \leq j \leq m+n-1, \end{cases} \tag{10}$$

where $P^{m,n}_{i,j}$ is the projective module associated with vertex (i,j) of $\overrightarrow{\mathbb{A}}_m \times \overrightarrow{\mathbb{A}}_n$.

Using a projective resolution of the simple modules, one deduces that

$$[\nabla^n_{m;i}(S^{m+n-1}_j)] = \begin{cases} \sum_{k=1}^n [S^{m,n}_{j,k}] & \text{if } 1 \leq j \leq i-1, \\ [S^{m,n}_{i,j-i+1}] & \text{if } i \leq j \leq i+n-1, \\ \sum_{k=1}^n [S^{m,n}_{j-n+1,k}] & \text{if } i+n \leq j \leq m+n-1, \end{cases} \tag{11}$$

where $S^{m,n}_{i,j}$ is the simple module at vertex (i,j) for $\overrightarrow{\mathbb{A}}_m \times \overrightarrow{\mathbb{A}}_n$.

Taking the linear dual basis e of the basis $[S]$, one finds that the linear dual maps $\circ$ to the ∇ maps are given by

$$\circ^n_{m;i}(e^m_j \otimes e^n_k) = \begin{cases} e^{m+n-1}_j & \text{if } i > j, \\ e^{m+n-1}_{i+k-1} & \text{if } i = j, \\ e^{m+n-1}_{j+n-1} & \text{if } i < j. \end{cases} \tag{12}$$

This is exactly the usual description of the Diassociative operad, in the usual basis e of $\mathrm{Dias}(n)$, see [Cha05, §3].

4 Cooperadic properties of ∇ functors

One has to check two different axioms to prove that the ∇ functors define a cooperad. Let us call them the commutativity axiom and the associativity axiom.

4.1 Commutativity axiom

Let m, n, p and i, j be integers such that $1 \leq i < j \leq m$.

Proposition 4.1. *The modules M have the following property : there is an isomorphism*

$$M^n_{m+p-1;i} \otimes_{\overrightarrow{\mathbb{A}}_{m+p-1}} M^p_{m;j} \simeq M^p_{m+n-1;j+n-1} \otimes_{\overrightarrow{\mathbb{A}}_{m+n-1}} M^n_{m;i}, \tag{13}$$

where both sides are $\overrightarrow{\mathbb{A}}^{op}_{m+n+p-2} \times \overrightarrow{\mathbb{A}}_m \times \overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_p$-modules.

Proof. As the modules M are standard and projective, their tensor products can be described using their supports. According to the description of tensor products in Prop. 2.1, one therefore has to compute and compare the sets $\mathrm{S}^p_{m;j} \times_{m+p-1} S^n_{m+p-1;i}$ and $\mathrm{S}^n_{m;i} \times_{m+n-1} S^p_{m+n-1;j+n-1}$.

By an elementary computation with boolean combinations of inequalities, one can show that both sides are given by the set of (γ, μ, ν, π) in $[1, m+n+p-2] \times [1, m] \times [1, n] \times [1, p]$ such that

$$\begin{aligned} &(\mu \leq i-1 \text{ and } \gamma \leq \mu) \\ \text{or } &(\mu = i \text{ and } \gamma \leq i+\nu-1) \\ \text{or } &(i+1 \leq \mu \leq j-1 \text{ and } \gamma \leq \mu+n-1) \\ \text{or } &(\mu = j \text{ and } \gamma \leq j+n+\pi-2) \\ \text{or } &(j+1 \leq \mu \text{ and } \gamma \leq \mu+n+p-2). \end{aligned}$$

∎

Corollary 4.2. *The functors ∇ have the following property : there is a natural transformation*

$$(\mathrm{Id}_m \times X)(\nabla^p_{m;j} \times \mathrm{Id}_n)\nabla^n_{m+p-1;i} \simeq (\nabla^n_{m;i} \times \mathrm{Id}_p)\nabla^p_{m+n-1;j+n-1}. \tag{14}$$

4.2 Associativity axiom

Let m, n, p and i, j be integers such that $1 \leq i \leq m$ and $1 \leq j \leq n$.

Proposition 4.3. *The modules M have the following property : there is an isomorphism*

$$M_{m;i}^{n+p-1} \otimes_{\overrightarrow{\mathbb{A}}_{n+p-1}} M_{n;j}^{p} \simeq M_{m+n-1;j+i-1}^{p} \otimes_{\overrightarrow{\mathbb{A}}_{m+n-1}} M_{m;i}^{n}, \tag{15}$$

where both sides are $\overrightarrow{\mathbb{A}}_{m+n+p-2}^{op} \times \overrightarrow{\mathbb{A}}_m \times \overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_p$-modules.

Proof. As in the previous section, one just has to compute the supports of these modules. One can check that they both give the set of (μ, ν, π, γ) in $[1, m] \times [1, n] \times [1, p] \times [1, m+n+p-2]$ such that

$$\begin{aligned}
&(\mu \leq i-1 \text{ and } \gamma \leq \mu)\\
\text{or } &(\mu = i \text{ and } \nu \leq j-1 \text{ and } \gamma \leq i+\nu-1)\\
\text{or } &(\mu = i \text{ and } \nu = j \text{ and } \gamma \leq i+j+\pi-2)\\
\text{or } &(\mu = i \text{ and } j+1 \leq \nu \text{ and } \gamma \leq i+\nu+p-2)\\
\text{or } &(i+1 \leq \mu \text{ and } \gamma \leq \mu+n+p-2).
\end{aligned}$$

■

Corollary 4.4. *The functors ∇ have the following property : there is a natural transformation*

$$(\mathrm{Id}_m \times \nabla_{n;j}^{p}) \nabla_{m;i}^{n+p-1} \simeq (\nabla_{m;i}^{n} \times \mathrm{Id}_p) \nabla_{m+n-1;j+i-1}^{p}. \tag{16}$$

5 Relations between ∇ and τ

Let us consider the functors $\nabla_{m;i}^{n}$ as the derived tensor product with $M_{m;i}^{n}$. As the modules $M_{m;i}^{n}$ are projective in every direction, this is just the usual tensor product. Therefore, we obtain a cooperad structure on the collection of derived categories $(D\mathcal{A}_n)_{n\geq 1}$.

In this section, we define an anticyclic cooperad structure on the collection of derived categories $(D\mathcal{A}_n)_{n\geq 1}$. This means that some compatibility properties hold between the functors ∇ and the Auslander-Reiten translations τ. We will rather work with the Nakayama functors ν, described as derived tensor product with the modules N.

There are two different axioms for the notion of anticyclic operad : let us call them the border axiom and the inner axiom.

5.1 Border axiom

Proposition 5.1. *The fiber-reversal of $\mathsf{S}_{m;1}^{n}$ in the direction of length $m+n-1$ is equal to the fiber-reversal in the direction of length n of the fiber-reversal in the direction of length m of $\mathsf{S}_{n;n}^{m}$. In terms of modules, this means that*

$$N_{m+n-1} \otimes_{\overrightarrow{\mathbb{A}}_{m+n-1}}^{L} M_{m;1}^{n} \simeq (M_{n;n}^{m} \otimes_{\overrightarrow{\mathbb{A}}_m}^{L} N_m) \otimes_{\overrightarrow{\mathbb{A}}_n}^{L} N_n. \tag{17}$$

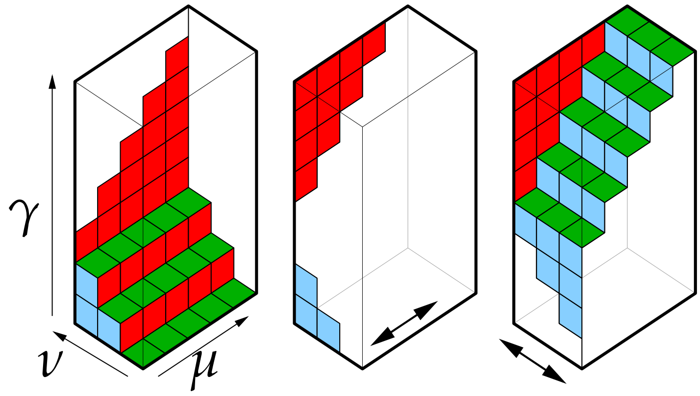

Figure 3: The module $M^6_{4;4}$, its fiber-reversal in the direction of length 6 and the fiber-reversal of the result in the direction of length 4

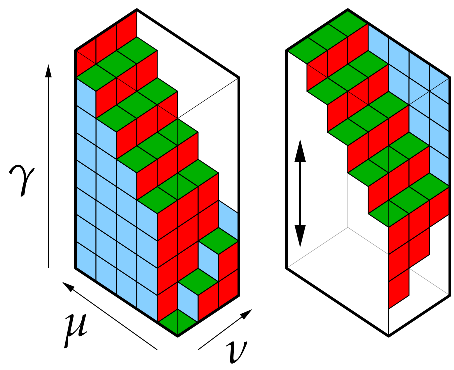

Figure 4: The module $M^4_{6;1}$, its top-bottom fiber-reversal

Proof. Let us first compute the fiber-reversal of $\mathrm{S}^n_{m;1}$ in the direction of γ of length $m+n-1$. One easily gets

$$\begin{aligned} &(\mu = 1 \text{ and } \gamma \geq \nu) \\ \text{or } &\gamma \geq \mu + n - 1. \end{aligned}$$

Let us then compute the fiber-reversal of $\mathrm{S}^m_{n;n}$ in the direction of μ of length m. One gets

$$\begin{aligned} &(\mu = 1 \text{ and } \gamma \leq \nu) \\ \text{or } &(\nu = n \text{ and } \gamma \geq n + \mu - 1). \end{aligned}$$

Then one can compute the fiber-reversal of this set in the direction of ν and check the expected result. ■

Corollary 5.2. *The functors* ∇ *satisfy*

$$\nabla^n_{m;1}\tau_{m+n-1} \simeq SX(\tau_n \times \tau_m)\nabla^m_{n;n}. \tag{18}$$

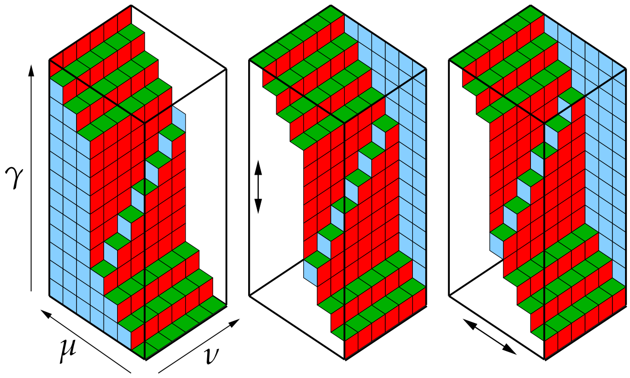

Figure 5: The module $M^7_{8;4}$, its top-bottom fiber-reversal and its fiber-reversal in the direction of length 8

5.2 Inner axiom

Let us assume here that $2 \leq i \leq m$.

Proposition 5.3. *The fiber-reversal of* $\mathsf{S}^n_{m;i}$ *in the direction of length* $m+n-1$ *coincides with the fiber-reversal of* $\mathsf{S}^n_{m;i-1}$ *in the direction of length* m*. In terms of modules, this means*

$$N_{m+n-1} \otimes^L_{\vec{\mathbb{A}}_{m+n-1}} M^n_{m;i} \simeq M^n_{m;i-1} \otimes^L_{\vec{\mathbb{A}}_m} N_m. \tag{19}$$

Proof. On the one hand, for the fiber-reversal of $\mathsf{S}^n_{m;i}$ in the direction γ of length $m+n-1$, one easily gets

$$\begin{aligned} &(\mu \leq i-1 \text{ and } \gamma \geq \mu)\\ \text{or } &(\mu = i \text{ and } \gamma \geq i+\nu-1)\\ \text{or } &(i+1 \leq \mu \text{ and } \gamma \geq \mu+n-1). \end{aligned}$$

On the other hand, using the alternative description (5) of $S^n_{m;i-1}$, the fiber-reversal of $\mathsf{S}^n_{m;i-1}$ in the direction μ of length m is

$$\begin{aligned} &(\gamma \leq i-1 \text{ and } \gamma \geq \mu)\\ \text{or } &(i \leq \gamma \leq i+n-2 \text{ and } \gamma \leq i+\nu-2) \text{ and } i-1 \geq \mu\\ \text{or } &(i \leq \gamma \leq i+n-2 \text{ and } i-1+\nu \leq \gamma) \text{ and } i \geq \mu\\ \text{or } &(i+n-1 \leq \gamma \text{ and } \gamma-n+1 \geq \mu). \end{aligned}$$

It is then a routine matter to check that they are indeed equal. ∎

Corollary 5.4. *The functors* ∇ *satisfy*

$$\nabla^n_{m;i}\tau_{m+n-1} \simeq (\tau_m \times \mathrm{Id})\nabla^n_{m;i-1}, \tag{20}$$

for $2 \leq i \leq m$.

Remerciements : Merci à Bernhard Keller pour ses conseils. Ce travail a été partiellement financé par le programme blanc ANR-06-BLAN-0380.

References

[Cha05] F. Chapoton. On some anticyclic operads. *Algebr. Geom. Topol.*, 5:53–69 (electronic), 2005.

[GK94] V. Ginzburg and M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.

[Lod95] J.-L. Loday. Algèbres ayant deux opérations associatives (digèbres). *C. R. Acad. Sci. Paris Sér. I Math.*, 321(2):141–146, 1995.

[Lod97] J.-L. Loday. Overview on Leibniz algebras, dialgebras and their homology. In *Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995)*, volume 17 of *Fields Inst. Commun.*, pages 91–102. Amer. Math. Soc., Providence, RI, 1997.

[Lod01] J.-L. Loday. Dialgebras. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 7–66. Springer, Berlin, 2001.

[Mar99] M. Markl. Cyclic operads and homology of graph complexes. *Rend. Circ. Mat. Palermo (2) Suppl.*, (59):161–170, 1999. The 18th Winter School "Geometry and Physics" (Srní, 1998).

[MSS02] M. Markl, S. Shnider, and J. Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[Smi01] V. A. Smirnov. *Simplicial and operad methods in algebraic topology*, volume 198 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the Russian manuscript by G. L. Rybnikov.

Université de Lyon ; Université Lyon 1 ;
CNRS, UMR5208, Institut Camille Jordan,
43 boulevard du 11 novembre 1918,
F-69622 Villeurbanne-Cedex, France