On natural representations of the symplectic group

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Abstract

Let V_k be the Weyl module of dimension $\binom{2n}{k} - \binom{2n}{k-2}$ for the group $G = \mathrm{Sp}(2n, \mathbb{F})$ arising from the k-th fundamental weight of the Lie algebra of G. Thus, V_k affords the grassmann embedding of the k-th symplectic polar grassmannian of the building associated to G. When $\mathrm{char}(\mathbb{F}) = p > 0$ and n is sufficiently large compared with the difference $n - k$, the G-module V_k is reducible. In this paper we are mainly interested in the first appearance of reducibility for a given $h := n - k$. It is known that, for given h and p, there exists an integer $n(h, p)$ such that V_k is reducible if and only if $n \geq n(h, p)$. Moreover, let $n \geq n(h, p)$ and R_k the largest proper non-trivial submodule of V_k. Then $\dim(R_k) = 1$ if $n = n(h, p)$ while $\dim(R_k) > 1$ if $n > n(h, p)$. In this paper we will show how this result can be obtained by an investigation of a certain chain of G-submodules of the exterior power $W_k := \wedge^k V$, where $V = V(2n, \mathbb{F})$.

1 Introduction

Let V be a $2n$-dimensional vector space over a field $\mathbb{F}$ and, for a given non-degenerate alternating form $\alpha(.,.)$ of V, let $G \cong \mathrm{Sp}(2n, \mathbb{F})$ be the symplectic group associated with it. Let Δ be the building associated to the group G. So, the elements of Δ of type $k = 1, 2, ..., n$ are the k-dimensional subspaces of V totally isotropic for the form α.

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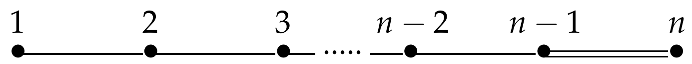

For $1 \leq k \leq n$, let $\mathcal{G}_k$ be the k-th grassmannian of $\mathrm{PG}(V)$, where the k-subspaces of V are taken as points. The lines of $\mathcal{G}_k$ are the sets $l_{X,Y} = \{Z \mid X \subset Z \subset Y,\ \dim(Z) = k\}$ for a $(k+1)$-subspace Y of V and a $(k-1)$-subspace X of Y. Put $W_k := \wedge^k V$ and let $\iota_k \colon \mathcal{G}_k \mapsto \mathrm{PG}(W_k)$ be the natural embedding of $\mathcal{G}_k$ in $\mathrm{PG}(W_k)$, sending a k-subspace $\langle v_1, ..., v_k \rangle$ of V to the 1-dimensional subspace $\langle v_1 \wedge ... \wedge v_k \rangle$ of W_k. Let Δ_k be the k-th grassmannian of Δ, elements of Δ of type k being taken as points of Δ_k. When $1 < k < n$ the lines of Δ_k are the lines $l_{X,Y}$ of $\mathcal{G}_k$ where X and Y are totally α-isotropic, while Δ_1 and Δ_n are respectively the polar space and the dual polar space associated to Δ. In any case, Δ_k is a full subgeometry of $\mathcal{G}_k$. The embedding ι_k induces an embedding $\varepsilon_k \colon \Delta_k \mapsto \mathrm{PG}(V_k)$, where V_k is the subspace of W_k spanned by $\iota_k(\Delta_k)$. We call ε_k the *grassmann embedding* of Δ_k. When $\mathrm{char}(\mathbb{F}) \neq 2$ the embedding ε_k is universal (Blok [3]).

The group G acts on V_k via ε_k. In the language of Chevalley groups, V_k is the Weyl module obtained from the irreducible module for the complex Lie algebra of type C_n whose highest weight is the k-th fundamental dominant weight, by tensoring a minimal admissible lattice with the field $\mathbb{F}$. Since this process does not reduce the dimension, it follows from Weyl's dimension formula that $\dim(V_k) = \binom{2n}{k} - \binom{2n}{k-2}$.

The G-module V_k is irreducible when $\mathrm{char}(\mathbb{F}) = 0$. On the other hand, when $\mathrm{char}(\mathbb{F}) = p > 0$ the module V_k can be reducible. Explicitly, V_k admits a unique maximal proper G-submodule R_k and the dimension $f_k := \dim(V_k/R_k)$ can be computed with the help of the following recursive formula (Premet and Suprunenko [15]; also Adamovich [1] for the case of $\mathrm{char}(\mathbb{F}) = 2$; see also Brouwer [8]):

$$f_k = \binom{2n}{k} - \binom{2n}{k-2} - \sum_{j \in J_p(k)} f_j, \tag{1}$$

where $J_p(k) := \{j \mid 0 \leq j < k,\ k - j \equiv 0 \pmod 2,\ n - j + 1 \geq_p (k-j)/2\}$ and, for two integers $a \geq b \geq 0$, expressed as $a = a_0 + a_1 p + \cdots + a_r p^r$ and $b = b_0 + b_1 p + \ldots b_s p^s$ to the base p, we write $a \geq_p b$ and say that *a contains b to the base p* if $s \leq r$ and for every $i = 1, \ldots, s$ either $b_i = a_i$ or $b_i = 0$.

The following is a corollary of the proof of (1) by Premet and Suprunenko [15]. Choose a nonnegative integer h and for $n > h$ put $k = n - h$. Let $N(h,p)$ be the smallest integer $n > h$ such that p divides $\binom{1+\lfloor (n+h)/2 \rfloor}{h+1}$. Then

Theorem 1.1. *The G-module V_k is reducible if and only if $n \geq N(h,p)$. If $n = N(h,p)$ then* $\dim(R_k) = 1$. *If $n > N(h,p)$ then* $\dim(R_k) > 1$.

In their investigation, Premet and Suprunenko (as well as Brouwer) focus on the structure of weight spaces of V_k. In doing this, they ultimately rely on the theory of Specht modules for symmetric groups. This approach is perfect in its own kind, but a geometry-oriented reader might want something else. The approach by Adamovich [1] is different, but even less geometric.

During a visit of the first author to the University of Siena in the summer of 2007, we laid down the project of developing a more geometric approach to this matter. We don't aim at a strikingly simple proof of (1). Rather, we believe there are still interesting facts that, concealed under the approach of [15], wait to be discovered. In a sense, our project is a pretext to enlight a few of them. This paper is a first contribution to our project. It is also a continuation of earlier papers as [11], [6], [7] and [9], which quite naturally fit with this project.

Throughout this paper we assume char($\mathbb{F}$) $\neq 2$ (as Premet and Suprunenko do in [15]). We will say a few words on this restriction at the end of this introduction.

The following characterization of R_k is crucial in our investigation. Note first that the grassmann embedding ε_k of Δ_k is *polarized*, in the following sense: for every point X of Δ_k, the ε_k-image $\varepsilon_k(H(X))$ of the set $H(X)$ of points of Δ_k at non-maximal distance from X spans a hyperplane $H(X)_k := \langle \varepsilon_k(H(X)) \rangle$ of $\mathrm{PG}(V_k)$. Then

$$R_k = \bigcap (H(X)_k \mid X \text{ point of } \Delta_k) \qquad (2)$$

This is proved by Blok [4] for arbitrary Lie incidence geometries associated to Chevalley groups by considering a certain contravariant form β, whose radical is exactly R_k. (See also [7], where (2) is proved for any embeddable dual polar space, provided it is defined over a commutative division ring.) According to (2), the subspace R_k defines a quotient ε_k/R_k of ε_k and the embedding ε_k/R_k is the minimal homogeneous embedding as well as the minimal polarized embedding of Δ_k. Equivalently, every homogeneous embedding of Δ_k is polarized. Moreover, as we will see in Section 2, a non-degenerate bilinear form $\alpha_k(.,.)$ can be defined on W_k such that, for any two points X and Y of Δ_k and any non-zero vectors $x \in \varepsilon_k(X), y \in \varepsilon_k(Y)$, we have $\alpha_k(x,y) = 0$ if and only if $X \in H(Y)$. By (2), R_k is precisely the radical of the restriction of α_k to $V_k \times V_k$. In other words, $R_k = V_k \cap V_k^{\perp_k}$, where $\perp_k$ stands for the orthogonality relation with respect to α_k. A relation certainly exists between α_k and the above mentioned form β but this point is not yet completely clear to us.

We call R_k the *radical* of ε_k. Our project amounts to investigate R_k. We shall firstly investigate the structure of the G-module W_k introducing what we call its *basic series*, namely a chain

$$V_k = W_k^{(k)} \subset W_{k-2}^{(k)} \subset W_{k-4}^{(k)} \subset ... \subset W_{k-2\lfloor k/2 \rfloor}^{(k)} = W_k$$

of G-invariant subspaces of W_k such that, for every integer i with $0 \leq i < k/2 - 1$, the quotient $W_{k-2i}^{(k)}/W_{k-2i+2}^{(k)}$ affords the embedding $\varepsilon_{k-2i} : \Delta_{k-2i} \to \mathrm{PG}(V_{k-2i})$ (see Theorem 3.5). When k is odd the clause $i < k/2 - 1$ is equivalent to $i < \lfloor k/2 \rfloor$. When k is even and $i = k/2 - 1$ then $W_2^{(k)}$ is a hyperplane of $W_0^{(k)} = W_k$. In this case $W_0^{(k)}/W_2^{(k)}$ is 1-dimensional (a trivial module for G).

In Section 4 we prove that if k is odd then G acts fixed-point-freely on $\mathrm{PG}(W_k)$ while when k is even G admits exactly one fixed-point P on $\mathrm{PG}(W_k)$, which we call the *pole* of G on W_k. Actually, $P = (W_2^{(k)})^{\perp_k}$. The pple P seems to be ultimately responsible for R_k being non-trivial.

Indeed, chosen a nonnegative integer h, let n range in the set of integers $n > h$ and put $k := n - h$, as we have done before. Let $p = \mathrm{char}(\mathbb{F})$. If $p = 0$ then $R_k = 0$ for every k. In this case, when k is even, the pole P always sits outside $W_2^{(k)}$. In particular, it sits out of the first member $V_k = W_k^{(k)}$ of the basic series. Let $p > 0$. Then, as we shall see in Section 5, there exists an even nonnegative integer $k(h,p)$, depending on h and p, such that $R_k = 0$ if and only if $n < n(h,p) := k(h,p) + h$. In fact, as long as $n < n(h,p)$ (and $k = n - h$ is even) P travels from one member of the basic series to another one, but keeping out of the deepest member $V_k = W_k^{(k)}$ of the series. As soon as $n = n(h,p)$ the pole P enters V_k. In this case $R_k = P$. If $n > n(h,p)$ then $\dim(R_k) > 1$ and R_k contains possibly improper submodules generated by poles of subspaces of V_k generated by ε_k-images of subgeometries of Δ_k corresponding to residues of certain elements of Δ. Our dream is to explain formula (1) in this perspective, but this goes far beyond what we can do at present, provided it can be done. In this paper, leaving that dream aside for the moment, we will mainly focus on $n(h,p)$.

In Section 5 (Theorem 5.7) we exploit our approach to prove that $n(h,p) \leq N(h,p)$, where $N(h,p)$ is as in Theorem 1.1. We also give a very simple explicit expression for $N(h,p)$. Moreover, we prove that, if $h + 1 \not\equiv 0 \pmod{p}$, then $n(h,p) = N(h,p)$ (Theorem 5.8). However $n(h,p) = N(h,p)$ for any value of h, as we know by Theorem 1.1, no matter if p divides $h+1$ or not. Regretfully, we are presently unable to squeeze this equality out of our approach when $h + 1 \equiv 0 \pmod{p}$.

We must mention that a different proof of Theorem 1.1 has also been obtained by De Bruyn [13]. The proof by De Bruyn is remarkable. It only exploits elementary linear algebra: no Lie algebras and almost no groups. However it goes on through rather complicated computations which, as we feel, do not help so much to catch the very substance of what is going on.

As we have said above, we assume $\mathrm{char}(\mathbb{F}) \neq 2$. The following is the main reason for this restriction. In a few arguments of ours we will exploit the fact that the embedding ε_k is universal, but this can be false when $\mathrm{char}(\mathbb{F}) = 2$. Most likely, those arguments can be modified so that to work in the case of characteristic 2 as well, but we prefer to keep this task for a future work.

2 Preliminaries

2.1 Notation

Let $\mathbb{F}$, n, V, $\alpha(.,.)$, G, Δ, Δ_k, $\mathcal{G}_k$, ι_k, ε_k, W_k and V_k be as in the introduction. As said in the introduction, we assume $\mathrm{char}(\mathbb{F}) \neq 2$. Also, $n \geq 2$.

A linear mapping $f : V \to V$ can be carried to W_k in two different ways. Given an ordered basis $\mathbb{E} = (e_1, ..., e_{2n})$ of V, which we may assume to be hyperbolic for the form α, consider the following basis of W_k:

$$\wedge^k\mathbb{E} := \{e_{i_1} \wedge e_{i_2} \wedge ... e_{i_k} \ : \ 1 \leq i_1 < i_2 < ... < i_k \leq 2n\}.$$

We denote by $(f)_k$ and $[f]_k$ the linear mappings of W_k defined as follows on the vectors of $\wedge^k\mathbb{E}$:

$$(f)_k \;:\; e_{i_1} \wedge ... \wedge e_{i_k} \mapsto f(e_{i_1}) \wedge ... \wedge f(e_{i_k}),$$
$$[f]_k \;:\; e_{i_1} \wedge ... \wedge e_{i_k} \mapsto \sum_{r=1}^{k} e_{i_1} \wedge ... \wedge e_{i_{r-1}} \wedge f(e_{i_r}) \wedge e_{i_{r+1}} \wedge ... \wedge e_{i_k}.$$

For every subgroup X of $GL(V)$, we denote by $(X)_k$ the image of X by the mapping $(.)_k$ sending $f \in X$ to $(f)_k$. Note that $(X)_k$ might be non-isomorphic to X. However, denoted by Z and Z_k the centers of $GL(V)$ and $GL(W_k)$ respectively, $(.)_k$ induces an isomorphism from $X/(X \cap Z)$ to $(X)_k/(X)_k \cap Z_k$. In other words, X and $(X)_k$ are projectively isomorphic. Let $\mathcal{L}(V)$ be the Lie algebra of all linear mappings of V. Then for every subalgebra X of $\mathcal{L}(V)$ the mapping $[.]_k$ sending f to $[f]_k$ induces an isomorphism from X to a subalgebra $[X]_k$ of the Lie algebra $\mathcal{L}(W_k)$ of all linear transformations of W_k.

The image $(G)_k$ of $G = \mathrm{Sp}(2n, \mathbb{F})$ by $(.)_k$ stabilizes the subspace V_k of W_k. Similarly, V_k is stabilized by the image $[L(G)]_k$ of the Lie algebra $L(G)$ of G. According to this, in the sequel we will freely regard $(.)_k$ and $[.]_k$ as mappings from G to $GL(V_k)$ and from $L(G)$ to the Lie algebra $\mathcal{L}(V_k)$ of linear mappings of V_k, whenever this point of view will be convenient.

When there will be no danger of confusion, for a subgroup X of G we will simply write X instead of $(X)_k$. In particular, if $(X)_k$ stabilizes a subspace K of W_k then we say that K is *X-invariant*.

2.2 Point-stabilizers and their unipotent radicals

In this subsection we recall a number of known facts on stabilizers of elements of Δ_k in $G = \mathrm{Sp}(2n, \mathbb{F})$. Steinberg [17], Carter [12] and Tits [18, Chapter 13] are our main sources for this matter.

Given a point S of Δ_k let G_S be its stabilizer in G. Then $G_S = U_S L$ (*Levi-decomposition*) where U_S (the *unipotent radical* of G_S) is the subgroup of G_S that stabilizes both S and $S^\perp/S$ elementwise (where $\perp$ stands for orthogonality with respect to α), while L (a *Levi complement*) is the stabilizer of S, S_1 and S_2, where S_1 is a complement of S in $S^\perp$ and S_2 is a complement of $S^\perp$ in V contained in $S_1^\perp$, namely a complement of S in $S_1^\perp$. The alternating form α of V induces a non-degenerate alternating form on S_1 while S_2 is totally isotropic for α.

The product $U_S L = G_S$ is semidirect with $U_S \trianglelefteq G_S$. The unipotent radical U_S acts regularly on the set of direct sums $S_1 \oplus S_2$ as above (whence on the set of Levi complements). The Levi complement L splits as a direct product $L_1 \times L_2$ where L_1 acts trivially on S_1 and induces $\mathrm{GL}(k, \mathbb{F})$ on S. The action of L_1 on S_2 is the dual of that on S. On the other hand, L_2 acts trivially on $S \oplus S_2$ and induces $\mathrm{Sp}(2n-2k, \mathbb{F})$ on S_1. The quotients $L_1/Z(L_1) \cong \mathrm{PGL}(k, \mathbb{F})$ and $L_2/Z(L_2) \cong \mathrm{PSp}(2n-2k, \mathbb{F})$ are the groups induced by G_S on the lower and upper residues of S and $U_S \cdot (Z(L_1) \times Z(L_2))$ is the kernel of the action of G_S on $\mathrm{Res}_\Delta(S)$.

The unipotent radical U_S acts trivially on each of S, $S^\perp/S$ and $V/S^\perp$. Let $\widehat{U}_S$ be the elementwise stabilizer of $S^\perp$ in G. Clearly $\widehat{U}_S \trianglelefteq U_S$. The quotient group $U_S/\widehat{U}_S$ acts regularly on the set of complements of S in V while U_S acts regularly on the set of complements of $S^\perp$ in V. The latters are just the points of Δ_k at maximal distance from S.

Both groups $U_S/\widehat{U}_S$ and $\widehat{U}_S$ are abelian. The quotient $U_S/\widehat{U}_S$ is isomorphic to the additive group of $V(2n-2k,\mathbb{F})$ while $\widehat{U}_S$ is isomorphic to the additive group of symmetric $(k\times k)$-matrices with entries in $\mathbb{F}$. Moreover, $Z(U_S)=\widehat{U}_S$.

Given a Borel subgroup B of G containing G_S, let R^+ be the set of positive roots associated to B and $\{U_\alpha\}_{\alpha\in R^+}$ the set of corresponding root subgroups of B. We recall that B is the stabilizer in G of a chamber C of Δ containing S and every root group U_α is a 1-parameter group $U_\alpha=\{x_\alpha(t)\}_{t\in\mathbb{F}}$ isomorphic to the additive group of $\mathbb{F}$. Chosen an apartment $\mathcal{A}$ of Δ containing C, we may regard R^+ as the set of roots of $\mathcal{A}$ containing C. Let R^+_S be the set of roots of $\mathcal{A}$ containing all chambers of $\mathcal{A}$ that contain S. Then $|R^+_S|=2(n-k)+(k+1)k/2$ and, for a root $\alpha\in R^+$, we have $U_\alpha\leq U_S$ if and only if $\alpha\in R^+_S$. We call the subgroups U_α for $\alpha\in R^+_S$ the *root subgroups* of U_S. They generate U_S. We state this fact explicitly, for further reference:

Fact 2.1. *Chosen an ordering $(\alpha_1,\alpha_2,...,\alpha_N)$ of R^+_S, where $N:=|R^+_S|$, every element $u\in U_S$ can be expressed as a product as follows for suitable scalars $t_1,t_2,...,t_N\in\mathbb{F}$:*

$$u=x_{\alpha_1}(t_1)\cdot x_{\alpha_2}(t_2)\cdot\ldots\cdot x_{\alpha_N}(t_N).$$

The ordering $(\alpha_1,\alpha_2,...,\alpha_N)$ can be chosen in such a way that $\{U_{\alpha_i}+\widehat{U}_S\}_{i=1}^{2(n-k)}$ is a minimal generating family of subgroups for the abelian group $U_S/\widehat{U}_S$ while $\{U_{\alpha_i}\}_{i=2(n-k)+1}^{N}$ is a minimal generating family of subgroups for $\widehat{U}_S=Z(U_S)$.

We now turn to the actions of U_S on V_k and W_k. We will only consider their actions on V_k, but everything we will say holds for W_k as well. The symbols $(.)_k$ and $[.]_k$ are defined as in Subsection 2.1. We use the symbol I to denote both the identity mapping of V and the identity mapping of V_k. Let $L(U_S)$ be the Lie algebra of U_S. The mapping sending $u\in U_S$ to $u-I$ is a bijection from U_S to $L(U_S)$ and it sends the commutator $[u,v]:=uvu^{-1}v^{-1}$ of two elements u and v of U_S to the Lie bracket of $u-I$ and $v-I$ in $L(U_S)$:

$$[u-I,v-I]:=(u-I)(v-I)-(v-I)(u-I)=uv-vu.$$

In particular, this mapping induces a bijection from the center $\widehat{U}_S$ of U_S to the center of $L(U_S)$. The following is well known (see for instance Premet and Suprunenko [15, Lemma 1]):

Fact 2.2. *For every root subgroup U_α of U_S and every $u\in U_\alpha$, the element $[u-I]_k$ is nilpotent of exponent ≤ 3 and $(u)_k=I+[u-I]_k+[u-I]_k^2/2$. Hence $(u)_k-I$ is nilpotent of exponent ≤ 3.*

2.3 Singular hyperplanes and the radical $R(\varepsilon_k)$ of ε_k

Let $\mathrm{diam}(\Delta_k)$ be the diameter of the collinearity graph of Δ_k. It is well known that $\mathrm{diam}(\Delta_k)=k+1$ if $k<n$ and $\mathrm{diam}(\Delta_n)=n$ (see Blok [3], for instance). Given a point S of Δ_k, let $H(S):=\{X\in\Delta_k\mid d(X,S)<\mathrm{diam}(\Delta_k)\}$ be the set of points of Δ_k at non-maximal distance from S in the collinearity graph of Δ_k and

$\overline{H(S)} := \{X \in \Delta_k \mid d(X,S) = \mathrm{diam}(\Delta_k)\}$ the complement of $H(S)$ in the set of points of Δ_k. Note that

$$\begin{array}{rcl} H(S) & = & \{X \in \Delta_k \mid \dim(S \cap X^\perp) > 0\}, \\ \overline{H(S)} & = & \{X \in \Delta_k \mid \dim(S \cap X^\perp) = 0\}. \end{array}$$

The first claim of the following lemma is a special case of a far more general result of Blok and Brouwer [5]. The second claim is implicit in [2].

Lemma 2.3. *$H(S)$ is a geometric hyperplane of Δ_k and $\overline{H(S)}$ spans Δ_k.*

The hyperplane $H(S)$ is called the *singular hyperplane* of Δ_k with S as the *deepest point*. The next lemma is proved by Blok [4] in a far more general setting:

Lemma 2.4. *For every point S of Δ_k, the image $\varepsilon_k(H(S))$ of $H(S)$ by ε_k spans a hyperplane $H(S)_k := \langle \varepsilon_k(H(S)) \rangle$ of V_k.*

In short, ε_k is polarized, where we say that an embedding $\varepsilon : \Delta_k \to \mathrm{PG}(U)$ for a vector space U is *polarized* if, for every point S of Δ_k, $\varepsilon(H(S))$ spans a hyperplane of U. Turning back to ε_k, put

$$R(\varepsilon_k) := \bigcap (H(S)_k \mid \ S \text{ point of } \Delta_k).$$

We call $R(\varepsilon_k)$ the *radical* of ε_k. It is not so difficult to see that $R(\varepsilon_k)$ defines a quotient $\varepsilon_k/R(\varepsilon_k)$ of ε_k and that $\varepsilon_k/R(\varepsilon_k)$ is polarized. Moreover, the embedding ε_k is relatively universal (Blok [3]; recall that we assume $\mathrm{char}(\mathbb{F}) \neq 2$). On the other hand, polar grassmannians admit the absolutely universal embedding (Kasikova and Shult [14]). Hence ε_k is absolutely universal, namely every embedding of Δ_k is a quotient of ε_k. This implies that every polarized embedding of Δ_k sits between ε_k and $\varepsilon_k/R(\varepsilon_k)$. In other words, $\varepsilon_k/R(\varepsilon_k)$ is the minimal polarized embedding of Δ_k (compare Cardinali, De Bruyn and Pasini [10], where this matter is settled for dual polar spaces). It is also clear that $R(\varepsilon_k)$ is G-invariant. The following is a special case of a more general result of Blok [4], valid for any Lie geometry associated to a Chevalley group (see also [7] for a similar theorem, valid for dual polar spaces).

Theorem 2.5. *The radical $R(\varepsilon_k)$ of ε_k is the largest proper G-submodule of V_k.*

In short, $R(\varepsilon_k) = R_k$ (notation as in the introduction). An embedding $\varepsilon : \Delta_k \to \mathrm{PG}(W)$, for an $\mathbb{F}$-vector space W, is said to be *G-homogeneous* if every $g \in G$ lifts through ε to a linear mapping of W stabilizing the image $\varepsilon(\Delta_k)$ of Δ_k. The next corollary is a rephrasing of Theorem 2.5.

Corollary 2.6. *Every G-homogeneous embedding of Δ_k is polarized.*

2.4 The fundamental form α_k

With α and V as in the introduction, let $\mathbb{E} := \{e_1, e_2, ..., e_{2n}\}$ be a hyperbolic basis for the form α of V, where $\alpha(e_i, e_j) = \alpha(e_{i+n}, e_{j+n}) = 0$, $\alpha(e_i, e_{j+n}) = \delta_{i,j}$

(Kronecker symbol) and $\alpha(e_{i+n}, e_j) = -\delta_{i,j}$ for $i, j = 1, 2, ..., n$. The form α is represented by the following matrix with respect to $\mathbb{E}$, where O_n and I_n are the null and identity matrix of order n respectively:

$$M = \begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix}.$$

Given two totally isotropic k-subspaces A and B of V let X and Y be $2n \times k$ matrices whose columns form bases of A and B respectively. By definition, A and B are at non-maximal distance in Δ_k precisely when $A^\perp \cap B \neq 0$. This happens if and only if the homogeneous linear system with matrix $X^T MY$ has non-trivial solutions, namely $\det(X^T MY) = 0$. By expanding the determinant $\det(X^T \cdot MY)$ according to the Cauchy-Binet formula we can rewrite the equation $\det(X^T MY) = 0$ as follows:

$$\sum_{J \in \binom{I}{k}} \det(X_{|J}) \cdot \det((MY)_{|J}) = 0 \tag{3}$$

where $I := \{1, 2, ..., 2n\}$, $\binom{I}{k}$ stands for the family of k-subsets of I and, for $J \in \binom{I}{k}$, $X_{|J}$ is the submatrix of X formed by the jth rows with $j \in J$ while $(MY)_{|J}$ is the submatrix of the jth rows of MY for $j \in J$. Put $X_J := \det(X_{|J})$ and $Y_J := \det(Y_{|J})$. The scalars X_J for $J \in \binom{I}{k}$ are the coordinates of a non-zero vector of $\iota_k(A) = \langle \sum_J X_J e_J \rangle$ relatively to the basis $\{e_J\}_{J \in \binom{I}{k}}$ of W_k, where $e_J := e_{j_1} \wedge ... \wedge e_{j_k}$ for $J = \{j_1, ..., j_k\}$, $j_1 < j_2 < \cdots < j_k$. Put $\theta(J) := |J \cap \{n+1, n+2, ..., 2n\}|$. Then

$$\det((MY)_{|J}) = (-1)^{\theta(J)} \cdot (-1)^{\theta(J)(k-1)} \det(Y_{|\rho(J)}) \tag{4}$$

where $\rho(j) = j + n$ if $j \leq n$ and $\rho(j) = j - n$ if $j > n$. The factor $(-1)^{\theta(J)}$ is contributed by multiplying $\theta(J)$ rows of Y by (-1)-entries of M while the factor $(-1)^{\theta(J)(k-1)}$ takes care of the cyclic permutation to apply in order to put the rows of MY which are involved in $\det((MY)_{|J})$ in the same natural order they had in Y. With $(MY)_J := \det((MY)_{|J})$ and $Y_{\rho(J)} := \det(Y_{|\rho(J)})$, we can rewrite (3) as follows:

$$\sum_{J \in \binom{I}{k}} (-1)^{\theta(J)k} X_J \cdot Y_{\rho(J)} = 0 \tag{5}$$

where $\rho(j) = j + n$ if $j \leq n$ and $\rho(j) = j - n$ if $j > n$.

We have established (5) thinking of singular subspaces of V, but the bilinear form considered in (5) is defined on the whole of W_k. We shall denote this form by α_k and call it the *fundamental form* of ε_k in W_k.

Proposition 2.7. *The fundamental form α_k is non-degenerate. If k is even then α_k is symmetric. If k is odd then α_k is alternating.*

Proof. Suppose first that k is even. Then for any two vectors x and y of W_k with coordinates X_J and Y_J ($J \in \binom{I}{k}$), we have

$$\alpha_k(x, y) = \sum_{J \in \binom{I}{k}} X_J \cdot Y_{\rho(J)}.$$

We may assume to have ordered the basis vectors e_J, $J \in \binom{I}{k}$ in such a way that if $J \neq \rho(J)$ then J and $\rho(J)$ occur one immediately after the other in that ordering, with no other k-subset of I in between. With this ordering of the basis the matrix representing α_k is block-diagonal with blocks either of order 1 and equal to the scalar 1 or of order 2 and as follows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Clearly, α_k is non-degenerate and symmetric.

Let now k be odd. We have $J \neq \rho(J)$ for every k-subset J of I. Moreover, $\theta(J) + \theta(\rho(J)) = k$, which is odd. For any two vectors x and y of W_k with coordinates X_J and Y_J ($J \in \binom{I}{k}$), we have

$$\alpha_k(x, y) = \sum_{J \in \binom{I}{k}} (-1)^{\theta(J)} X_J \cdot Y_{\rho(J)}.$$

If we order the basis vectors e_J such that J and $\rho(J)$ appear one immediately after the order then the matrix representing α_k is block-diagonal with blocks as follows:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

So, α_k is a non-degenerate alternating form. ■

Proposition 2.8. *The group $(G)_k$ preserves the form α_k.*

Proof. Let $\perp_k$ be the orthogonality relation with respect to α_k. Then $(G)_k$ preserves $\perp_k$ when restricted to pairs of vectors of W_k corresponding to k-spaces of V, namely points of $\mathcal{G}_k$. We shall firstly prove that $(G)_k$ preserves $\perp_k$ over the whole of W_k. Equivalently, every element of $(G)_k$ preserves α_k modulo a scalar.

Let A be the matrix representing the form α_k as in equation (5) and, for $g \in G$, let M_g be the matrix representing $(g)_k$. Let $a \in W_k \setminus \{0\}$ with $\langle a \rangle \in \mathcal{G}_k$. By assumption $a^T A x = 0$ if and only if $a^T M_g^T A M_g x = 0$, for any $x \in W_k \setminus \{0\}$ with $\langle x \rangle \in \mathcal{G}_k$. Consider the following linear functionals $f_1^{(a)} : x \to a^T A x$ and $f_2^{(a)} : x \to a^T M_g^T A M_g A x$. Clearly $\ker(f_1^{(a)}) \cap \mathcal{G}_k = \ker(f_2^{(a)}) \cap \mathcal{G}_k$. Since $\ker(f_1^{(a)}) \cap \mathcal{G}_k$ and $\ker(f_2^{(a)}) \cap \mathcal{G}_k$ are maximal subspaces of $\mathcal{G}_k$ and $\mathcal{G}_k$ spans $\mathrm{PG}(W_k)$, they span hyperplanes of W_k. Hence $\ker(f_1^{(a)}) = \ker(f_2^{(a)})$, namely $f_1^{(a)}$ and $f_2^{(a)}$ are proportional. So, for all $a \in W_k \setminus \{0\}$ with $\langle a \rangle \in \mathcal{G}$ there exists a non-zero scalar $\lambda_{a,g}$ such that $f_2^{(a)} = \lambda_{a,g} f_1^{(a)}$. Clearly, $\lambda_{ta,g} = t\lambda_{a,g}$.

Let $\langle a \rangle$ and $\langle b \rangle$ be collinear points of $\mathcal{G}_k$ and $c = ta + sb$ for $(s,t) \neq (0,0)$. As $\mathcal{G}_k$ is a full subgeometry of $\mathrm{PG}(W_k)$, $\langle c \rangle$ is a point of $\mathcal{G}_k$. Then $f_2^{(c)} = t f_2^{(a)} + s f_2^{(b)} = t\lambda_{a,g} f_1^{(a)} + s\lambda_{b,g} f_1^{(b)}$. However, $f_2^{(c)} = \lambda_{ta+sb,g} f_1^{(c)} = \lambda_{ta+sb,g}(t f_1^{(a)} + s f_1^{(b)})$. So,

$$t\lambda_{a,g} f_1^{(a)} + s\lambda_{b,g} f_1^{(b)} = \lambda_{ta+sb,g} t f_1^{(a)} + \lambda_{ta+sb,g} s f_1^{(b)}.$$

Suppose that there exists $h \in \mathbb{F} \setminus \{0\}$ such that $f_1^{(b)} = hf_1^{(a)}$. Then $b^T Ax = ha^T Ax$, hence $b^T - ha \in \mathrm{Rad}(\alpha_k)$. This is impossible since α_k is non-degenerate (Proposition 2.7). Hence $f_1^{(a)}$ and $f_1^{(b)}$ are non-proportional. Therefore $\lambda_{a,g} = \lambda_{ta+sb,g} = \lambda_{b,g}$. Put $\lambda_g := \lambda_{a,g}$ $(= \lambda_{b,g} = \lambda_{ta+sb,g})$. As $\mathcal{G}_k$ is connected, $a^T M_g^T A M_g x = \lambda_g a^T Ax$ for all $x \in W_k$ and all $a \in W_k \setminus \{0\}$ such that $\langle a \rangle \in \mathcal{G}_k$. However $\langle \mathcal{G}_k \rangle = W_k$. Hence $M_g^T A M_g = \lambda_g A$, namely $(g)_k$ preserves α_k modulo a scalar λ_g.

If λ_g depended on the choice of $g \in G$, then G would admit a quotient isomorphic to a subgroup $\mathbb{F}_0^*$ of the multiplicative group $\mathbb{F}^*$ of $\mathbb{F}$. However $G/Z(G)$ is simple while $|Z(G)| = 2$. Hence 1 and -1 are the only admissible values for λ_g and, if z is the non-trivial element of $Z(G)$ and $\lambda_g = -1$ for some $g \in G$ then $\lambda_z = -1$. This immediately rules out the case of k even since in this case $(z)_k$ is the identity mapping of W_k. However, even if k is odd, the effect of z on $\alpha_k(x, y)$ amounts to multiply the coordinates X_J and Y_J by -1. This has no effect on the value $\alpha_k(x, y)$ itself. Hence $(G)_k$ preserves α_k. ■

While α_k is non-degenerate as a form of W_k, its restriction to V_k can be degenerate, with radical $V_k^\perp \cap V_k$, where now $\perp$ stands for the orthogonality relation with respect to α_k, denoted by $\perp_k$ in the proof of Proposition 2.8. (We have previously used the symbol $\perp$ to denote orthogonality with respect to α, but this notational ambiguity is harmless, since α and α_k live in different environments.)

Proposition 2.9. $R(\varepsilon_k) = V_k^\perp \cap V_k$.

Proof. For two points X and Y of Δ_k we have $Y \in H(X)$ if and only if $\alpha_k(\varepsilon_k(X), \varepsilon_k(Y)) = 0$, namely $H(X) = \varepsilon_k(X)^\perp$. Hence $R(\varepsilon_k) = \cap(\varepsilon_k(X)^\perp \mid X \text{ point of } \Delta_k)$. The latter equals $V_k^\perp \cap V_k$, as $\varepsilon_k(\Delta_k)$ spans V_k. ■

Remark 2.10. When $k = 1$, $\alpha_k = \alpha$. When $k = n$, a formula different from (5) was considered in [10], namely the following:

$$\sum_{J \in \binom{I}{n}} (-1)^{\zeta(J)} X_J Y_{I \setminus J} = 0 \tag{6}$$

where $\zeta(J) = n(n+1)/2 + \sum_{j \in J} j$. One can prove that (5) with $k = n$ and (6) are in fact equivalent on V_n, but they are not equivalent on the whole of W_n.

2.5 G-invariant equivalence relations

We finish this section with an elementary lemma, to be used several times in this paper. Let Θ be an equivalence relation on the set of points of Δ_k. We say that Θ is *G-invariant* if $(g(X), g(Y)) \in \Theta$ for every $g \in G$ and every pair $(X, Y) \in \Theta$.

Lemma 2.11. *Let Θ be a G-invariant equivalence relation on the set of points of Δ_k. Then Θ is either the identity relation or trivial.*

Proof. By way of contradiction, suppose that Θ is neither the identity nor trivial. Then it admits at least two classes, say C and C', and at least one of them, say C, contains at least two elements. Let X and Y be distinct elements of C. As Θ is G-invariant, the stabilizer G_X of X in G also stabilizes C. Moreover, G contains at least one element g mapping X onto Y, since G is transitive on the set of points of Δ_k. Hence the setwise stabilizer G_C of C in G is larger than G_X. This is a contradiction, because G_X is a maximal subgroup of G. ∎

3 The basic series of G in W_k

Throughout this section $\perp$ stands for orthogonality with respect to α. For $0 \leq i \leq \lfloor k/2 \rfloor$ (where $\lfloor k/2 \rfloor$ is the integral part of $k/2$), we denote by $V^{(k)}_{k-2i}$ the subspace of W_k spanned by the vectors $\iota_k(X)$ for a k-subspace X of V with $\dim(X \cap X^{\perp}) \geq k-2i$. In particular, $V^{(k)}_k = V_k$. Clearly, $V^{(k)}_{k-2i}$ is G-invariant and $V^{(k)}_{k-2i} \subseteq V^{(k)}_{k-2j}$ for $0 \leq i \leq j \leq \lfloor k/2 \rfloor$.

Note that $k - 2\lfloor k/2 \rfloor$ is equal to 0 or 1 according to whether k is even or odd. In any case, $V^{(k)}_{k-2\lfloor k/2 \rfloor} = W_k$. We put $V^{(k)}_{k+2} := 0$, by convention. The series of the G-submodules of W_k defined above will be called the *basic series* of G in W_k:

$$0 = V^{(k)}_{k+2} \subseteq V^{(k)}_{k} \subseteq V^{(k)}_{k-2} \subseteq ... \subseteq V^{(k)}_{k-2\lfloor k/2 \rfloor} = W_k.$$

We denote by $\widetilde{V}^{(k)}_{k-2i}$ the set of non-zero vectors $w \in W_k$ such that $\langle w \rangle = \iota_k(X)$ for a k-subspace X of V with $\dim(X \cap X^{\perp}) = k - 2i$. Clearly, $\widetilde{V}^{(k)}_{k-2i} \subseteq V^{(k)}_{k-2i} \setminus V^{(k)}_{k-2i+2}$.

Lemma 3.1. $\langle \widetilde{V}^{(k)}_{k-2i} \rangle = V^{(k)}_{k-2i}$ *for every* $i = 0, 1, ..., \lfloor k/2 \rfloor$.

Proof. By induction on i. Let $i = 0$. Then $\widetilde{V}^{(k)}_{k}$ is just the image of the set of points of Δ_k by ι_k. In this case the equality $\langle \widetilde{V}_{k-2i} \rangle = V^{(k)}_{k-2i}$ rephrases the definition of V_k.

Assume $i > 0$. Let A be a k-dimensional subspace of V with $\dim(A \cap A^{\perp}) = k - 2i + 2$ and $\mathbb{E} = \{e_1, ..., e_n, f_1, ..., f_n\}$ a hyperbolic basis of V, with $\alpha(e_i, f_j) = \delta_{i,j}$ for $i, j = 1, 2, ..., n$. We may assume to have chosen $\mathbb{E}$ in such a way that $A = \langle \mathbb{E}_A \rangle$ where $\mathbb{E}_A = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \{e_{k-2i+2}\}$,

$$\begin{aligned} \mathbb{E}_1 &= \{e_1, e_2, ..., e_{k-2i+1}\}, \\ \mathbb{E}_2 &= \{e_{k-2i+3}, f_{k-2i+3}, e_{k-2i+4}, f_{k-2i+4}, ..., e_{k-i+1}, f_{k-i+1}\}. \end{aligned}$$

Put $\mathbb{E}_B = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \{f_{k-2i+1}\}$, $\mathbb{E}_C = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \{e_{k-2i+2} + f_{k-2i+1}\}$, $B = \langle \mathbb{E}_B \rangle$ and $C = \langle \mathbb{E}_C \rangle$. Then $\dim(B \cap B^{\perp}) = \dim(C \cap C^{\perp}) = k - 2i$. Both $\wedge^k \mathbb{E}_B$ and $\wedge^k \mathbb{E}_C$ belong to $\widetilde{V}^{(k)}_{k-2i}$. Moreover $\wedge^k \mathbb{E}_A = \wedge^k \mathbb{E}_C - \wedge^k \mathbb{E}_B$. Hence $\wedge^k \mathbb{E}_A \in \langle \widetilde{V}^{(k)}_{k-2i} \rangle$. So far, we have proved that $\widetilde{V}^{(k)}_{k-2i+2} \subseteq \langle \widetilde{V}^{(k)}_{k-2i} \rangle$. However $\langle \widetilde{V}^{(k)}_{k-2i+2} \rangle = V^{(k)}_{k-2i+2}$ by the inductive hypothesis. Therefore $\langle \widetilde{V}^{(k)}_{k-2i} \rangle \supseteq V^{(k)}_{k-2i+2}$. The equality $\langle \widetilde{V}^{(k)}_{k-2i} \rangle = V^{(k)}_{k-2i}$ follows. ∎

Let Z be a totally isotropic subspace of V of dimension $d = k - 2i$, $i > 0$. Define the following subsets of $\mathcal{G}_k$

$$\mathcal{G}_k(Z) = \{X \mid \dim(X) = k \text{ and } Z \subseteq X^{\perp} \cap X\},$$

$$H_k(Z) = \{X \in \mathcal{G}_k(Z) \mid Z \subset X^\perp \cap X\}.$$

Lemma 3.2. *The set $\mathcal{G}_k(Z)$ is a subspace of $\mathcal{G}_k$ and it is isomorphic to the $2i$-grassmann geometry of $\mathrm{PG}(Z^\perp/Z)$.*

Proof. For every k-subspace X of V, we have $X^\perp \cap X \supseteq Z$ if and only if $Z \subseteq X \subseteq Z^\perp$. The lemma immediately follows from this. ∎

Note also that $\langle \iota_k(\mathcal{G}_k(Z))\rangle \subseteq V^{(k)}_{k-2i}$, $\langle \iota_k(H_k(Z))\rangle \subseteq V^{(k)}_{k-2i+2}$ and $\iota_k(\mathcal{G}_k(Z) \setminus H_k(Z)) \subseteq \widetilde{V}^{(k)}_{k-2i}$. (The inclusion $\iota_k(\mathcal{G}_k(Z) \setminus H_k(Z)) \subseteq \widetilde{V}^{(k)}_{k-2i}$ is literally incorrect, since $\iota_k(\mathcal{G}_k(Z) \setminus H_k(Z))$ is a set of 1-dimensional subspaces while $\widetilde{V}^{(k)}_{k-2i}$ is a set of vectors, but we take this way of writing as a shortening for the correct formulation; little abuses like this will be freely committed henceforth.)

Lemma 3.3. *The set $H_k(Z)$ is a geometric hyperplane of $\mathcal{G}_k(Z)$ and its complement $\mathcal{G}_k(Z) \setminus H_k(Z)$ is connected. Moreover, $\iota_k(H_k(Z))$ spans a projective hyperplane of $\langle \iota_k(\mathcal{G}_k(Z))\rangle$.*

Proof. Suppose $Z = 0$, to begin with. Hence k is even. We firstly prove that $H_k = H_k(0)$ is a subspace of $\mathcal{G}_k = \mathcal{G}_k(0)$. Let X and Y be two distinct collinear points of H_k. Hence $\dim(X \cap Y) = k-1$. Put $R_X := X \cap X^\perp$ and $R_Y := Y \cap Y^\perp$. Note that $R_X \neq 0 \neq R_Y$, as $X, Y \in H_k$. Since X and Y have even dimension, R_X and R_Y have dimension at least 2. Hence $R_X \cap Y \neq 0 \neq R_Y \cap X$. Let $r_X \in R_X \cap Y$ and $r_Y \in R_Y \cap X$, $r_X \neq 0 \neq r_Y$.

Suppose that $r_X \neq r_Y$. Take a point Z in the line of $\mathcal{G}_k$ spanned by X and Y. Hence $Z = \langle X \cap Y, z\rangle$ for a vector $z \notin X \cap Y$. There exists at least one point $\langle r\rangle$ in the projective line $\langle r_X, r_Y\rangle$ orthogonal to z. Since $X \cap Y$ is orthogonal to both r_X and r_Y, $X \cap Y$ is also orthogonal to r. Therefore $r \in Z \cap Z^\perp$. Hence $Z \in H_k$. So, the line spanned by X and Y is contained in H_k.

Let $r_X = r_Y = r$, say. Let $Z := \langle X \cap Y, z\rangle$ for a non-zero vector $z \in \langle x, y\rangle$ where $x \in X \setminus Y$ and $y \in Y \setminus X$. Since $r \perp x, y$ the vector r is orthogonal to every point of $\langle x, y\rangle$. In particular, $r \perp z$. Moreover $r \in Z \cap Z^\perp$ since r is also orthogonal to $X \cap Y$. As above, the line of $\mathcal{G}_k$ spanned by X and Y is contained in H_k. We have proved that H_k is a subspace of $\mathcal{G}_k$.

Take now a line L of $\mathcal{G}_k$ not contained in H_k. As H_k is a subspace of $\mathcal{G}_k$, at most one point of L belongs to H_k. Let X and Y be two distinct points of L not in H_k. Then $X \cap X^\perp = Y \cap Y^\perp = 0$. Since $X \cap Y$ has odd dimension, there exists a non-zero vector $r \in (X \cap Y) \cap (X \cap Y)^\perp$. Take $x \in X \setminus Y$, $y \in Y \setminus X$ and let $z \in r^\perp \cap \langle x, y\rangle$, $z \neq 0$. Put $Z := \langle X \cap Y, z\rangle$. Then $r \in Z \cap Z^\perp$. Hence $Z \in H_k$. We have proved that H_k is a hyperplane of $\mathcal{G}_k$.

By Shult [16], the hyperplane H_k arises from the embedding $\iota_k \colon \mathcal{G}_k \mapsto \mathrm{PG}(W_k)$ and the complement $\mathcal{G}_k \setminus H_k$ is simply connected (whence connected).

Finally, let $Z \neq 0$. Put $V_Z := Z^\perp/Z$ and let $\mathcal{G}_{2i,Z}$ be the $2i$-grassmann geometry of $\mathrm{PG}(V_Z)$. By Lemma 3.2, $\mathcal{G}_k(Z) \cong \mathcal{G}_{2i,Z}$. Moreover, the embedding $\iota_{k,Z} : \mathcal{G}_k(Z) \to \langle \iota_k(\mathcal{G}_k(Z))\rangle$ induced by ι_k on $\mathcal{G}_k(Z)$ is isomorphic to the natural embedding $\iota_{2i} : \mathcal{G}_{k,Z} \to \mathrm{PG}(\wedge^{2i} V_Z)$. So, we can replace Z by the null space of V_Z and k by $2i$ and we obtain the conclusion by the first part of the proof. ∎

For $0 \leq i \leq \lfloor k/2 \rfloor$ we define a mapping $f_{k-2i}\colon \Delta_{k-2i} \to \mathrm{PG}(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2})$ as follows: for every point Z of Δ_{k-2i} we put

$$\begin{aligned} f_{k-2i}(Z) &= (\langle \iota_k(\mathcal{G}_k(Z))\rangle + V^{(k)}_{k-2i+2})/V^{(k)}_{k-2i+2} = \\ &= \iota_k(X) + V^{(k)}_{k-2i+2} \text{ for any } X \in \mathcal{G}_k(Z) \setminus H_k(Z). \end{aligned}$$

The latter equality holds because $\iota_k(H_k(Z)) \subseteq V^{(k)}_{k-2i+2}$ and $\langle \iota_k(H_k(Z))\rangle$ is a hyperplane of $\langle \iota_k(\mathcal{G}_k(Z))\rangle$, by Lemma 3.3. The following lemma is obvious:

Lemma 3.4. *$(g)_k(f_{k-2i}(Z)) = f_{k-2i}(g(Z))$ for every $g \in G$ and every point Z of Δ_{k-2i}.*

Theorem 3.5. *Let $i < k/2$. Then the above defined mapping*

$$f_{k-2i}\colon \Delta_{k-2i} \to \mathrm{PG}(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2})$$

is a projective embedding and it is isomorphic to the natural embedding $\varepsilon_{k-2i} : \Delta_{k-2i} \to \mathrm{PG}(V_{k-2i})$. Moreover $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$ and V_{k-2i} are isomorphic as G-modules.

Proof. We split the proof in a number of steps.

(1) *One of the following holds:*

(1.a) *$f_{k-2i}(Z)$ is a point of $\mathrm{PG}(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2})$ for every point Z of Δ_{k-2i};*

(1.b) $V^{(k)}_{k-2i+2} = V^{(k)}_{k-2i}$.

Let Z be a point of Δ_{k-2i}. By Lemma 3.3, $\iota_k(H_k(Z))$ spans a hyperplane of $\langle \iota_k(\mathcal{G}_k(Z))\rangle$. If $\langle \iota_k(\mathcal{G}_k(Z))\rangle \subset V^{(k)}_{k-2i+2}$ then $(\langle \iota_k(\mathcal{G}_k(Z))\rangle + V^{(k)}_{k-2i+2})/V^{(k)}_{k-2i+2}$ is the null space. Otherwise, $V^{(k)}_{k-2i+2}$ is a hyperplane of $\langle \iota_k(\mathcal{G}_k(Z))\rangle + V^{(k)}_{k-2i+2}$, hence $(\langle \iota_k(\mathcal{G}_k(Z))\rangle + V^{(k)}_{k-2i+2})/V^{(k)}_{k-2i+2}$ is a point of $\mathrm{PG}(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2})$. If the latter case occurs for every point $Z \in \Delta_{k-2i}$ then (1.a) holds. Suppose that $\langle \iota_k(\mathcal{G}_k(Z))\rangle \subset V^{(k)}_{k-2i+2}$ for at least one point $Z \in \Delta_{k-2i}$. By the transitivity of G on the set of points of Δ_{k-2i} and by Lemma 3.4 we obtain that $\langle \iota_k(\mathcal{G}_k(Z))\rangle \subseteq V^{(k)}_{k-2i+2}$ for every point $Z \in \Delta_{k-2i}$. However, the image $\iota_k(\Delta_{k-2i})$ of the point-set of Δ_{k-2i} is just the set of 1-dimensional linear spaces contained in $\widetilde{V}^{(k)}_{k-2i}$ and the latter spans $V^{(k)}_{k-2i}$ by Lemma 3.1. Hence $V^{(k)}_{k-2i} = V^{(k)}_{k-2i+2}$ as in (1.b).

(2) *Assume* (1.a). *Then one of the followings holds:*

(2.a) *f_{k-2i} is injective;*

(2.b) *$V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$ is a point.*

Suppose that f_{k-2i} is not injective. Since G permutes the fibers of f_{k-2i}, Lemma 2.11 implies that $f_{k-2i}(Z) = f_{k-2i}(Z')$ for any two points $Z, Z' \in \Delta_{k-2i}$. Hence the image of f_{k-2i} is just a point, as in (2.b).

(3) *If both* (1.a) *and* (2.a) *hold then f_{k-2i} is a projective embedding of Δ_{k-2i} in $\mathrm{PG}(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2})$.*

Assume (1.a) and (2.a). Let L be a line of Δ_{k-2i}, $L = \{Z \mid L_1 \subset Z \subset L_2, \dim(Z) = k-2i\}$ for two isotropic subspaces $L_1 \subset L_2$ of V of dimensions $k-2i-1$ and $k-2i+1$ respectively. Take a $2i$-dimensional non-singular space Y of V orthogonal to L_2 and disjoint from L_2. (To see that such a subspace exists, consider a complement of L_2 in $L_2^\perp$.) Put $\overline{Z} := \langle Y, Z\rangle$ for $Z \in L$, $\overline{L}_1 := \langle Y, L_1\rangle$ and $\overline{L}_2 := \langle Y, L_2\rangle$. Then $\overline{L} := \{X \mid \overline{L}_1 \subset X \subset \overline{L}_2, \dim(X) = k\}$ is a line of $\mathcal{G}_k$ and $\overline{Z} \in \overline{L}$ for every $Z \in L$. For $Z \in L$ we have $\overline{Z} \cap \overline{Z}^\perp = Z$, whence $\overline{Z} \in \mathcal{G}_k(Z) \setminus H_k(Z)$. Therefore $f_{k-2i}(Z) = (\langle \iota_k(\mathcal{G}_k(Z))\rangle + V^{(k)}_{k-2i+2})/V^{(k)}_{k-2i+2} = \iota_k(\overline{Z}) + V^{(k)}_{k-2i+2}$. Since, according to (2.a), f_{k-2i} is injective, the image of L by f_{k-2i} is the line of $\mathrm{PG}(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2})$ whose points are represented by the spaces $\iota(\overline{Z})$ for $Z \in L$. It remains to prove that $f_{k-2i}(\Delta_{k-2i})$ spans $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$, but this immediately follows from Lemma 3.1.

(4) *For every* $i = 0, 1, ..., \lfloor k/2\rfloor$, *the mapping* f_{k-2i} *is a projective embedding of* Δ_{k-2i} *with vector dimension*

$$\dim(f_{k-2i}) = \dim(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}) = \binom{2n}{k-2i} - \binom{2n}{k-2i-2}.$$

Given $i \in \{0, 1, ..., \lfloor k/2\rfloor\}$, suppose that (1.a) and (2.a) hold. Then, by (3), f_{k-2i} is an embedding of Δ_{k-2i}. By Blok [3], $\dim(f_{k-2i}) \leq \binom{2n}{k-2i} - \binom{2n}{k-2i-2}$, namely $\dim(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}) \leq \binom{2n}{k-2i} - \binom{2n}{k-2i-2}$. The latter inequality trivially holds in cases (1.b) and (2.b). On the other hand,

$$\sum_{i=0}^{\lfloor k/2\rfloor}\left[\binom{2n}{k-2i} - \binom{2n}{k-2i-2}\right] = \binom{2n}{k} = \dim(W_k) =$$

$$= \sum_{i=0}^{\lfloor k/2\rfloor} \dim(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}).$$

This forces $\dim(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}) = \binom{2n}{k-2i} - \binom{2n}{k-2i-2}$ for every $i = 0, 1, ..., \lfloor k/2\rfloor$.

(5) f_{k-2i} *is isomorphic to the natural embedding* $\varepsilon_{k-2i} : \Delta_{k-2i} \to \mathrm{PG}(V_{k-2i})$.

By Kasikova and Shult [14] (§§4.6–4.8) Δ_{k-2i} admits the absolutely universal embedding which, by Blok [3], is the natural embedding ε_{k-2i} of dimension $\binom{2n}{k-2i} - \binom{2n}{k-2i-2}$. Hence $f_{k-2i} \cong \varepsilon_{k-2i}$ and $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2} \cong V_{k-2i}$. ■

When k is even Theorem 3.5 implies that

$$\dim(V_2^{(k)}) = \sum_{i=0}^{k/2-1}\binom{2n}{k-2i} - \binom{2n}{k-2i-2} = \binom{2n}{k} - 1.$$

So, $V_0^{(k)}/V_2^{(k)} \cong W_0$ is 1-dimensional and f_0 affords the trivial representation of G. In other words,

Corollary 3.6. *Let k be even. Then $V_2^{(k)}$ is a hyperplane of W_k.*

Lemma 3.7. *Let A be a G-invariant proper subspace of W_k. Then there exists an index $i \geq 0$ such that $V^{(k)}_{k-2j} \subseteq A$ for $j < i$ and $\widetilde{V}^{(k)}_{k-2r} \cap A = \emptyset$ for $r \geq i$.*

Proof. Let i be the largest index j such that $V^{(k)}_{k-2j+2} \subseteq A$. We must show that $\widetilde{V}^{(k)}_{k-2r} \cap A = \emptyset$ for every $r \geq i$. Suppose the contrary. Then $A \supseteq \langle \widetilde{V}^{(k)}_{k-2r} \rangle$ as G acts transitively on $\widetilde{V}_{k-2r}$ and, by assumption, A is stabilized by G. However $\widetilde{V}^{(k)}_{k-2r}$ spans $V^{(k)}_{k-2r}$ by Lemma 3.1. Hence $A \supseteq V^{(k)}_{k-2r}$, contrary to our choice of i. ■

Corollary 3.8. *Let A be a G-invariant proper subspace of W_k and suppose that $V^{(k)}_{k-2i} \not\subseteq A + V^{(k)}_{k-2i+2}$. Then the subspace $(V^{(k)}_{k-2i+2} + A \cap V^{(k)}_{k-2i})/V^{(k)}_{k-2i+2}$ of $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$ defines a homogeneous quotient of the embedding $f_{k-2i} : \Delta_{k-2i} \to \mathrm{PG}(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2})$.*

Proof. By Lemmas 3.1 and 3.7 applied to $A' = V^{(k)}_{k-2i+2} + A \cap V^{(k)}_{k-2i}$, either $A' \supseteq V^{(k)}_{k-2i}$ or $A' \cap \widetilde{V}^{(k)}_{k-2i} = \emptyset$. The first case is excluded by assumption. So, the latter holds. We have $\{\langle w \rangle + V^{(k)}_{k-2i+2} \mid w \in \widetilde{V}^{(k)}_{k-2i}\} = f_{k-2i}(\Delta_{k-2i})$. In order to prove that $A'/V^{(k)}_{k-2i+2}$ defines a (necessarily homogeneous) quotient of f_{k-2i} we only must prove that $\langle w_1, w_2 \rangle \cap A' = 0$ for every choice of vectors $w_1, w_2 \in \widetilde{V}^{(k)}_{k-2i}$. Suppose the contrary and let $\langle w_1, w_2 \rangle \cap A' \neq 0$ for two given vectors $w_1, w_2 \in \widetilde{V}^{(k)}_{k-2i}$. The vectors w_1 and w_2 correspond to two totally isotropic $(k-2i)$-dimensional subspaces X_0 and Y_0 of V and $f_{k-2i}(X_0) + A' = f_{k-2i}(Y_0) + A'$. By Lemma 2.11, $f_{k-2i}(X) + A' = f_{k-2i}(Y) + A'$ for any two points X and Y of Δ_{k-2i}. In particular, this also holds if X and Y are collinear in Δ_{k-2i}. On the other hand, f_{k-2i} maps lines of Δ_{k-2i} onto lines of $\mathrm{PG}(V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2})$. Therefore $f_{k-2i}(Z) \in A'$ for a point Z of Δ_{k-2i} belonging to the line spanned by X and Y. This contradicts the fact that $A' \cap \widetilde{V}_{k-2i} = \emptyset$. ■

4 The pole of G in W_k

Throughout this section $\perp$ is the orthogonality relation associated to the fundamental form α_k, defined in Subsection 2.4. Suppose there exists a point P of $\mathrm{PG}(W_k)$ fixed by G. As G preserves α_k, G also stabilizes the hyperplane $P^\perp$ of W_k. Let i be the largest index such that $P^\perp \supseteq V^{(k)}_{k-2i+2}$ (compare Lemma 3.7). Then $(P^\perp \cap V^{(k)}_{k-2i})/V^{(k)}_{k-2i+2}$ is a G-invariant hyperplane of $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$. However, if $i < k/2$ then $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$ hosts the embedding ε_{k-2i} (by Theorem 3.5), hence it cannot admit any G-invariant hyperplane (compare Corollary 3.8). Therefore k is even and $i = k/2$. In this case $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2} = V^{(k)}_0/V^{(k)}_2$ is 1-dimensional and $P^\perp = V^{(k)}_2$, namely $P = (V^{(k)}_2)^\perp$. We have proved the following:

Theorem 4.1. *If k is odd then G acts fixed-point freely on $\mathrm{PG}(W_k)$. If k is even then $P := (V^{(k)}_2)^\perp$ is the only point of $\mathrm{PG}(W_k)$ fixed by G.*

For the rest of this section k is assumed to be even. The unique fixed point P of G in $\mathrm{PG}(W_k)$ will be called the *pole* of G.

Let $\{e_1,...,e_n,f_1,...,f_n\}$ be a hyperbolic basis of V as in the proof of Lemma 3.1, namely a basis as in Section 2.4 but with $f_i := e_{i+n}$. For $J = \{j_1,...,j_{k/2}\} \subseteq I := \{1,2,...,n\}$ we put $e_J = e_{j_1} \wedge ... \wedge e_{j_{k/2}}$ and $f_J = f_{j_1} \wedge ... \wedge f_{j_{k/2}}$. Consider the following vector:

$$v_P := \sum_{1\leq j_1<\cdots<j_{k/2}\leq n} e_{j_1} \wedge \cdots \wedge e_{j_{k/2}} \wedge f_{j_1} \wedge \cdots \wedge f_{j_{k/2}} = \sum_{J\in\binom{I}{k/2}} e_J \wedge f_J$$

Lemma 4.2. $P = \langle v_P \rangle$.

Proof. We only must prove that G fixes v_P. Recall that $G = \langle U^+, U^- \rangle$ where U^+ and U^- are the unipotent radicals of two 'opposite' Borel subgroups B^+ and B^- of G, stabilizing two opposite chambers of Δ. In its turn U^+ is generated by root subgroups U_α with $\alpha \in R^+$ (notation as in Subsection 2.2) while U^- is generated by the root subgroups $U_{-\alpha}$ for $\alpha \in R^+$. So, if Π^+ is the basis of simple roots associated to the given set R^+ of positive roots then $U^+ = \langle U_\alpha \rangle_{\alpha\in\Pi^+}$ and $U^- = \langle U_{-\alpha} \rangle_{\alpha\in\Pi^+}$. In order to prove that G fixes v_P we only need to prove that v_P is fixed by U_α and $U_{-\alpha}$ for every $\alpha \in \Pi^+$. Put $\{\alpha_1,\alpha_2,...,\alpha_n\} = \Pi^+$ and let $x_{\alpha_i}(t)$ be the generic element of U_{α_i} (see Subsection 2.2). Then $x_{\alpha_i}(t)$, regarded as a linear transformation of V, is represented by a matrix as follows, where I_n and O_n are the identity and null matrices of order n and the symbol $E_{i,j}$ denotes the square matrix of order n with only null entries except for the (i,j)-entry, which is 1:

$$\underset{(\text{for } x_{\alpha_i}(t),\ i=1,...,n-1)}{\begin{bmatrix} I_n + tE_{i,i+1} & O_n \\ O_n & I_n - tE_{i+1,i} \end{bmatrix}}, \qquad \underset{(\text{for } x_{\alpha_n}(t))}{\begin{bmatrix} I_n & tE_{n,n} \\ O_n & I_n \end{bmatrix}}.$$

The element $x_{-\alpha_i}(t)$ of U^- is represented by the transpose $x_{\alpha_i}(t)^T$ of the matrix $x_{\alpha_i}(t)$. Having recalled this, it is straightforward to check that v_p is fixed by $(x_{\alpha_i}(t))_k$ and $(x_{-\alpha_i}(t))_k$ for every $i = 1,2,...,n$. Fact 2.2 can be profitably used here to speed up computations. We leave them for the reader. ∎

We call v_P the *polar vector* of G. Clearly $V_2^{(k)} = v_P^\perp$, since $P = \langle v_P \rangle$ and $V_2^{(k)} = P^\perp$. Recall that α_k is expressed by the left hand side of formula (3) of Subsection 2.4. Notice also that, if Y_J is the J-coordinate of v_P for $J \in \binom{I}{k}$, then $Y_J = 1$ if $\rho(J) = J$ and $Y_J = 0$ otherwise. (Here $I = \{1,2,...,2n\}$ and ρ permutes j with $j+n$ for every $j = 1,2,...,n$, as in formula (3) of Subsection 2.4). It is now clear that $V_2^{(k)}$ is described by the following equation:

$$\sum_{J\in\binom{I}{k},\rho(J)=J} X_J = 0. \tag{7}$$

Theorem 4.3. *We have $v_P \in V_2^{(k)}$ if and only if* $\mathrm{char}(\mathbb{F})$ *is positive and divides $\binom{n}{k/2}$.*

Proof. In view of the condition $\rho(J) = J$, the k-subsets $J \in \binom{I}{k}$ to consider in (7) are those of the form $J = J' \cup \rho(J')$ for a $(k/2)$-subset J' of $I' = \{1,2,...,n\}$. Hence they bijectively correspond to the $(k/2)$-sets $J' \in \binom{I'}{k/2}$. Therefore $v_P \in V_2^{(k)}$ if and only if $\sum_{J' \in \binom{I'}{k/2}} 1 = 0$, namely $\binom{n}{k/2} = 0$. The latter holds if and only if $\mathrm{char}(\mathbb{F})$ is positive and divides $\binom{n}{k/2}$. ■

For the rest of this section we assume that $\mathrm{char}(\mathbb{F}) = p > 0$ and k is even. Suppose that p divides $\binom{n}{k/2}$. Then $v_P \in V_2^{(k)}$ by Theorem 4.3. If we knew that $v_P \in V_k^{(k)}$ then, by Theorem 2.5 we could conclude that $R(\varepsilon_k) \neq 0$. However Theorem 4.3 does not give any information on where v_P is placed inside $V_2^{(k)}$. Suppose that $v_P \in V_{k-2i}^{(k)} \setminus V_{k-2i+2}^{(k)}$. By Theorem 3.5, $V_{k-2i}^{(k)}/V_{k-2i+2}^{(k)}$ and $V_{k-2i}^{(k-2i)}$ are isomorphic G-modules. Moreover G fixes the vector $v_P^{(k-2i)} := v_P + V_{k-2i+2}^{(k)}$ of $V_{k-2i}^{(k)}/V_{k-2i+2}^{(k)} \cong V_{k-2i}^{(k-2i)}$, as it fixes v_P. Hence $v_P^{(k-2i)}$ is the polar vector of G in W_{k-2i}, and it belongs to $V_{k-2i}^{(k-2i)}$. Therefore p divides $\binom{n}{(k-2i)/2}$ by Theorem 4.3.

Suppose that for some $j = 0, 1, ..., k/2-1$ the polar vector $v_P^{(k-2j)}$ of G in W_{k-2j} belongs to $V_{k-2j}^{(k-2j)}$. Then we say that $v_P^{(k-2j)}$, regarded as a vector of the quotient $V_{k-2j}^{(k)}/V_{k-2j+2}^{(k)}$, is a *virtual polar vector* of G in the basic series of G and we call $\langle v_P^{(k-2j)} \rangle$ a *virtual pole* of G. So, if the pole P of G in W_k is contained in $V_2^{(k)}$ then it must appear among the virtual poles. However, it is possible that several virtual poles exist but none of them arises from the true pole. It is also possible that no virtual pole exists. Note also that, if a virtual pole appears in a section $V_{k-2j}^{(k)}/V_{k-2j+2}^{(k)}$, then p divides $\binom{n}{k/2-j}$, but in general the converse does not hold. The next lemma immediately follows from the previous discussion.

Lemma 4.4. *Suppose that p divides $\binom{n}{k/2}$ but does not divide $\binom{n}{k/2-i}$ for every $i = 1, 2, ..., k/2-1$. Then $v_P \in V_k^{(k)}$.*

Lemma 4.5. *The basic series of G in W_k admits at least one virtual pole if and only if p divides $\binom{n}{k/2-i}$ for some $i = 0, 1, ..., k/2-1$.*

Proof. In the previous discussion we have already remarked that the 'only if' part of the lemma holds. Let us prove the 'if' part. Suppose that p divides $\binom{n}{k/2-i}$. Then $v_P^{(k-2i)} \in V_2^{(k-2i)}$. Hence $v_P^{(k-2i)}$ appears among the virtual poles of the basic series of G in W_{k-2i}. However every section of this series is isomorphic to a section of the basic series of G in W_k. Indeed

$$V_{k-2i-2j}^{(k-2i)}/V_{k-2i-2j+2}^{(k-2i)} \cong V_{k-2i-2j}^{(k)}/V_{k-2i-2j+2}^{(k)} \cong V_{k-2i-2j}^{(k-2i-2j)}$$

for every $j = 0, 1, ..., k/2-i-1$. Therefore at least one virtual pole appears in the basic series of G in W_k. ■

The next corollary easily follows from the previous two lemmas.

Corollary 4.6. *Suppose that p divides $\binom{n}{k/2-i}$ for some $i = 0, 1, ..., k/2-1$ and let i_0 be the maximal value of i such that p divides $\binom{n}{k/2-i}$. Then the section $V_{k-2i_0}^{(k)}/V_{k-2i_0+2}^{(k)}$ contributes a virtual pole.*

In the next two propositions we state a few results that, in certain situations, can help to locate virtual poles and possibly the true pole among them.

Proposition 4.7. *Suppose that a virtual pole $P^{(k-2i)}$ appears in $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$ for an index $i \in \{1, 2, ..., k/2 - 1\}$. Then one of the following holds:*

(1) *p divides neither $\binom{n-k+2i}{i}$ nor $\binom{k/2}{k/2-i}$ and the virtual pole $P^{(k-2i)}$ arises from the true pole P of G;*

(2) *p divides $\binom{n-k+2i}{i}$ but does not divide $\binom{k/2}{k/2-i}$. In this case the true pole of G is contained in $V^{(k)}_{k-2i+2}$, whence it appears as a virtual pole in a section $V^{(k)}_{k-2j}/V^{(k)}_{k-2j+2}$ for some $j < i$.*

(3) *p divides both $\binom{n-k+2i}{i}$ and $\binom{k/2}{k/2-i}$.*

Proof. By assumption, $v^{(k-2i)}_P \in V^{(k-2i)}_{k-2i}$, namely $v^{(k-2i)}_P = \sum_{S\in\mathcal{S}} a_S v_S$ where $\mathcal{S}$ is a suitable set of singular $(k-2i)$-spaces of V, a_S is a scalar and $v_S = u_1 \wedge ... \wedge u_{k-2i}$ for a given basis $\{u_1, ..., u_{k-2i}\}$ of S. We may also assume to have chosen $\mathcal{S}$ in such a way that the vectors v_S are linearly independent in $V^{(k-2i)}_{k-2i}$. Now choose for every $S \in \mathcal{S}$ a $2i$-subspace X_S such that $X_S \cap S = 0$ and $(X_S + S)^\perp \cap (X_S + S) = S$ (here $\perp$ stands for orthogonality with respect to α). Choose also a basis $\{v_1, ..., v_{2i}\}$ of X_S and put $v_{X_S} = v_1 \wedge ... \wedge v_{2i}$. Note that the form α of V induces a non-degenerate form on X_S. Thus, we may always choose a hyperbolic basis as $\{v_1, ..., v_{2i}\}$. Put

$$\overline{v}^{(k-2i)}_P := \sum_{S\in\mathcal{S}} a_S v_S \wedge v_{X_S} + V^{(k)}_{k-2i+2}.$$

Namely, $\overline{v}^{(k-2i)}_P$ is the vector of $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$ corresponding to $v^{(k-2i)}_P$. Note that $v_S \wedge v_{X_S} + V^{(k)}_{k-2i+2}$ does not depend on the particular choice of X_S but for a scalar (see Theorem 3.5; recall that, however, a scalar is also involved in the choice of a basis of X_S). We are now going to choose the complements X_S (actually, several of them for every S) in a standard way. Consider the polar vector $v^{(2i)}_P$ of G in W_{2i},

$$v^{(2i)}_P = \sum_{J\in\binom{I}{i}} e_J \wedge f_J.$$

Then $v^{(k-2i)}_P \wedge v^{(2i)}_P = \sum_{S\in\mathcal{S}} a_S(v_S \wedge v^{(2i)}_P)$. We may assume $S = \langle e_1, ..., e_{k-2i}\rangle$ and $v_S = e_1 \wedge ... \wedge e_{k-2i}$. (This amounts to change the given basis of V, which can be done safely since G fixes both $v^{(2i)}_P$ and $v^{(k-2i)}_P$.) So the only summands $e_J \wedge f_J$ of $v^{(2i)}_P$ such that $v_S \wedge e_J \wedge f_J \neq 0$ are those where $J \subseteq \{k-2i+1, ..., n\}$ (and $|J| = i$). Exactly $\binom{n-k+2i}{i}$ choices are possible for such a set. We claim the following:

$(*)$ $v_S \wedge e_J \wedge f_J - v_S \wedge e_{J'} \wedge f_{J'} \in V^{(k)}_{k-2i+2}$ for any two i-subsets J and J' of $\{k-2i+1, ..., n\}$.

Suppose firstly that the symmetric difference of J and J' has size 2. Let $J = \{k-2i+1, k-2i+3, ..., k-i+1\}$ and $J' = \{k-2i+2, k-2i+3, ..., k-i+1\}$, to fix ideas. In this case $(*)$ immediately follows by observing that

$$\begin{aligned} & e_{k-2i+1} \wedge f_{k-2i+1} - e_{k-2i+2} \wedge f_{k-2i+2} = \\ & \quad = (e_{k-2i+1} + e_{k-2i+2}) \wedge (f_{k-2i+1} - f_{k-2i+2}) + \\ & \quad + e_{k-2i+1} \wedge f_{k-2i+2} - e_{k-2i+2} \wedge f_{k-2i+1}. \end{aligned}$$

Turning to the general case, we can always find a sequence $J_0, J_1, ..., J_m$ of i-subsets of $\{k-2i+1, ..., n\}$ such that $J_0 = J$, $J_m = J'$ and the symmetric difference of J_{j-1} and J_j has size 2 for $j = 1, ..., m$. Hence all differences $v_S \wedge e_{J_{j-1}} \wedge f_{J_{j-1}} - v_S \wedge e_{J_j} \wedge f_{J_j}$ belong to $V^{(k)}_{k-2i+2}$. The difference $v_S \wedge e_J \wedge f_J - v_S \wedge e_{J'} \wedge f_{J'}$ is the sum of these differences, hence it also belongs to $V^{(k)}_{k-2i+2}$. Claim $(*)$ is proved.

In view of $(*)$ and since there are exactly $\binom{n-k+2i}{i}$ possible choices for J, we obtain that $v_S \wedge v^{(2i)}_P = \binom{n-k+2i}{i} \cdot v_S \wedge e_J \wedge f_J$ for a given $J \subset \{k-2i+1, ..., n\}$ with $|J| = i$. Therefore

$$v^{(k-2i)}_P \wedge v^{(2i)}_P + V^{(k)}_{k-2i+2} = \binom{n-k+2i}{i} \cdot \overline{v}^{(k-2i)}_P. \tag{8}$$

However,

$$v^{(k-2i)}_P \wedge v^{(2i)}_P = \binom{k/2}{k/2-i} v^{(k)}_P. \tag{9}$$

Suppose firstly that p divides $\binom{k/2}{k/2-i}$. Then the left side of (8) is the null vector of $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$. If p does not divide $\binom{n-k+2i}{i}$ then we obtain $\overline{v}^{(k-2i)}_P = 0$, a contradiction to our hypotheses. Hence p divides $\binom{n-k+2i}{i}$ and we have case (3) of the proposition.

Suppose now that p does not divide $\binom{k/2}{k/2-i}$. Then we can rewrite (8) as follows:

$$v^{(k)}_P + V^{(k)}_{k-2i+2} = \frac{\binom{n-k+2i}{i}}{\binom{k/2}{k/2-i}} \cdot \overline{v}^{(k-2i)}_P. \tag{10}$$

If p divides $\binom{n-k+2i}{i}$ then (10) implies $v^{(k)}_P \in V^{(k)}_{k-2i+2}$, which is the situation considered in case (2). Finally, let p do not divide $\binom{n-k+2i}{i}$. Then (10) says that $v^{(k)}_P$ belongs to $V^{(k)}_{k-2i} \setminus V^{(k)}_{k-2i+2}$. We have case (1). ∎

Corollary 4.8. *Assume that there is an integer $0 \leq r \leq k/2-1$ such that p divides $\binom{n}{k/2-i}$ for some $i \geq r$ but does not divide $\binom{n-2j}{k/2-j}$ for any $j \geq r$. Then exactly one of the sections $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$ contains a virtual pole and that virtual pole is contributed by the real pole P of G. In particular, if p divides $\binom{n}{k/2-i}$ for some i but it does not divide $\binom{n-2j}{k/2-j}$ for any j, then exactly one virtual pole appears in the basic series of G and it is contributed by the real pole of G.*

Proof. Let i_0 be the largest i for which p divides $\binom{n}{k/2-i}$. By Corollary 4.6, a virtual pole $P^{(k-2i_0)}$ appears in $V^{(k)}_{k-2i_0}/V^{(k)}_{k-2i_0+2}$. If $i_0 = 0$ then $P \subseteq V^{(k)}_k$ by Lemma 4.4 and P is also the unique virtual pole of G, since in this case p does not divide $\binom{n}{k/2-i}$ for any $i > 0 = i_0$. On the other hand, let $i_0 > 0$ and let i be any index with $r \leq i \leq i_0$ for which a virtual pole $P^{(k-2i)}$ appears in $V^{(k)}_{k-2i}/V^{(k)}_{k-2i+2}$. Since p does not divide $\binom{n-2j}{k/2-j}$ for any $j \geq r$, case (1) of Proposition 4.7 holds. Hence $P \in V^{(k)}_{k-2i} \setminus V^{(k)}_{k-2i+2}$. Consequently, i takes only one value, namely $i = i_0$. ■

5 First appearance of reducibility

The following is well known (e.g., see Premet and Suprunenko, Corollary at page 1317).

Lemma 5.1. *A subspace S of W_k is G-invariant if and only if it is $L(G)$-invariant.*

The next lemma is also well known. It is a variation of a celebrated theorem of Lie.

Lemma 5.2. *Let S be G-invariant. Given a Borel subgroup B of G, let U be its unipotent radical. Then U stabilizes at least one maximal flag $0 = S_0 \subset S_1 \subset S_2 \subset ... \subset S_d = S$ of subspaces of S, where $d = \dim(S)$. Moreover, $[U, S_{i+1}] \subseteq S_i$ for every $i = 0, 1, ..., d$. The same holds if U is replaced by any of its subgroups, in particular by the unipotent radical of a parabolic subgroup containing B.*

Let A be a 1-dimensional subspace of V, namely a 1-element of Δ. We may assume that $A = \langle e_1 \rangle$ for a given hyperbolic basis $\mathbb{E} = \{e_1, ..., e_n, f_1, ..., f_n\}$ of V (notation as in Section 4). We denote by U_A the unipotent radical of the stabilizer G_A of A in G and we put $W_A := \langle e_1 \wedge x \,:\, x \in \bigwedge^{k-1} W_{e_1} \rangle$ where $W_{e_1} := \langle e_2, ..., e_k, f_2, ..., f_k \rangle$.

Lemma 5.3. *The group U_A acts trivially on W_A.*

Proof. Let us give $\mathbb{E}$ the following ordering: $(e_1, e_2, ..., e_n, f_2, ..., f_n, f_1)$. The group U_A is generated by the root subgroups $X_i := \{x_i(t)\}_{t\in\mathbb{F}}$ for $i = 2, ..., n$, $Y_i := \{y_i(t)\}_{t\in\mathbb{F}}$ for $i = 2, ..., n$ and $Z := \{z(t)\}_{t\in\mathbb{F}}$, where the elements $x_i(t), y_i(t)$ and $z(t)$ are represented by matrices as follows with respect to the above ordering of $\mathbb{E}$:

$$\begin{bmatrix} I_n + tE_{1,i} & O_n \\ O_n & I_n - tE_{i-1,n} \end{bmatrix} \text{ for } x_i(t), \quad \begin{bmatrix} I_n & tE_{1,i-1} + tE_{i,n} \\ O_n & I_n \end{bmatrix} \text{ for } y_i(t),$$

$$\begin{bmatrix} I_n & tE_{1,n} \\ O_n & I_n \end{bmatrix} \text{ for } z(t).$$

(Notation as in the proof of Lemma 4.2.) The reader may check that $x_i(t), y_i(t)$ and $z(t)$ fix all vectors of W_A. Exploit Fact 2.2 to speed up computations. ■

If n is even we denote by P the pole of G (defined as in Section 4). If n is odd we put $P = 0$.

Lemma 5.4. *The following are equivalent for a vector $v \in W_k$:*
(1) $U_A(v) = v$;
(2) $U_A(\langle v\rangle) = \langle v\rangle$;
(3) $v \in W_A + P$.

Proof. Obviously (1) implies (2), while (3) implies (1) by Lemma 5.3. It remains to prove that (2) implies (3).

Let $\mathbb{E} = \{e_1, ..., e_n, f_1, ..., f_n\}$ be a hyperbolic basis as above. We turn back to the notation of Section 2.4, thus writing e_{i+n} for f_i. So, the ordering $(e_1, ..., e_n, f_2, ..., f_n, f_1)$ of $\mathbb{E}$ considered in the proof of Lemma 5.3 is now written as $(e_1, ..., e_n, e_{n+2}, ..., e_{2n}, e_{n+1})$. For $i = 0, 1, 2$ let $\mathcal{I}_i$ be the collection of all subsets of $\{2, ..., n, n+2, ..., 2n\}$ of size $k - i$. With this notation, every vector $x \in W_k$ can be written as a sum $x = y_x + u_x + v_x + w_x$ where

$$\begin{aligned} y_x &= e_1 \wedge \textstyle\sum_{J\in\mathcal{I}_1} y_J e_J, & u_x &= e_{n+1} \wedge \textstyle\sum_{J\in\mathcal{I}_1} \eta_J u_J e_J, \\ v_x &= e_1 \wedge e_{n+1} \wedge \textstyle\sum_{J\in\mathcal{I}_2} \eta_J v_J e_J, & w_x &= \textstyle\sum_{J\in\mathcal{I}_0} w_J e_J \end{aligned}$$

for suitable scalars y_J, u_J, v_J and w_J and with $\eta_J := (-1)^{|J\cap\{2,3,...,n\}|}$. The factor η_J takes into account how many transpositions we must perform in order to move e_{n+1} to right in $e_{n+1} \wedge e_J$ so that to place it after all factors e_j with $j \in J \cap \{2, 3, ..., n\}$, consistently with the natural ordering $(e_1, ..., e_n, e_{n+1}, ..., e_{2n})$ of $\mathbb{E}$.

We must prove that, if $(u)_k(\langle x\rangle) = \langle x\rangle$ for every $(u)_k \in U_A$, then $x \in W_A + P$. Note that $y_x \in W_A + P$. Thus we may safely assume that $y_x = 0$.

As remarked in the proof of Lemma 5.3, U_A is generated by root elements $x_2(t), ..., x_n(t), y_2(t), ..., y_n(t), z(t)$. In the proof of Lemma 5.3 we have also described matrices that represent those elements with respect to the ordering $(e_1, e_2, ..., e_n, e_{n+2}, ..., e_{2n}, e_{n+1})$ of $\mathbb{E}$. We want to see how the vectors u_x, v_x and w_x should be chosen for $\langle x\rangle$ to be stabilized by each of $x_i(t)$, $y_i(t)$ and $z(t)$. (Here and henceforth we take the liberty of writing $x_i(t)$, $y_i(t)$ and $z(t)$ for short instead of $(x_i(t))_k$, $(y_i(t))_k$ and $(z(t))_k$.)

Let us start with $z(t)$, $t \neq 0$. By exploiting Fact 2.2 and the matrix given for $z(t)$ in the proof of Lemma 5.3, it is not difficult to see that $\langle x\rangle$ is stabilized by $z(t)$ if and only if $te_1 \wedge \sum_{J\in\mathcal{I}_1} \eta_J u_J e_J = 0$. Therefore, if $z(t)$ stabilizes $\langle x\rangle$ and $t \neq 0$ then $u_J = 0$ for every $J \in \mathcal{I}_1$. Hence $u_x = 0$.

Next we consider $x_2(t)$, $t \neq 0$. By exploiting Fact 2.2 and the matrix given for $x_i(t)$ in the proof of Lemma 5.3 we can see that $x_2(t)$ stabilizes $\langle x\rangle$ (recall that $u_x = 0$) if and only if

$$-te_1 \wedge e_{n+2} \wedge \sum_{n+2\notin J\in\mathcal{I}_2} \eta_J v_J e_J + te_1 \wedge \sum_{2\in J\in\mathcal{I}_0} w_J e_{J\setminus\{2\}} = 0.$$

Therefore

Claim 1. For $J \in \mathcal{I}_0$ we have $w_J = 0$ if $2 \in J$ but $n + 2 \notin J$ while for $J \in \mathcal{I}_2$ we have $v_J = 0$ if $2 \in J$ but $n + 2 \notin J$.

On the other hand, if $\{2, n + 2\} \subseteq J$ for a set $J \in \mathcal{I}_0$ then

$$tv_{J\setminus\{2,n+2\}} \cdot e_1 \wedge e_{n+2} \wedge \eta_{J\setminus\{2,n+2\}} e_{J\setminus\{2,n+2\}} = tw_J \cdot e_1 \wedge e_{J\setminus\{2\}}.$$

Note that $|(J \setminus \{2, n+2\}) \cap \{2, 3, ..., n\}|$ is equal to the number of transpositions to perform in order to move e_{n+2} to the right in $e_1 \wedge e_{n+2} \wedge e_{J\setminus\{2,n+2\}}$ so that to place it in its natural position. Hence,

Claim 2. If $\{2, n+2\} \subseteq J \in \mathcal{I}_0$ then $w_J = v_{J\setminus\{2,n+2\}}$.

Turning to $y_2(t)$ with $t \neq 0$ and recalling that $u_x = 0$, one can see that $y_2(t)$ stabilizes $\langle x \rangle$ if and only if

$$te_1 \wedge e_{n+2} \wedge \sum_{2\notin J\in\mathcal{I}_2} \eta_J v_J e_J + te_1 \wedge \sum_{n+2\in J\in\mathcal{I}_0} (-1)^{|J\cap\{2,3,\ldots,n\}|} w_J e_{J\setminus\{n+2\}} = 0.$$

Therefore,

Claim 3. For $J \in \mathcal{I}_0$ we have $w_J = 0$ if $2 \notin J$ but $n+2 \in J$ while for $J \in \mathcal{I}_2$ we have $v_J = 0$ if $2 \notin J$ but $n+2 \in J$.

If $\{2, n+2\} \subseteq J \in \mathcal{I}_0$ then we get just the same conclusions as in Claim 2. Claims 1 and 3 can be fused as follows:

Claim 4. For $J \in \mathcal{I}_0$ we have $w_J \neq 0$ only if J contains either both of 2 and $n+2$ or none of them. Similarly, for $J \in \mathcal{I}_2$ we have $v_J \neq 0$ only if J contains either both of 2 and $n+2$ or none of them.

If we replace $x_2(t)$ and $y_2(t)$ by $x_i(t)$ and $y_i(t)$ for any choice of $i = 3, 4, ..., n$ we obtain claims similar to (2) and (4). Thus, for every $i = 2, 3, ..., n$ and $J \in \mathcal{I}_2$ we have $v_J \neq 0$ only if J contains either both i and $n+i$ or none of them. In other words, J is a union of $k/2 - 1$ pairs $\{i, n+i\}$. (Clearly, this can happen only if k is even.) Similarly, for $J \in \mathcal{I}_0$ we have $w_J \neq 0$ only if J contains either both i and $n+i$ or none of them, namely J is a union of $k/2$ pairs $\{i, n+i\}$. Moreover, if $\{i, i+n\} \subseteq J \in \mathcal{I}_0$ then $w_J = v_{J\setminus\{i,n+i\}}$.

Let $\overline{\mathcal{I}}_2$ and $\overline{\mathcal{I}}_0$ be the subfamilies of $\mathcal{I}_2$ and $\mathcal{I}_0$ formed by those sets that are unions of pairs $\{i, n+i\}$. By the above, $v_x = e_1 \wedge e_{n+1} \wedge \sum_{J\in\overline{\mathcal{I}}_2} \eta_J v_J e_J$ and $w_x = \sum_{J\in\overline{\mathcal{I}}_0} w_J e_J$. Clearly $\overline{\mathcal{I}}_2 = \overline{\mathcal{I}}_0 = \emptyset$ when n is odd. So, if n is odd then $x = u_x + v_x + w_x = 0$. In this case we are done. Let n be even. We also know that if $J \in \overline{\mathcal{I}}_0$ with $\{i, n+i\} \subseteq J$ then $v_{J\setminus\{i,n+i\}} = w_J$. Suppose firstly that $n > 2$. Then, given $J \in \mathcal{I}_0$ with $i, n+i, j, n+j \in J$, we have $v_{J\setminus\{i,n+i\}} = w_J = v_{J\setminus\{j,n+j\}}$. As the pairs $\{i, n+i\}$ and $\{j, n+j\}$ can be chosen arbitrarily, we eventually obtain that $v_K = v_{K'}$ for any two sets $K, K' \in \overline{\mathcal{I}}_2$. Put $\lambda := v_K$. As $w_J = v_{J\setminus\{i,n+i\}}$ if $J \in \overline{\mathcal{I}}_0$ and $\{i, n+i\} \subseteq J$, we also obtain $w_J = \lambda$ for every $J \in \overline{\mathcal{I}}_0$. Therefore $x = u_x + v_x + w_x = v_x + w_x = \lambda \sum_{K\in\binom{I'}{k/2}} e_K \wedge f_K$, where $I' := \{1, 2, ..., n\}$. However $\sum_{K\in\binom{I'}{k/2}} e_K \wedge f_K$ is just the polar vector v_P of G. So, $x = \lambda v_P$, namely $x \in P$.

The case of $n = 2$ remains to examine. In this case $\overline{\mathcal{I}}_2 = \emptyset$. It is straightforward to check that $x = \lambda v_P$ in this case too. ∎

In view of the next theorem we need to modify our notation a little. We write $\Delta_{k,n}$ instead of Δ_k, to remind that $\Delta_{k,n}$ is built by subspaces of $V(2n, \mathbb{F})$. Accordingly, we write $\varepsilon_{k,n}$ instead of ε_k, $W_{k,n}$ instead of W_k, $V_{k,n}$ instead of V_k and $V^{(k,n)}_{k-2i}$ instead of $V^{(k)}_{k-2i}$. However, in order to avoid a too heavy notation if not strictly necessary, we keep the symbols P and v_P for the pole and the polar vector of G in $W_{k,n}$ when k is even (Theorem 4.1). We also keep the symbol Δ to denote the building associated to $\mathrm{Sp}(2n, \mathbb{F})$, from which $\Delta_{k,n}$ arises.

We recall that, according to Theorem 2.5, the radical $R(\varepsilon_{k,n})$ of $\varepsilon_{k,n}$ is the largest G-invariant proper subspace of $W_{k,n}$.

Theorem 5.5. *Suppose that* $R(\varepsilon_{k,n}) \neq 0$ *but* $R(\varepsilon_{k-1,n-1}) = 0$. *Then:*
(1) $R(\varepsilon_{k,n}) = P$;
(2) $\dim(R(\varepsilon_{k+r,n+r})) > 1$ *for* $r = 1, 2, 3, ...$;
(3) $R(\varepsilon_{k-s,n-s}) = 0$ *for* $1 \leq s < k$.

Proof. Let $R(\varepsilon_{k,n}) \neq 0 = R(\varepsilon_{k-1,n-1})$. By Lemma 5.2, given a 1-element A of Δ, the group U_A stabilizes a 1-dimensional subspace R_1 of $R := R(\varepsilon_{k,n})$. We recall that $W_A = e_1 \wedge W(e_1) = \{e_1 \wedge x \mid x \in W_{e_1}\}$ where $W_{e_1} := \wedge^{k-1} S(e_1)$, $S(e_1) := \langle e_2, ..., e_n, e_{n+2}, ..., e_{2n}\rangle$. Moreover, U_A acts trivially on W_A (Lemma 5.3). Clearly $W_A \cong W_{k-1,n-1}$ as modules for $\overline{G}_A := (G_A/U_A)' \cong \mathrm{Sp}(2n-2, \mathbb{F})$, where $(G_A/U_A)'$ is the commutator subgroup of G_A/U_A. (Note that the group induced by G_A/U_A on W_A is slightly larger than $(G_A/U_A)'$, since it also involves multiplications of e_1 by scalars, but we have ruled them out by considering $(G_A/U_A)'$ instead of G_A/U_A.) The isomorphism $W_A \cong W_{k-1,n-1}$ maps $V_{k,n} \cap W_A$ onto $V_{k-1,n-1}$ and $R_1 \cap W_A$ onto a proper subspace of $V_{k-1,n-1}$ stabilized by $\overline{G}_A$. However $R(\varepsilon_{k-1,n-1}) = 0$ by assumption. Hence $R_1 \cap W_A = 0$ by Theorem 2.5. Therefore $R_1 = P$, by Lemma 5.4. It follows that n is even.

Assume that $P \subset R$. By Lemma 5.2, U_A stabilizes a 2-dimensional subspace R_2 of R containing P. As above, $R_2 \cap W_A = 0$ since $R(\varepsilon_{k-1,n-1}) = 0$ by assumption. Pick a vector $x \in R_2 \setminus P$ and let v_P be the polar vector of G in $W_{k,n}$. So, $P = \langle v_P \rangle$. If $u \in U_A$ then $(u)_k(x) = x + \lambda_u v_P$ for a scalar λ_u because R_2 is stabilized by U_A. As in the proof of Lemma 5.4, we write x as $x = y_x + u_x + v_x + w_x$ where

$$\begin{array}{ll} y_x = e_1 \wedge \sum_{J \in \mathcal{I}_1} y_J e_J, & u_x = e_{n+1} \wedge \sum_{J \in \mathcal{I}_1} \eta_J u_J e_J, \\ v_x = e_1 \wedge e_{n+1} \wedge \sum_{J \in \mathcal{I}_2} \eta_J v_J e_J, & w_x = \sum_{J \in \mathcal{I}_0} w_J e_J \end{array}$$

(notation as in the proof of Lemma 5.4) and we consider the effect of applying $z(t)$, $x_i(t)$ and $y_i(t)$ to x. We firstly apply $z(t)$. We obtain $z(t)(x) = x + te_1 \wedge \sum_{J \in \mathcal{I}_1} u_J e_J$. Hence

$$\lambda_{z(t)} v_P = te_1 \wedge \sum_{J \in \mathcal{I}_1} \eta_J u_J e_J. \tag{11}$$

However the vector at the right side of (11) belongs to W_A while $v_P \in P$ and $W_A \cap P = 0$. Therefore $\lambda_{z(t)} = 0$ and $u_J = 0$ for every $J \in \mathcal{I}_1$. So $u_x = 0$, as in the proof of Lemma 5.4. If we now apply $x_2(t)$, recalling that $u_x = 0$ we obtain

$$x_2(t)(x) = x - te_1 \wedge e_{n+2} \wedge \sum_{n+2 \notin J \in \mathcal{I}_2} \eta_J v_J e_J + te_1 \wedge \sum_{2 \in J \in \mathcal{I}_0} w_J e_{J \setminus \{2\}}.$$

Hence

$$\lambda_{x_2(t)} v_P = -te_1 \wedge e_{n+2} \wedge \sum_{n+2 \notin J \in \mathcal{I}_2} \eta_J v_J e_J + te_1 \wedge \sum_{2 \in J \in \mathcal{I}_0} w_J e_{J \setminus \{2\}}. \tag{12}$$

The factor e_1 appears in each of the summands at the right side of (12) while e_{n+1} does not appear in any of them. On the other hand, $e_1 \wedge e_{n+1}$ is involved in v_P. Hence both sides of (12) are null, namely $\lambda_{x_2(t)} = 0$ and the coefficients v_J and w_J occurring in (12) satisfy certain conditions that ensure the right side of (12) to be null. However we will not exploit these latter conditions in the sequel. Finally, apply $y_2(t)$ and recall that $u_x = 0$. We obtain

$$y_2(t)(x) = x + te_1 \wedge e_2 \wedge \sum_{2 \notin J \in \mathcal{I}_2} \eta_J v_J e_J + te_1 \wedge \sum_{n+2 \in J \in \mathcal{I}_0} \eta_J w_J e_{J \setminus \{n+2\}}.$$

Therefore

$$\lambda_{y_2(t)} v_P = te_1 \wedge e_2 \wedge \sum_{2 \notin J \in \mathcal{I}_2} \eta_J v_J e_J + te_1 \wedge \sum_{n+2 \in J \in \mathcal{I}_0} \eta_J w_J e_{J \setminus \{n+2\}}. \qquad (13)$$

Again, e_1 occurs in each summand at the right side of (13) but e_{n+1} doesn't, while $e_1 \wedge e_{n+1}$ occurs in v_P. Therefore both sides of (13) are null. Hence $\lambda_{y_2(t)} = 0$.

Similarly, $\lambda_{x_i(t)} = \lambda_{y_i(t)} = 0$ for every $i = 3, 4, ..., n$. It follows that U_A fixes x. Hence $x \in P$ by Theorem 4.1. This contradicts the choice of $x \in R_2 \setminus P$ and the fact that $W_A \cap R_2 = 0$. Therefore $P = R$. Claim (1) is proved.

We now turn to claim (2). Given $r \geq 1$, put $\widehat{V} := V(2n + 2r, \mathbb{F})$, $\widehat{G} := \mathrm{Sp}(2n + 2r, \mathbb{F})$ and let $\hat{\alpha}$ be the alternating form of $\widehat{V}$ and $\widehat{\Delta}$ the building associated to $\widehat{G}$. So, $W_{k+r,n+r} = \wedge^{k+r}\widehat{V}$ and $\Delta_{k+r,n+r}$ is the $(k+r)$-grassmannian of $\widehat{\Delta}$.

Let $\widehat{\mathbb{E}} = \{e_1, ..., e_{2n+2r}\}$ be a hyperbolic basis of $\widehat{V}$, with indices chosen so that $\hat{\alpha}(e_i, e_j) = \hat{\alpha}(e_{i+n}, e_{j+n}) = 0$ for $i, j \in \{1, 2, ..., n+r\}$, $\hat{\alpha}(e_i, e_{j+n+r}) = \delta_{i,j}$ and $\hat{\alpha}(e_{i+n+r}, e_j) = -\delta_{i,j}$ for $i, j \in \{1, 2, ..., n+r\}$, as usual. Turning back to the notation introduced at the beginning of this section, we put $f_i := e_{i+n+r}$ for $i = 1, 2, ..., n+r$. Also $\hat{e}_i := e_{i+n}$ and $\hat{f}_i := f_{i+n}$ for $i = 1, 2, ..., r$. Given a totally isotropic r-subspace X of $\widehat{V}$ and a basis $\{x_1, ..., x_r\}$ of X, let $v_X := x_1 \wedge ... \wedge x_r$ and $\widehat{W}_X := v_X \wedge \widehat{W}(X)$ where $\widehat{W}(X) := \wedge^k S$ for a complement S of X in $X^\perp$. We warn that v_X is defined modulo a scalar, but this has no effect on the definition of $\widehat{W}_X$. Note also that $\widehat{W}(X)$ depends on the choice of S, but this choice has no effect on the wedge product $v_X \wedge \widehat{W}(X)$.

Let A and B be two totally isotropic r-subspaces of $\widehat{V}$ such that $A^\perp \cap B = 0$, namely A and B have maximal distance in the r-grassmannian $\Delta_{r,n+r}$ of $\widehat{\Delta}$. We may assume to have chosen $\widehat{\mathbb{E}}$ so that $A = \langle \hat{e}_1, ..., \hat{e}_r \rangle$ and $B = \langle \hat{f}_1, ..., \hat{f}_r \rangle$. Accordingly, $v_A = \hat{e}_1 \wedge \hat{e}_2 \wedge ... \wedge \hat{e}_r$, $v_B = \hat{f}_1 \wedge \hat{f}_2 \wedge ... \wedge \hat{f}_r$, $\widehat{W}_A = v_A \wedge \widehat{W}(A)$ and $\widehat{W}_B = v_B \wedge \widehat{W}(B)$ where $\widehat{W}(A) = \widehat{W}(B) = \wedge^k S$, $S := \langle e_1, ..., e_n, f_1, ..., f_n \rangle$. Moreover, $\widehat{V}_A := \widehat{W}_A \cap V_{k+r,n+r} = v_A \wedge V_{k,n}$, where we regard V as the same thing as the linear subspace $\langle e_1, ..., e_n, f_1, ..., f_n \rangle$ of $\widehat{V}$, whence $V_{k,n}$ as a linear subspace of $\wedge^k \widehat{V}$.

Let $\widehat{G}_A$ be the stabilizer of A in $\widehat{G}$ and let $\widehat{U}_A$ be the unipotent radical of $\widehat{G}_A$. The group $\widehat{U}_A$ acts trivially on $\widehat{W}_A$ (Lemma 5.3). Moreover $\widehat{W}_A \cong W_{k,n}$ as modules for $(\widehat{G}_A/\widehat{U}_A)' \cong G$ $(= \mathrm{Sp}(2n, \mathbb{F}))$. By the first part of the proof, $\widehat{V}_A$ contains a 1-dimensional subspace $P_A = \langle v_{P_A} \rangle$ corresponding to the pole of G in its action on $W_{k,n}$. We may also assume that the vector v_{P_A} chosen to generate P_A corresponds to the polar vector v_P of G. So,

$$v_{P_A} = v_A \wedge \sum_{J \in \binom{I}{k/2}} e_J \wedge f_J \tag{14}$$

where $I = \{1, 2, ..., n\}$. Note that $v_{P_A} \in \widehat{V}_A$ because $v_P \in V_{k,n}$ by the first part of the proof. Let $\alpha_{k+r,n+r}$ be the fundamental form of $\varepsilon_{k+r,n+r}$ in $W_{k+r,n+r}$. By comparing (14) with (5) of Subsection 2.4 it is easy to see that $\alpha_{k+r,n+r}(v_{P_A}, x) = 0$ for every vector $x \in \widehat{W}_B$. However v_{P_A} only depends on the choice of A whereas B can be any totally isotropic r-subspace of $\widehat{V}$ at maximal distance from A in $\Delta_{r,n+r}$. By the second claim of Lemma 2.3, these subspaces span $V_{r,k+n}$. Moreover, $V_{k+r,n+r}$ is the union of the subspaces $\widehat{W}_X$ where X ranges in the family of totally isotropic r-subspaces of $\widehat{V}$. It follows that $v_{P_A} \in V_{k+r,n+r}^{\perp}$, where $\perp$ denotes orthogonality with respect to $\alpha_{k+r,n+r}$. On the other hand $v_{P_A} \in \widehat{V}_A \subseteq V_{k+r,n+r}$. Hence $v_{P_A} \in R(\varepsilon_{k+r,n+r})$. As any totally isotropic r-subspace can be chosen as A, $\dim(R(\varepsilon_{k+r,n+r})) > 1$, as claimed in (2).

Finally, $R(\varepsilon_{k-s,n-s}) = 0$ for $1 \leq s < k$ because, if otherwise, by claim (2) with n and k replaced by $n-s$ and $k-s$ we obtain $R(\varepsilon_{k-1,n-1}) \neq 0$, contrary to our hypotheses. ∎

Corollary 5.6. *If* $\mathrm{char}(\mathbb{F}) = 0$ *then* $R(\varepsilon_{k,n}) = 0$*, namely the G-module* $V_{k,n}$ *is irreducible for every choice of n and* $1 \leq k \leq n$*.*

Let $\mathrm{char}(\mathbb{F}) = p > 0$ *and assume that* $R(\varepsilon_{k,n}) \neq 0$*. Then p divides* $\binom{n-i}{(k-i)/2}$ *for some* $i < k$ *with* $k - i$ *even.*

Proof. Suppose that $V_{k,n}$ is reducible as a G-module. Then $R(\varepsilon_{k,n}) \neq 0$ by Theorem 2.5. Let i be the largest integer ($< k$) for which $R(\varepsilon_{k-i,n-i}) \neq 0$. Then $R(\varepsilon_{k-i-1,n-i-1}) = 0$. By Theorem 5.5, $R(\varepsilon_{k-i,n-i})$ is the pole of G on $W_{k-i,n-i}$. By Theorem 4.3, $\mathrm{char}(\mathbb{F}) = p > 0$, $k - i$ is even and p divides $\binom{n-i}{(k-i)/2}$. In particular, if $i = 0$ then k is even and p divides $\binom{n}{k/2}$. ∎

For the rest of this section we assume $\mathrm{char}(\mathbb{F}) = p > 0$. Given a nonnegative integer h, we denote by $n(h,p)$ the smallest integer $n \geq h$ such that $R(\varepsilon_{n-h,n}) \neq 0$ if such an integer exists, otherwise $n(h,p) := \infty$ (but $n(h,p) < \infty$ in any case, as we will see in a few lines). Note that, in view of Theorem 5.5, if $n = n(h,p) < \infty$ and $k := n - h$ then $R(\varepsilon_{k,n})$ is 1-dimensional, whence k is even and $R(\varepsilon_{k,n}) = (V_2^{(k,n)})^{\perp}$.

In view of the next theorem we need one more definition. Let $h = \sum_{j=0}^{\infty} h_j p^j$ be the expansion of h to the base p. (Needless to say, only finitely many of the coefficients h_j are > 0.) Let e the smallest j such that $h_j < p - 1$. So,

$$h = [(p-1) \cdot \sum_{j=0}^{e-1} p^j] + h_e p^e + h_{e+1} p^{e+1} + ...$$

with $0 \leq h_e < p - 1$. Note that $e = 0$ is allowed. In this case $h_0 < p - 1$, namely $h + 1 \not\equiv 0 \pmod{p}$. With this convention, we define:

$$N(h,p) = 2(p - 1 - h_e)p^e + h. \tag{15}$$

Theorem 5.7. $n(h, p) \leq N(h, p)$.

Proof. Put $n := N(h, p)$ and $k = N(h, p) - h$. We firstly prove the following:

$(*)$ *The prime p divides $\binom{n}{k/2}$ but it does not divide $\binom{n}{k/2-i}$, for any positive integer $i < k/2$.*

For a positive integer m, let $\mathrm{ord}_p(m)$ be the largest exponent f such that p^f divides m. It is well known that $\mathrm{ord}_p(m!) = \sum_{j\geq 1}\lfloor m/p^j\rfloor$, where $\lfloor m/p^j\rfloor$ is the integral part of m/p^j (see [15, page 1336] for instance). Therefore

$$\mathrm{ord}_p(\binom{m}{r}) = \sum_{j\geq 1}\left(\lfloor\frac{m}{p^j}\rfloor - \lfloor\frac{r}{p^j}\rfloor - \lfloor\frac{m-r}{p^j}\rfloor\right). \tag{16}$$

(Note that all summands of the right hand side of (16) are nonnegative.) By straightforward calculations, which we leave to reader, one can check that

$$\lfloor\frac{n}{p^{e+1}}\rfloor - \lfloor\frac{k/2}{p^{e+1}}\rfloor - \lfloor\frac{n-k/2}{p^{e+1}}\rfloor = 1$$

while

$$\lfloor\frac{n}{p^j}\rfloor - \lfloor\frac{k/2-i}{p^j}\rfloor - \lfloor\frac{n-(k/2-i)}{p^j}\rfloor = 0$$

for every j and every $i = 1, 2, ..., k/2-1$. Claim $(*)$ follows from this with the help of (16).

By $(*)$ and Lemma 4.4, the pole of G in $W_{k,n}$ belongs to $V_{k,n}$. Hence $R(\varepsilon_{k,n}) \neq 0$. By Theorem 5.5, $n(h, p) \leq n$. ∎

Theorem 5.8. *Suppose that p does not divide $h+1$, namely $h = h_0 + h_1p + h_2p^2 + ...$ with $h_0 < p-1$. Then $n(h, p) = N(h, p)$.*

Proof. Put $n = n(p, h)$ and $k = n - h$, for short. Also, $h = h_0 + \chi p$ where $\chi := h_1 + h_2p + h_3p^2 + ...$ We recall that the pole P of G belongs to $V_k^{(k,n)}$, whence k is even and p divides $\binom{n}{k/2}$. By Theorem 5.7, $n = N(h, p) - 2\xi = 2(p - h_0 - 1) + h - 2\xi$ for a nonnegative integer ξ. According to this, $k = 2(p - h_0 - 1) - 2\xi$. Hence $\xi \leq p - h_0 - 2$ since $k/2 \geq 1$. We want to prove that $\xi = 0$. To a contradiction suppose that $\xi > 0$. We shall firstly prove the following:

$$p \text{ divides } \binom{n}{k/2-1}. \tag{17}$$

We know that p divides

$$\binom{n}{k/2} = \frac{n-k/2+1}{k/2}\cdot\binom{n}{k/2-1}.$$

So, if p does not divide $\binom{n}{k/2-1}$ then p divides $n - k/2 + 1$ to a higher power than $k/2$. However $n - k/2 + 1 = (p - 1 - h_0) + h - \xi + 1 = p + \chi p - \xi$, which is prime to p because $0 < \xi \leq p - h_0 - 2 < p$. We have reached a contradiction. Claim (17) is proved.

Let now r be the smallest even integer $i \geq 2$ such that p divides $\binom{n}{i/2}$. By (17), $r < k$. By assumption, p divides $\binom{n}{r/2}$ but it does not divide $\binom{n}{r/2-1}$. We can repeat the argument used to prove (17), now obtaining that p divides $n - r/2 + 1$ to a higher power than $r/2$. In particular, p divides $n - r/2 + 1$. We now have

$$\begin{aligned} n - r/2 + 1 &= 2(p - 1 - h_0) + h - 2\xi - r/2 + 1 \\ &= 2p + \chi p - 1 - h_0 - 2\xi - r/2. \end{aligned}$$

As p divides $n - r/2 + 1$, we obtain $1 + h_0 + 2\xi + r/2 \equiv 0 \pmod p$. However $r/2 \leq k/2 - 1 = (p - h_0 - 1) - \xi - 1$ and $\xi \leq p - h_0 - 2$. Therefore $1 + h_0 + 2\xi + r/2 < 2p$. It follows that $1 + h_0 + 2\xi + r/2 = p$, namely

$$r/2 = p - 1 - h_0 - 2\xi. \tag{18}$$

Consequently,

$$\left.\begin{aligned} n - r &= 2(p - 1 - h_0) + h - 2\xi - r = h + 2\xi, \\ (k - r)/2 &= \xi, \\ n - r - (k - r)/2 &= h + \xi. \end{aligned}\right\} \tag{19}$$

Moreover $2\xi \leq p - 1$ because $2\xi = p - 1 - h_0 - r/2$ by (18). By combining this inequality with (19) we see that

$$\lfloor \frac{n - r}{p^j} \rfloor - \lfloor \frac{(k - r)/2}{p^j} \rfloor - \lfloor \frac{n - r - (k - r)/2}{p^j} \rfloor = 0$$

for every positive integer j. By (16),

$$p \text{ does not divide } \binom{n - r}{(k - r)/2}. \tag{20}$$

On the other hand, we have chosen r in such a way that p divides $\binom{n}{r/2}$ but it does not divide $\binom{n}{r/2-i}$ for any positive integer $i < r/2$. Hence a virtual pole appears in $V_r^{(k,n)}/V_{r+2}^{(k,n)}$, by Corollary 4.6. By (20), we are in case (i) of Proposition 4.7: the polar vector v_P of G belongs to $V_r^{(k,n)} \setminus V_{r+2}^{(k,n)}$. This contradicts the hypothesis that $v_P \in V_k^{(k,n)}$. ∎

Remark 5.9. It is not difficult to see that the number $N(h, p)$ defined in (15) is indeed the smallest $n \geq h$ such that $n - h$ is even and $(n - h)/2 \leq_p n + 1$. Equivalently, $N(h, p)$ is the smallest $n \geq h$ such that p divides $\binom{1+\lfloor (n+h)/2 \rfloor}{h+1}$, which also is called $N(h, p)$ in Theorem 1.1. Hence Theorem 1.1 implies that $n(h, p) = N(h, p)$ for any choice of h, while our method has allowed us to prove this equality only when $h + 1 \not\equiv 0 \pmod p$. We believe it is possible to exploit our methods to prove that $n(h, p) = N(h, p)$ in any case, but at present we are not yet able to do that.

Remark 5.10. So far, we have regarded h as given, letting n and k to vary subject to the condition $n - k = h$. Conversely, assume to have chosen k and let n and h vary subject to the restriction $n - h = k$. Put

$$f := \mathrm{ord}_p(n - k + 1), \quad \nu := \frac{n + k - 1}{p^f} - p \cdot \lfloor \frac{n + k - 1}{p^{f+1}} \rfloor.$$

By exploiting the equality $n(h,p) = N(h,p)$ one can see that $R(\varepsilon_{k,n}) \neq 0$ if and only if

$$\lfloor k/2 \rfloor \geq p^f, \quad p-1 \geq \nu \geq \max(1, p - \frac{k}{2p^f}). \tag{21}$$

Moreover, $\dim(R(\varepsilon_{k,n})) = 1$ if and only if k is even and $\nu = p - k/(2p^f)$. The proof of the above claims is straightforward. We leave it for the reader.

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