A formula that maps elements to proper classes in an arbitrary $\in$-universe

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Abstract

In this paper we construct a formula $\varphi(x, a)$ on the language of set theory $\mathcal{L} : (\in, =)$ such that $\{x \mid \varphi(x, a)\}$ is a proper class for each element a and such that if $a \neq a'$, the classes $\{x \mid \varphi(x, a)\}$ and $\{x \mid \varphi(x, a')\}$ are different. This formula works for any structure with the exception of 2 structures with two elements each. This formula "maps" elements to proper classes injectively.

Introduction

In any structure for the language $\mathcal{L} : (\in, =)$ of set theory, the Russell class $R = \{x \mid x \notin x\}$ is always proper. In [1] Roland Hinnion studies the question if there are other ones and how many. He studied the cases where the classes are definable with and without parameters. One of his results states that there always as many proper classes (definable with parameters) as elements in the structure. In his paper, Roland Hinnion considers structures for the language of set theory and work with these in the meta theory ZF. In an oral question he asks if we could do this uniformly by finding a formula $\varphi(x, a)$ for the language $\mathcal{L}$ such that $\{x \mid \varphi(x, a)\}$ is a proper class for each element a and such that if $a \neq a'$, the classes $\{x \mid \varphi(x, a)\}$ and $\{x \mid \varphi(x, a')\}$ are different.

In this paper, we construct such a formula that works for any structure with the exception of exactly 2 structures, each having exactly 2 elements (for which we show that no such formula exists).

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1 Some notations and definitions

In this paper, we will work in the language $\mathcal{L}:(\in,=)$ of set theory. By "structure", we always mean a non-empty structure. We will consider an arbitrary structure for this language; so $\in$ is just any binary relation on the universe. Before going further, we will need some definitions and notations.

For elements $a_1,\ldots,a_n$ and for a formula $\varphi(x,a_1,\ldots,a_n)$, we will say that $\{x \mid \varphi(x,a_1,\ldots,a_n)\}$ ***is a set*** if there is an element e such that $\forall x\ \big(x \in e \Leftrightarrow \varphi(x,a_1,\ldots,a_n)\big)$. Notice that due to the lack of the axiom of extensionality, this element e is not necessarily unique.

For elements $a_1,\ldots,a_n$, and for a formula $\varphi(x,a_1,\ldots,a_n)$, we will say that $\{x \mid \varphi(x,a_1,\ldots,a_n)\}$ ***is a proper class*** if it is not a set. If there are no a_i in the formula φ, we will say that $\{x \mid \varphi(x)\}$ is a proper class definable ***without parameters*** otherwise we will say that it is a proper class definable ***with parameters***.

We will use some usual abuse of language using classes. For example, consider the parameters $a_1,\ldots,a_n$ and a formula $\varphi(x,a_1,\ldots,a_n)$ we may "consider" $A = \{x \mid \varphi(x,a_1,\ldots,a_n)\}$ and write $x \in A$ as meaning $\varphi(x,a_1,\ldots,a_n)$. If B is another class, defined with the formula ψ and the parameters $b_1,\ldots,b_n$: $B = \{x \mid \psi(x,b_1,\ldots,b_n)\}$, we write $A = B$ as meaning $\forall x\ \big(\varphi(x,a_1,\ldots,a_n) \Leftrightarrow \psi(x,a_1,\ldots,a_n)\big)$ and similarly for $A \neq B$. Notice that no axiom of extensionality is used here, we just use syntactical replacements. We will avoid to write $A = e$ for a class A and an element e because, due to the lack of the axiom of extensionality, we could have several elements e having the same elements as the class A.

If e and f are elements, we say that e is a ***singleton of*** f if $\forall x\ (x \in e \Leftrightarrow x = f)$. For an element e, we say that e is a ***singleton*** if there is an element f such that e is a singleton of f. We say that e is an ***antisingleton of*** f if $\forall x\ (x \notin e \Leftrightarrow x = f)$ and we say that e is an ***antisingleton*** if there is an element f such that e is the antisingleton of f. The previous notions will also be used for classes.

2 The formula

As mentioned in the introduction, we propose to give a formula $\varphi(x,a)$ such that $\forall a\ \{x \mid \varphi(x,a)\}$ is a proper class and such that if $a \neq a'$, $\{x \mid \varphi(x,a)\} \neq \{x \mid \varphi(x,a')\}$ (the last "$\neq$" should be understood as not equal as classes, i.e. not having the same elements). We shall say that a formula satisfying the previous requirement ***maps elements to proper classes injectively***.

This formula will work for any structure with the exception of exactly 2 structures each of which having exactly 2 elements (for which no such formula exists).

For the presentation, we will define a "mapping" F sending each element e of the structure to a proper class $F(e)$ such that $e \neq e' \Rightarrow F(e) \neq F(e')$ (we say that F sends elements to proper classes injectively). As we will use only first order formulas to define this mapping, this could easily be translated as a formula $\varphi(x,a)$ mapping elements to proper classes injectively as above. Three cases will be needed and we will define the mapping F separately for the three cases. As the three cases are distinguishable by first order formulas, all of these could be combined in a single first order formula $\varphi(x, a)$ mapping elements to proper classes injectively.

In what follows V denotes the universal class: $V = \{x \mid x = x\}$. R denotes the Russell class: $R = \{x \mid x \notin x\}$. Obviously R is always a proper class.

Case I. The class R is not empty and does not consist of exactly two singletons.

We consider the following definition for an element $e \in V$:

$$R_e = \begin{cases} \{e\} \quad \text{if it is a proper class} & \text{(a)} \\ \{x \mid x \notin x\} \cup e \quad \text{if } \forall t\ (t \in e \Leftrightarrow t = e) & \text{(b)} \\ \{x \mid x \notin x\} \setminus \{x \mid \forall t\ (t \in x \Leftrightarrow t = e)\} \quad \text{otherwise} & \text{(c)} \end{cases}$$

Lemma 1. *If for an element e, we are in the case (b) of the previous definition, then R_e is not a singleton. There is at most one element such that we are in the case (c) of the definition of R_e and such that R_e is a singleton.*

Proof. The fact that the case (b) never gives a singleton is due to the fact that we have supposed R not empty (case I). As the case (b) is simply R with an additional element, it could not give a singleton.

Suppose that e and e' are two distinct elements such that for e and e', we are in the case (c) for the definition of R_e and $R_{e'}$ and that R_e and $R_{e'}$ are both singletons. For an element e, the class defined in the case (c) is the class $R = \{x \mid x \notin x\}$ minus all elements that are singletons of e. So if R_e is a singleton, we deduce that there is only one element that is not a singleton of e that belongs to R; similarly for $R_{e'}$. As no element can be a singleton of e and e' at the same time, we conclude that R has at most two elements. Moreover there is an element that is a singleton of e (otherwise, we would be in the case (a) of the definition of R_e) and this element belongs to R (otherwise we would be in the case (b) of the definition of R_e), similarly for e'. So we conclude that R consists of exactly two singletons; this contradicts the hypothesis of case I. ∎

We are now ready to define the mapping F sending elements to proper classes injectively.

Definition. We define the mapping F as follows for an element e:

$$F(e) = \begin{cases} \{x \mid x \notin x\} & \text{if in the definition of } R_e \text{, we are in the case (c) and if } R_e \text{ is a singleton.} \quad (1) \\ R_e & \text{otherwise} \quad (2) \end{cases}$$

Lemma 2. *The mapping F defined previously sends elements to proper classes injectively.*

Proof. We first show that the mapping F is injective. In order to show this, we will show that e can be univoquely found from $F(e)$.

Let us first notice that if there is an element f for which we are in the case (1) of the definition of $F(f)$, then we conclude that the Russell class R is not a singleton (because in this case R_f is a singleton and $R_f \subsetneq R$). So in this case we are sure that for all elements e, $R_e \neq R$ (since clearly the case (b) and (c) of the definition of R_e does not give R).

Let us describe e from $F(e)$. If $F(e)$ is the Russell class R and if there is an element e such that we are in the case (1) of the definition of $F(e)$, then it is the element we are looking for (the element e is necessarily unique by lemma 1). Otherwise we have $F(e) = R_e$ and we are sure that the case (b) and (c) of the definition of R_e does not give a singleton. So if $F(e)$ is a singleton, we are in the case (a) of the definition of R_e and e is the unique element for which $F(e)$ is the singleton of e. Otherwise we are in the case (b) or (c) of the definition of R_e. We easily check that all elements belonging to the symmetrical difference $R \triangle R_e$ are singletons of a single element e and this is the element we are looking for.

Let us now check that for all elements e, $F(e)$ is a proper class. It is obvious if we are in the case (1) of the definition of $F(e)$ as well as if we are in the case (a) of the definition of R_e. If we are in the case (2) of the definition of $F(e)$ and in one of the cases (b) or (c) of the definition of R_e; then R_e is not a singleton. The only elements that are in the symmetric difference $R \triangle R_e$ are singletons; so we could reproduce the Russell paradox with R_e: if R_e would be a set, denoting r a set having the same elements as R_e, we would have $r \in r \Leftrightarrow r \notin r$. ∎

Case II. Case I does not hold and the complement $R^{\complement} = \{x \mid x \in x\}$ of the Russell class is not empty and does not consists of exactly two antisingletons.

We will use the following lemma.

Lemma 3. *Consider a formula $\varphi(x, a)$ for the language $(\notin, =)$ that maps elements to proper classes injectively for the structure $(V, \notin)$, then $\neg\varphi(x, a)$ maps elements to classes injectively for the structure $(V, \in)$.*

Proof. Easy ∎

It is straightforward to check that the hypotheses of case II just say that the structure $(V, \notin)$ satisfies the hypotheses of case I. So let $\psi(x, a)$ be the formula for the language $(\notin, =)$ that maps elements to proper classes injectively for the structure $(V, \notin)$ given by the case I. The formula $\varphi(x, a) \overset{\text{def}}{\equiv} \neg\psi(x, a)$ is thus the formula we are looking for.

Case III. Cases I and II do not hold and there are not exactly 2 elements.

In this case, we can see that the universe consists of 4 elements in which there are exactly 2 singletons and 2 antisingletons. Any class having exactly two elements is proper. For an element e, let us denote $S_e = \{x \mid x \text{ is a singleton of } e\}$ and $\widehat{S}_e = \{x \mid x \text{ is an antisingleton of } e\}$. For an element e, consider the classes: $A_1 = \{e\}$, $A_2 = \{x \mid x \neq e\}$, $A_3 = S_e$, $A_4 = \widehat{S}_e$, $A_5 = S_e \cup \widehat{S}_e$. We easily see that if A_1 and A_2 are not proper classes then one of the classes A_3, A_4, A_5 consists of 2 elements and is thus proper. For $F(e)$, we take A_1 if it is proper; we take A_2 if A_2 is proper and A_1 is not; otherwise we take the first of the classes A_3, A_4, A_5 that consists of two elements. F can straightforwardly be shown injective.

We easily show that the cases I, II and III cover every structure with the only exception of the following 2 structures, each of which having 2 elements: $(V_1, \in_1)$ with $V_1 = \{a, b\}$ and $\in_1 = \{(a, a), (b, b)\}$ and $(V_2, \in_2)$ with $V_2 = \{a, b\}$ and

$\in_2 = \{(a,b),(b,a)\}$; so we can summarize the previous results in the following theorem:

Main theorem. *There exists a formula $\varphi(x,a)$ on the language of set theory $\mathcal{L} : (\in, =)$ that maps elements to proper classes injectively in any $\mathcal{L}$-structure with the exception of the two structures $(V_1, \in_1)$ and $(V_2, \in_2)$ described above.*

We can show that no formula $\varphi(x,a)$ mapping elements to proper classes injectively exists for the two structures $(V_1, \in_1)$ and $(V_2, \in_2)$. In both of these structures there is an automorphism exchanging a and b so that a and b satisfy the same formulas. The only proper classes of these structures are $\emptyset$ and $\{a,b\}$. So if we could define a mapping F sending elements to proper classes injectively, one of the elements would be sent to the empty class; the other one to the class $\{a,b\}$. But we could then use this mapping to find a formula distinguishing a and b; a contradiction.

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