Diameter preserving linear bijections and $\mathcal{C}_0(L)$ spaces

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Abstract

We study diameter preserving linear bijections from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, \mathcal{C}_0(L))$ where X, Y are compact Hausdorff spaces, L is a locally compact Hausdorff space and V is a Banach space. In the case when X and Y are infinite and $\mathcal{C}_0(L)^*$ has the Bade property we prove that there is a diameter preserving linear bijection from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, \mathcal{C}_0(L))$ if and only if X is homeomorphic to Y and V is linearly isometric to $\mathcal{C}_0(L)$. Similar results are obtained in the case when X and Y are not compact but locally compact spaces.

1 Background and notation

The Banach-Stone theorem has been thoroughly studied and generalized to different settings in the last decades. A very instructive and readable starting point is the classic monograph by E. Behrends ([3]). The general question is when, given two compact Hausdorff spaces X, Y and two Banach spaces V, Z, the existence of a surjective linear isometry T from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, Z)$ implies any of the following:

(a) V is linearly isometric to Z.

(b) X is homeomorphic to Y.

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(c) Both (a) and (b) are satisfied and T can be expressed as $Tf(y) = \rho(y)(f(t(y)))$ with $t : Y \to X$ a homeomorphism and $\rho : Y \to \mathcal{G}(V, Z)$ a continuous mapping, $\mathcal{G}(V, Z)$ being the set of surjective linear isometries from V onto Z endowed with the strong operator topology.

If we focus on a concrete Banach space V then it is commonly said that V has the (strong/weak) Banach-Stone property or solves (strongly/weakly) the Banach-Stone problem depending on whether properties (a), (b), (c), (a)⇒(c), etc... hold or not for every X, Y, Z. The terminology is far from being standardized so we will not go further into the precise definitions. These are detailed in [8], [9] and [10], to mention only a few.

This problem can be studied considering linear bijections that preserve the diameter of the range of the functions instead of surjective linear isometries, and the results obtained are still very rich, sometimes even better (e.g. see [1], [2], [4], [5], [6], [7]). Such mappings are usually called diameter preserving linear bijections, we will frequently say dplb(s) for brevity.

Trying to keep some analogy with the Banach-Stone problem we introduce the following definitions. The symbol # stands for the cardinal of a set.

Definitions 1.1. *Let Z be a Banach space. If for all compact Hausdorff spaces X, Y with $\#X, \#Y \geq 3$ and every Banach space V, the existence of a diameter preserving linear bijection $T : \mathcal{C}(X, V) \to \mathcal{C}(Y, Z)$ implies that V is linearly isometric to Z and that*

1. *X is homeomorphic to Y then we say that Z solves the dplb problem.*

2. *There exists a homeomorphism $t : Y \to X$, a surjective linear isometry $G : V \to Z$ and a linear mapping $L : \mathcal{C}(X, V) \to Z$ so that $Tf(y) = G(f(t(y))) + L(f)$ for every $f \in \mathcal{C}(X, V)$ and $y \in Y$ then we say that Z solves strongly the dplb problem.*

If for all compact Hausdorff spaces X, Y with $\#X, \#Y \geq 3$ and every Banach space V which is linearly isometric to Z, the existence of a diameter preserving linear bijection $T : \mathcal{C}(X, V) \to \mathcal{C}(Y, Z)$ implies that the previous $1/2$ is satisfied then we say that Z has the dplb property / the strong dplb property, respectively.

Concerning the above properties, in the article [1] we proved: (1) If there is a dplb from $\mathcal{C}(X, V)$ to $\mathcal{C}(Y, Z)$, $\#X \geq 4$, $\#Y \geq 3$, V is linearly isometric to Z and Z^* has no isometric copy of $(\mathbb{R}^2, \| \|_\infty)$ then X is homeomorphic to Y; (2) there exist Banach spaces which do not solve the dplb problem; (3) every $\mathcal{C}_0(L)$ space, where L is a locally compact Hausdorff space, has the dplb property. In the following lines we give examples of $\mathcal{C}_0(L)$ spaces which lack the strong dplb property but we prove that if $\mathcal{C}_0(L)^*$ has the Bade property and X and Y are infinite then the dplbs from $\mathcal{C}(X, V)$ onto $\mathcal{C}(Y, \mathcal{C}_0(L))$ can still be expressed by a composition operator using one surjective linear isometry $G : V \to \mathcal{C}_0(L)$ and a family of homeomorphisms $(t_p)_{p \in L} : Y \to X$. We would like to remark that our examples here contradict theorem 2 in the article [11].

When it makes sense, we shall use the following notation:

B_V and S_V are, respectively, the closed unit ball and the unit sphere of the Banach space V. Besides, $\mathrm{ex}(B_V)$ is the set of extreme points of B_V.

hom(Y, X) is the set of homeomorphisms between the topological spaces Y and X.

δ_t is defined by $\delta_t(f) = f(t)$ for every f.

ξ_v is defined by $\xi_v(t) = v$ for every t.

By γL we denote the Alexandroff (i.e., one-point) compactification of the locally compact, noncompact Hausdorff space L, and by ∞ the point added.

When dealing with nets, the symbol $\rightarrow$ refers to norm or topological convergence depending on whether we are on a Banach or on a topological space. For the weak-star convergence we use $\xrightarrow{w^*}$.

The Choquet Boundary of a subspace M of $\mathcal{C}_0(L)$ is the set $\mathrm{ch}(M) = \{p \in L : \delta_p|_M \in \mathrm{ex}(B_{M^*})\}$. Besides, it is well known that every extreme point of B_{M^*} must be a $\delta_p|_M$ for some $p \in L$.

The following proposition summarizes some of the results in [1] and will be very useful in the next section.

Theorem 1.2. *Let X, Y be compact Hausdorff spaces with at least three points and V, Z Banach spaces. Suppose that there exists a diameter preserving linear bijection $T : \mathcal{C}(X, V) \rightarrow \mathcal{C}(Y, Z)$ and $H_V = \varnothing$. Then there exist:*

- *A mapping $t : \mathrm{ex}(B_{Z^*}) \rightarrow \mathrm{hom}(Y, X)$ which is w^*-pointwise continuous (in particular, Y and X are homeomorphic)*
- *A $w^* - w^*$ homeomorphism $F : \mathrm{ex}(B_{Z^*}) \cup \{0\} \rightarrow \mathrm{ex}(B_{V^*}) \cup \{0\}$ with $F(0) = 0$*

so that for every $f \in \mathcal{C}(X, V)$, $y, y' \in Y$ and $z^* \in \mathrm{ex}(B_{Z^*})$ we have

$$z^*(Tf(y) - Tf(y')) = F(z^*)(f(t_{z^*}(y)) - f(t_{z^*}(y')))$$

where $t_{z^*} = t(z^*)$. If X and Y are infinite then F can be extended to a surjective linear isometry $\overline{F} : \mathcal{L}(\mathrm{ex}(B_{Z^*})) \rightarrow \mathcal{L}(\mathrm{ex}(B_{V^*}))$.

If instead of those T, X, Y we have two locally compact, noncompact spaces X_0, Y_0 and a diameter preserving linear bijection $T : \mathcal{C}_0(X_0, V) \rightarrow \mathcal{C}_0(Y_0, Z)$, exactly the same can be said, with $f \in \mathcal{C}_0(X_0, V)$ and $X = \gamma X_0$ and $Y = \gamma Y_0$ everywhere else.

The set H_V is a subset of $\mathrm{ex}(B_{V^*})$ that depends on T and is defined in the aforementioned article, however we only need the fact that if $H_V \neq \varnothing$ then there exist pairwise linearly independent $z_1^*, z_2^*, z_3^* \in \mathrm{ex}(B_{Z^*})$ so that $z_1^* + z_2^* = z_3^*$. Therefore, if $Z = \mathcal{C}_0(L)$ then $H_V = \varnothing$, since $\mathrm{ex}(B_{\mathcal{C}_0(L)^*}) = \{\lambda\delta_p : p \in L, \lambda \in S_{\mathbb{K}}\}$.

2 Results

Theorem 2.1. *Let X, Y be compact Hausdorff spaces with at least three points, L a locally compact space and V a Banach space.*

1. *If there exists a diameter preserving linear bijection $T : \mathcal{C}(X, V) \rightarrow \mathcal{C}(Y, \mathcal{C}_0(L))$ then X and Y are homeomorphic and there exist*
    - *A diameter preserving linear isometry $\phi : \mathcal{C}(X, V) \rightarrow \mathcal{C}(Y, \mathcal{C}_0(L))$*

- *A mapping* $t : L \to \hom(Y, X)$ *which is continuous for the pointwise convergence in* $\hom(Y, X)$
- *A linear mapping* $\alpha : \mathcal{C}(X, V) \to \mathcal{C}_0(L)$
- *A linear isometry* $G : V \to \mathcal{C}_0(L)$

so that for every $p \in L$, $y \in Y$ *and* $f \in \mathcal{C}(X, V)$ *we have*

$$T(f)(y) = \phi(f)(y) + \alpha(f)$$

and

$$\phi(f)(y)(p) = G(f(t_p(y)))(p)$$

where $t_p = t(p)$*. In addition,* $\mathrm{ch}(G(V)) = L$*.*

2. *If* $T : \mathcal{C}(X, V) \to \mathcal{C}(Y, \mathcal{C}_0(L))$ *is a mapping expressed by* ϕ, t, α *and* G *as above then* T *is a diameter preserving linear mapping.* T *is injective if and only if the mapping* $D : V \to \mathcal{C}_0(L)$ *defined by* $D(v) = G(v) + \alpha(\xi_v)$ *is injective. If* T *is surjective then* D *is surjective. If* D *and* ϕ *are surjective then* T *is surjective. If* G *is surjective then* ϕ *is surjective.*

3. *In the situation of point (1.), if* X *and* Y *are infinite and* $\mathcal{C}_0(L)^*$ *has the Bade property then* G *is surjective.*

Proof. 1. From theorem 1.2 we know of the existence of the mapping t as well as a $w^* - w^*$ homeomorphism $F : \mathrm{ex}(B_{\mathcal{C}_0(L)^*}) \to \mathrm{ex}(B_{V^*})$ satisfying

$$(Tf(y) - Tf(y'))(p) = F(\delta_p)(f(t_p(y)) - f(t_p(y')))$$

for every $f \in \mathcal{C}(X, V)$, $y, y' \in Y$ and $p \in L$.

Define $G : V \to \mathcal{C}_0(L)$ by $G(v)(p) = F(\delta_p)(v)$. Then G is well defined by the properties of F, and $\|G(v)\| = \sup\{|F(\delta_p)(v)| : p \in L\} = \|v\|$. So G is a linear isometry.

Define $\phi : \mathcal{C}(X, V) \to \mathcal{C}(Y, \mathcal{C}_0(L))$ by $\phi f(y)(p) = F(\delta_p)(f(t_p(y)))$. Let us see that $\phi f(y) \in \mathcal{C}_0(L)$ for every $y \in Y$, $f \in \mathcal{C}(X, V)$. If $p_\alpha \to p$ is a convergent net in L then $t_{p_\alpha}(y) \to t_p(y)$ for every $y \in Y$, thus $f(t_{p_\alpha}(y)) \to f(t_p(y))$ for every $y \in Y$ and applying G we obtain that $\phi f(y)(p_\alpha) \to \phi f(y)(p)$ for every $y \in Y$. If L is noncompact and $p_\alpha \to \infty$ then $\delta_{p_\alpha} \xrightarrow{w^*} 0$, so $F(\delta_{p_\alpha}) \xrightarrow{w^*} 0$ and thus $F(\delta_{p_\alpha})(f(x)) \to 0$ for every f and uniformly on x, since X is compact. Therefore $\phi f(y)(p_\alpha) = F(\delta_{p_\alpha})(f(t_{p_\alpha}(y))) \to 0$.

If $f \in \mathcal{C}(X, V)$ and $y_\alpha \to y$ then $t_p(y_\alpha) \to t_p(y)$ for every $p \in L$, so $f(t_p(y_\alpha)) \to f(t_p(y))$ uniformly on p. Applying G we obtain that $\phi f(y_\alpha)(p) \to \phi f(y)(p)$ uniformly on p, thus $\phi f(y_\alpha) \to \phi f(y)$. Therefore, ϕf is continuous.

We shall see now that ϕ is a diameter preserving isometry. On one hand, $\rho(\phi f) = \sup\{\|\phi f(y) - \phi f(y')\| : y, y' \in Y\} = \sup\{\|Tf(y) - Tf(y')\| : y, y' \in Y\} = \rho(Tf) = \rho(f)$. On the other hand, $\|\phi f\| = \sup\{|\phi f(y)(p)| : y \in Y, p \in L\} = \sup\{|G(f(t_p(y)))(p)| : y \in Y, p \in L\} = \sup\{|G(f(x))(p)| : x \in X, p \in L\} = \|f\|$ since G is an isometry.

As we have already mentioned, $(Tf(y) - Tf(y'))(p) = (\phi f(y) - \phi f(y'))(p)$ for every $f \in \mathcal{C}(X,V)$, $y, y' \in Y$ and $p \in L$. This implies that $(Tf(y))(p) - \phi f(y)(p)$ does not depend on $y \in Y$, so we can define $\alpha(f) = Tf(y) - \phi f(y)$, now it is trivial that $\alpha : \mathcal{C}(X,V) \to \mathcal{C}_0(L)$ is a linear, well-defined mapping.

Besides, for every $p \in L$ we have $\delta_p \circ G = F(\delta_p)$, thus $\delta_p \circ G \in \mathrm{ex}(B_{V^*})$. Since G from V onto $G(V)$ is a surjective linear isometry, $\delta_p|_{G(V)} \in \mathrm{ex}(B_{G(V)^*})$. Therefore $\mathrm{ch}(G(V)) = L$.

2. T is clearly linear and diameter preserving. For every $v \in V, h \in \mathcal{C}_0(L)$, the mapping D satisfies $Dv = h$ if and only if $T\xi_v = \xi_h$; therefore, if T is injective/surjective then so is D. Now suppose D is injective and $Tf = 0$. This implies $\phi f(y) + \alpha(f) = 0$ for every $y \in Y$, thus $\phi f(y) - \phi f(y') = 0$ for every $y, y' \in Y$ and so $\rho(\phi f) = \rho(f) = 0$, which means $f = \xi_v$ for certain $v \in V$. Thus $Tf = \xi_{Dv} = 0$ and so $v = 0$. Suppose now that D and ϕ are surjective. For every $g \in \mathcal{C}(Y, \mathcal{C}_0(L))$ there exists $f \in \mathcal{C}(X,V)$ so that $\phi(f) = g$. Then $T(f - \xi_{D^{-1}(\alpha(f)}) = \phi(f) + \xi_{\alpha(f)} - \xi_{\alpha(f)} = g$. Finally, assume G is surjective. Take $g \in \mathcal{C}(Y, \mathcal{C}_0(L))$, for every $x \in X$ define a mapping $h_x : L \to X$ by $h_x(p) = g(t_p^{-1}(x))(p)$. Since Y is compact and g is continuous, $h_x \in \mathcal{C}_0(L)$. Now let $f : X \to V$ be the continuous mapping defined by $f(x) = G^{-1}(h_x)$, then $\phi f(y)(p) = G(f(t_p(y)))(p) = g(y)(p)$ and thus $\phi f = g$.

3. Suppose $\mu \in \mathcal{L}(\mathrm{ex}(B_{\mathcal{C}_0(L)^*}))$, then $\mu = \sum_{i=1}^n \alpha_i \delta_{p_i}$ and $\mu G(v) = \sum_{i=1}^n \alpha_i G(v)(p_i) = \sum_{i=1}^n \alpha_i F(\delta_{p_i})(v) = \overline{F}(\mu)(v)$. Now take $\mu \in B_{\mathcal{C}_0(L)^*}$ with $\mu G = 0$; since $\mathcal{C}_0(L)^*$ has the Bade property there exists a net $(\mu_\alpha)_{\alpha \in \Lambda} \subseteq \mathrm{co}(\mathrm{ex}(B_{\mathcal{C}_0(L)^*})$ with $\mu_\alpha \to \mu$, which implies $\overline{F}\mu_\alpha = \mu_\alpha G \to 0$, therefore $\mu_\alpha \to 0$ and $\mu = 0$. This proves that G^* is injective and G is surjective. ∎

By means of an almost identical proof we obtain an analogous theorem for the locally compact, noncompact case. The main difference is that now the mapping D disappears, simplifying slightly the problem.

Theorem 2.2. *Let X_0, Y_0 be locally compact, noncompact Hausdorff spaces with at least two points, L a locally compact space and V a Banach space. Consider $X = \gamma X_0$ and $Y = \gamma Y_0$.*

1. *If there exists a diameter preserving linear bijection $T : \mathcal{C}_0(X_0, V) \to \mathcal{C}_0(Y_0, \mathcal{C}_0(L))$ then X and Y are homeomorphic and there exist*

   - *A diameter preserving linear isometry $\phi : \mathcal{C}_0(X_0, V) \to \mathcal{C}_0(Y_0, \mathcal{C}_0(L))$*
   - *A mapping $t : L \to$ $\mathrm{hom}(Y, X)$ which is continuous for the pointwise convergence in $\mathrm{hom}(Y, X)$*
   - *A linear mapping $\alpha : \mathcal{C}_0(X_0, V) \to \mathcal{C}_0(L)$*
   - *A linear isometry $G : V \to \mathcal{C}_0(L)$*

so that for every $p \in L$, $y \in Y$ and $f \in \mathcal{C}_0(X_0, V)$ we have

$$T(f)(y) = \phi(f)(y) + \alpha(f)$$

and

$$\phi(f)(y)(p) = G(f(t_p(y)))(p)$$

where $t_p = t(p)$. In addition, $\mathrm{ch}(G(V)) = L$.

2. *If $T : \mathcal{C}_0(X_0, V) \to \mathcal{C}_0(Y_0, \mathcal{C}_0(L))$ is a mapping expressed by ϕ, t, α and G as above then T is a diameter preserving linear injection. If ϕ is surjective then T is surjective. If G is surjective then ϕ is surjective.*

3. *In the situation of point (1.), if X and Y are infinite and $\mathcal{C}_0(L)^*$ has the Bade property then G is surjective.*

In the particular case when $V = \mathcal{C}_0(J)$ (indeed, this works also with V linearly isometric to $\mathcal{C}_0(J)$), the mapping F yields a homeomorphism $s : L \to J$ given by $s(p) = q$ if and only if $F(\delta_p) = \delta_q$. Then $\delta_p(G(v)) = G(v)(p) = F(\delta_p)(v) = \delta_q(v)$, therefore $\delta_p \circ G = \delta_{s(p)}$ and from this it is easy to deduce that G is surjective and consequently ϕ is also surjective.

One could wonder whether, as tends to happen in Banach-Stone-like theorems, the family of homeomorphisms $(t_p)_p$ is indeed a unique homeomorphism. Next we give an example showing that this does not always happen.

EXAMPLE 2.3

Consider $X = Y = \gamma\mathbb{N}$, $V = \mathcal{C}_0(L) = (\mathbb{R}^2, \| \|_\infty)$ and the homeomorphisms $t_1, t_2 : Y \to X$ defined by $t_1(n) = \{n+1$ if n is odd $, n-1$ if n is even$\}$, $t_2(n) = n$. Everybody knows that necessarily $t_1(\infty) = t_2(\infty) = \infty$.

Define $T : \mathcal{C}(X, V) \to \mathcal{C}(Y, \mathcal{C}_0(L))$ by $Tf(y) = ((Tf)_1(y), (Tf)_2(y)) = (f_1(t_1(y)), f_2(t_2(y)))$, then T is easily seen to be a linear bijection and $\rho(Tf) = \max\{\rho((Tf)_1), \rho((Tf)_2)\} = \max\{\rho(f_1), \rho(f_2)\} = \rho(f)$. We have proved this only because of its simplicity, since theorem 2.1 allows us to say directly that T is a diameter preserving linear bijection.

Now suppose that there exist a linear isometry $G : (\mathbb{R}^2, \| \|_\infty) \to (\mathbb{R}^2, \| \|_\infty)$, a linear mapping $\alpha : \mathcal{C}(X, \mathbb{R}^2) \to \mathbb{R}^2$ and a homeomorphism $t : Y \to X$, so that $Tf(y) = G(f(t(y))) + \alpha(f)$ for every $f \in \mathcal{C}(X, V)$ and $y \in Y$. Then we have two expressions for $Tf(1) - Tf(3)$ that must be equal, so we obtain $G(f(t(1)) - f(t(3))) = (f_1(2) - f_1(4), f_2(1) - f_2(3))$ which is a clear contradiction since f is arbitrary. •

The previous example could be easily generalized to construct a diameter preserving linear bijection $T : \mathcal{C}(\gamma\mathbb{N}, c_0) \to \mathcal{C}(\gamma\mathbb{N}, c_0)$ that needs an infinite number of homeomorphisms $t_i : \gamma\mathbb{N} \to \gamma\mathbb{N}$ to be expressed. Note that both of these examples contradict theorem 2 in the article [11], since that result affirms that if E, F are Banach spaces such that the linear span of every extreme point of the dual unit ball is an L-summand (which is the case of $\mathcal{C}_0(L)$ spaces) then every dplb $T : \mathcal{C}(X, E) \to \mathcal{C}(X, F)$ can be expressed as $Tf(x) = G(f(t(x))) + \alpha(f)$, where X is a compact Hausdorff space, $t : X \to X$ is a homeomorphism, $G : E \to F$ is a surjective linear isometry and $\alpha : \mathcal{C}(X, E) \to F$ is a linear mapping.

3 Questions

Now we pose a few questions intended to sharpen theorems 2.1 and 2.2.

Question 3.1. *Can the hypothesis of the surjectivity of ϕ (used to deduce the surjectivity of T) be removed in point (2.) of: (a) theorem 2.1? (b) theorem 2.2? (thus obtaining that T is automatically surjective)*

Question 3.2. *Can the hypothesis of $\mathcal{C}_0(L)^*$ having the Bade property be removed or weakened in point (3.) of: (a) theorem 2.1? (b) theorem 2.2?*

Question 3.3. *Does every $\mathcal{C}_0(L)$ space, L being a locally compact Hausdorff space, solve the dplb problem?*

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