Remarks on the convergence of Bleimann, Butzer and Hahn type operators

Zbigniew Walczak

Abstract

In the present paper we introduce generalized Bleimann, Butzer and Hahn operators $F_n(f;p;x)$. The form of the operator discussed, makes results helpful from the computational point of view. Here we have studied the rate of convergence of $F_n(f;p;x)$. The current work extends the similar results of V. Gupta, H. M. Srivastava, A. Lupas, O. Dogru.

1 Introduction

In 1980 Bleimann, Butzer and Hahn ([5]) constructed for any real function f on the interval $R_0 := [0, +\infty]$ a sequence of positive linear operators $L_n^{[1]}$ defined by

$$(1)\quad L_n^{[1]}(f;x) = (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n-k+1}\right), \qquad n \in N := \{1, 2, \cdots\}.$$

These operators, called Bleimann, Butzer and Hahn operators, possess many remarkable properties. We present only several of them. Bleimann, Butzer and Hahn proved that, for bounded and continuous f on R_0,

$$(2)\qquad \lim_{n\to\infty} L_n^{[1]}(f;x) = f(x),$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

Received by the editors April 2007.
Communicated by J. Mawhin.
1991 *Mathematics Subject Classification* : 41A36.
Key words and phrases : Bleimann, Butzer and Hahn operators, rate of convergence.

Totik [15] derived the Voronovskaja type result

$$\lim_{n\to\infty} n(L_n^{[1]}(f;x) - f(x)) = \frac{x(1+x)^2}{2} f''(x) \tag{3}$$

for all $f \in C^2(R_0)$ with $f(x) = O(x)$.

Abel [1] extended this result by giving the complete asymptotic expansion for the Bleimann, Butzer and Hahn operators. For every function f on R_0 satisfying $f(x) = O(x)$ and possessing all derivatives in x Abel exhibited the following formula

$$L_n^{[1]}(f;x) = f(x) + \sum_{k=1}^{m} c_k(f;x)(n+1)^{-k} + o(n^{-m}), \quad m \in N,\ n \to \infty. \tag{4}$$

Approximation of continuous functions of two variables by Bleimann, Butzer and Hahn operators defined by

$$L_n^{[2]}(f(t,s);x,y) = \sum_{k=0}^{n} \sum_{i=0}^{n-k} f\left(\frac{k}{n-k+1}, \frac{i}{n-i+1}\right) \tag{5}$$

$$\times \frac{n!}{k!i!(n-k-i)!} \left(\frac{x}{1+x}\right)^k \left(\frac{y}{1+y}\right)^i \left(\frac{1-xy}{(1+x)(1+y)}\right)^{n-k-i}, \qquad n \in N,$$

have been investigated by several authors. A careful analysis of such operators, was carried out by Abel in [2].

In view of [1-2] it is known that:

A.

$$L_n^{[1]}(1;x) = 1, \tag{6}$$

B. $L_n^{[2]}(f;x,0) = L_n^{[1]}(g;x)$ for $g(x) = f(x,0)$,

C.

$$L_n^{[2]}\left((t-x)^{2p}(s-y)^{2q};x,y\right) = O(n^{-r}), \tag{7}$$

$$p+q = r,\ p,q \in R_0,\ r \in N_0 := \{0,1,2,\dots\}, \quad n \to \infty.$$

The Bleimann, Butzer and Hahn operators and their connections with different branches of analysis, such as convex and numerical analysis have been studied intensively. Basic facts on the Bleimann, Butzer and Hahn operators can be found above. Moreover, we refer the readers to U. Abel, M. Ivan [3-4], J. De La Cal, V. Gupta [6], O. Dogru, V. Gupta [7], C. Jayasri and Y. Sitaraman [10], R. A. Khan [11-12], H. M. Srivastava, V. Gupta [14]. Their results improve other related results in the literature.

In the paper [18] it was examined similar operators in polynomial weighted spaces.

In this paper we propose a new family of linear operators. The form of the operator makes results, given in the present paper, more helpful from the computational point of view. Thus the operators (9), may play an important role in the applications to actual approximation schemes.

Denote by D the space of all real-valued functions f, uniformly continuous and bounded on R_0 with the norm

$$\|f(\cdot)\| := \sup_{x \in R_0} |f(x)|. \tag{8}$$

Let D_p, $p \in N_0$, be the set of all $f \in D$ with derivatives $f^{(k)}$, $k = 1, 2 \cdots, p$ belonging also to D with the norm (8) ($D_0 \equiv D$).

We introduce the following class of operators in D_p, $p \in N_0$.

DEFINITION. *We define the class of operators F_n by the formula*

$$F_n(f;p;x) := (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k}{n-k+1}\right)}{j!} \left(x - \frac{k}{n-k+1}\right)^j, \tag{9}$$

$x \in R_0$, $n \in N$, $p \in N_0$.

Obviously for $p = 0$, the operators (9) reduce to the well-known Bleimann, Butzer and Hahn operators $L_n^{[1]}$.

In this paper we shall study a relation between the rate of approximation by $F_n(f)$ and the smoothness of the functions f. We shall show that the operators F_n give better rate of convergence than some other known operators.

2 Main results

In this section we study the properties of F_n. We give theorems on the rate of approximation of $f \in D_p$, $p \in N$, by these operators.

We apply the method used in [8, 16, 17].

We may observe here that if $f(x) = x^q$, $x \in R_0$, $q \in N_0$, then by Taylor's formula it follows that

$$f(x) = \sum_{j=0}^{q} \frac{f^{(j)}(y)}{j!} (x-y)^j, \qquad y \in R_0.$$

This fact and (9) yield

LEMMA. *Let $f(x) = x^q$, $x \in R_0$, $q \in N_0$. Then for every fixed $q \leq p \in N$ we have*

$$F_n\left(t^q;p;x\right) = x^q, \qquad n \in N.$$

THEOREM 1. *Fix $p \in N$. Then for every $f \in D_p$ we have*

$$F_n(f;p;x) - f(x) = o(n^{-p/2}), \qquad x \in R_0,\ n \to \infty. \tag{10}$$

Proof. We first suppose that $f \in D_p$. Using the modified Taylor formula we get

$$f(x) = \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k}{n-k+1}\right)}{j!}\left(x - \frac{k}{n-k+1}\right)^j$$
$$+\frac{\left(x-\frac{k}{n-k+1}\right)^p}{(p-1)!}\int_0^1 (1-t)^{p-1}\left\{f^{(p)}\left(\frac{k}{n-k+1}+t\left(x-\frac{k}{n-k+1}\right)\right)\right.$$
$$\left.-f^{(p)}\left(\frac{k}{n-k+1}\right)\right\}dt.$$

This implies that

$$|F_n(f;p;x) - f(x)|$$
$$\le (1+x)^{-n}\sum_{k=0}^{n}\binom{n}{k}x^k\left|\sum_{j=0}^{p}\frac{f^{(j)}\left(\frac{k}{n-k+1}\right)}{j!}\left(x-\frac{k}{n-k+1}\right)^j - f(x)\right|$$
$$\le (1+x)^{-n}\sum_{k=0}^{n}\binom{n}{k}x^k\frac{\left|x-\frac{k}{n-k+1}\right|^p}{(p-1)!}$$
$$\times\int_0^1(1-t)^{p-1}\left|f^{(p)}\left(\frac{k}{n-k+1}+t\left(x-\frac{k}{n-k+1}\right)\right)-f^{(p)}\left(\frac{k}{n-k+1}\right)\right|dt.$$

Observe that

$$\left|f^{(p)}\left(\frac{k}{n-k+1}+t\left(x-\frac{k}{n-k+1}\right)\right)-f^{(p)}\left(\frac{k}{n-k+1}\right)\right|$$
$$\le \omega\left(f^{(p)};t\left|x-\frac{k}{n-k+1}\right|\right)$$
$$\le\left(1+t\left|x-\frac{k}{n-k+1}\right|n^{1/2}\right)\omega\left(f^{(p)};n^{-1/2}\right),$$

where $\omega(f;\cdot)$ is the modulus of continuity of function $f \in D$. Using the above inequality and (1), we obtain

$$|F_n(f;p;x)-f(x)| \le (1+x)^{-n}\sum_{k=0}^{n}\binom{n}{k}x^k$$
$$\times\left(\left|x-\frac{k}{n-k+1}\right|^p + n^{1/2}\left|x-\frac{k}{n-k+1}\right|^{p+1}\right)\omega\left(f^{(p)};n^{-1/2}\right)$$
$$\le\left(L_n^{[1]}\left(|x-t|^p;x\right)+n^{1/2}L_n^{[1]}\left(|x-t|^{p+1};x\right)\right)\omega\left(f^{(p)};n^{-1/2}\right).$$

Applying now the results $A - C$, we can write

$$L_n^{[1]}\left(|x-t|^p;x\right) \le \left(L_n^{[1]}\left((x-t)^{2p};x\right)L_n^{[1]}(1;x)\right)^{1/2} = \left(L_n^{[2]}\left((x-t)^{2p};x,0\right)\right)^{1/2} = O(n^{-p/2}), \quad n\to\infty.$$

Analogously we obtain

$$L_n^{[1]}\left(|x-t|^{p+1};x\right) = O(n^{-(p+1)/2}), \quad n\to\infty. \tag{11}$$

Combining these, we derive

$$|F_n(f;p;x) - f(x)| = o(n^{-p/2}), \quad n\to\infty.$$

This ends the proof of (10).

For $F_n(f;0;x) = L_n^{[1]}(f;x)$ the assertion (10) is well known.

For these operators we can prove the Voronovskaja type theorem.

THEOREM 2. *Fix $p \in N$. Then for every $f \in D_{p+2}$ we have*

$$F_n(f;p;x) - f(x) = \frac{(-1)^p f^{(p+1)}(x) L_n^{[1]}\left((t-x)^{p+1};x\right)}{(p+1)!} + \frac{(-1)^p (p+1) f^{(p+2)}(x) L_n^{[1]}\left((t-x)^{p+2};x\right)}{(p+2)!} + o(n^{-1-p/2}), \qquad n\to\infty. \tag{12}$$

Proof. Fix $p \in N$ and $x \in R_0$. If $f \in D_{p+2}$ then $f^{(j)} \in D_{p+2-j}$, $0 \le j \le p$. Hence, for every $f^{(j)}$ we can write Taylor's formula

$$f^{(j)}(t) = \sum_{i=0}^{p+2-j} \frac{f^{(j+i)}(x)}{i!}(t-x)^i + \varphi_j(t;x)(t-x)^{p+2-j}, \quad 0 \le j \le p,$$

for $t \in R_0$, where $\varphi_j(t) \equiv \varphi_j(t;x)$ is a function such that $\varphi_j(t)t^{p+2-j}$ belongs to D_{p+2-j} and $\lim_{t\to x}\varphi_j(t) = 0$. From this we get

$$F_n(f;p;x) = (1+x)^{-n}\sum_{k=0}^{n}\binom{n}{k}x^k\sum_{j=0}^{p}\frac{\left(x-\frac{k}{n-k+1}\right)^j}{j!} \times \sum_{i=0}^{p+2-j}\frac{f^{(j+i)}(x)}{i!}\left(\frac{k}{n-k+1}-x\right)^i + (1+x)^{-n}\sum_{k=0}^{n}\binom{n}{k}x^k\sum_{j=0}^{p}\frac{\left(x-\frac{k}{n-k+1}\right)^j}{j!} \times \varphi_j\left(\frac{k}{n-k+1}\right)\left(\frac{k}{n-k+1}-x\right)^{p+2-j} := T_1 + T_2. \tag{13}$$

Observe that

$$T_1 = (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{j=0}^{p} \frac{\left(x - \frac{k}{n-k+1}\right)^j}{j!} \sum_{l=j}^{p+2} \frac{f^{(l)}(x)}{(l-j)!} \left(\frac{k}{n-k+1} - x\right)^{l-j}$$

$$= (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{j=0}^{p} \frac{(-1)^j}{j!} \sum_{l=j}^{p+2} \frac{f^{(l)}(x)}{(l-j)!} \left(\frac{k}{n-k+1} - x\right)^{l}$$

$$= (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{j=0}^{p} \frac{(-1)^j}{j!} \left(\sum_{l=j}^{p} \frac{f^{(l)}(x)}{(l-j)!} \left(\frac{k}{n-k+1} - x\right)^{l} \right.$$

$$\left. + \frac{f^{(p+1)}(x)}{(p+1-j)!} \left(\frac{k}{n-k+1} - x\right)^{p+1} + \frac{f^{(p+2)}(x)}{(p+2-j)!} \left(\frac{k}{n-k+1} - x\right)^{p+2} \right)$$

$$= (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{l=0}^{p} \frac{f^{(l)}(x)}{l!} \left(\frac{k}{n-k+1} - x\right)^{l} \sum_{j=0}^{l} \binom{l}{j} (-1)^j$$

$$+ \frac{f^{(p+1)}(x)}{(p+1)!} (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \left(\frac{k}{n-k+1} - x\right)^{p+1} \sum_{j=0}^{p} \binom{p+1}{j} (-1)^j$$

$$+ \frac{f^{(p+2)}(x)}{(p+2)!} (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \left(\frac{k}{n-k+1} - x\right)^{p+2} \sum_{j=0}^{p} \binom{p+2}{j} (-1)^j .$$

From this and by elementary calculations for $p \in N$

$$\sum_{j=0}^{p} \binom{p}{j} (-1)^j = 0, \qquad p \in N,$$

$$\sum_{j=0}^{p} \binom{p+1}{j} (-1)^j = (-1)^p,$$

$$\sum_{j=0}^{p} \binom{p+2}{j} (-1)^j = (p+1)(-1)^p,$$

and (1) we get

$$T_1 = f(x) + (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{l=1}^{p} \frac{f^{(l)}(x)}{l!} \left(\frac{k}{n-k+1} - x\right)^{l} \sum_{j=0}^{l} \binom{l}{j} (-1)^j$$

$$+ \frac{(-1)^p f^{(p+1)}(x) L_n^{[1]}\left((t-x)^{p+1}; x\right)}{(p+1)!} + \frac{(-1)^p (p+1) f^{(p+2)}(x) L_n^{[1]}\left((t-x)^{p+2}; x\right)}{(p+2)!}$$

$$= f(x) + \frac{(-1)^p f^{(p+1)}(x) L_n^{[1]}\left((t-x)^{p+1}; x\right)}{(p+1)!}$$

$$+ \frac{(-1)^p (p+1) f^{(p+2)}(x) L_n^{[1]}\left((t-x)^{p+2}; x\right)}{(p+2)!}.$$

Observe that

$$T_2 = (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \sum_{j=0}^{p} \frac{\left(x - \frac{k}{n-k+1}\right)^j}{j!} \varphi_j \left(\frac{k}{n-k+1}\right) \left(\frac{k}{n-k+1} - x\right)^{p+2-j}$$
$$= (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k \left(\frac{k}{n-k+1} - x\right)^{p+2} \sum_{j=0}^{p} \frac{(-1)^j}{j!} \varphi_j \left(\frac{k}{n-k+1}\right)$$
$$= L_n^{[1]} \left((t-x)^{p+2} \Phi_p(t); x\right),$$

where $\Phi_p(t) \equiv \Phi_p(t;x) := \sum_{j=0}^{p} \frac{(-1)^j}{j!} \varphi_j(t)$ is a function belonging to D and $\lim_{t\to x} \Phi_p(t) = \Phi_p(x) = 0$. Arguing as in the second part of the proof of Theorem 1 and by (2), we obtain

$$|T_2| \le \left(L_n^{[1]}\left((t-x)^{2p+4}; x\right)\right)^{1/2} \left(L_n^{[1]}\left(\Phi_p^2(t); x\right)\right)^{1/2} = o(n^{-(1+p/2)}), \qquad n \to \infty.$$

This ends the proof of (12).

The assertion (12) for Bleimann, Butzer and Hahn operators $F_n(f;0;x)$ and $f \in D_2$ is given in (3).

Theorems proved in this paper show that operators F_n, give better the rate of convergence of functions $f \in D_p$ than $L_n^{[1]}$ and some other known operators.

References

[1] U. Abel, On the asymptotic approximation with operators of Bleimann, Butzer and Hahn, *Indag. Math.(N.S.)* **7** (1996), 1-9.

[2] U. Abel, On the asymptotic approximation with bivariate operators of Bleimann, Butzer and Hahn, *J. Approx. Th.* **97**(1999), 181-198.

[3] U. Abel, M. Ivan, Best constant for a Bleimann-Butzer-Hahn moment estimation, *East J. Approx.* **6**, No.3, 349-355 (2000).

[4] U. Abel, M. Ivan, Some identities for the operator of Bleimann, Butzer and Hahn involving divided differences, *Calcolo* **36**, No.3, 143-160 (1999).

[5] G. Bleimann, P. L. Butzer, and L. Hahn, A Bernstein-type operator approximating continuous functions on the semi-axis, *Indag. Math.* **42**(1980), 255-262.

[6] J. De La Cal, V. Gupta, On the approximation of locally bounded functions by operators of Bleimann, Butzer and Hahn, *J. Inequal. Pure Appl. Math.* **6**, No.1, Paper No.4, 10 p., electronic only (2005).

[7] O. Dogru, V. Gupta, Monotonicity and the asymptotic estimate of Bleimann Butzer and Hahn operators based on q-integers, *Georgian Math. J.* **12**, No.3, 415-422 (2005).

[8] Z. Finta, On approximation by modified Kantorovich polynomials, *Math. Balcanica (New Ser.)* **13**(1999)3-4, 205-211.

[9] Vijay Gupta and Alexandru Lupas, Rate of Approximation for the Bezier variant of Bleimann-Butzer and Hahn operators, *General Mathematics* **13** (1) (2005), 41-54.

[10] C. Jayasri and Y. Sitaraman, On a Bernstein-type operator of Bleimann, Butzer and Hahn, *J. Comput. Appl. Math.* **47**(1993), 267-272.

[11] R. A. Khan, A note on a Bernstein-type operator of Bleimann, Butzer and Hahn, *J. Approx. Theory* **53** (1988), 295-303.

[12] R. A. Khan, Reverse martingales and approximation operators, *J. Approx. Theory* **80** (1995), 367-377.

[13] L. Rempulska, Z. Walczak, Modified Szasz-Mirakyan operators, *Mat. Balcanica (New Ser.)* **18** (1-2) (2004), 53-63.

[14] H. M. Srivastava, Vijay Gupta, Rate of convergence for Bezier variant of the Bleimann-Butzer-Hahn operators, *Appl. Math. Lett.* **18** (8) (2005), 849-857.

[15] V. Totik, Uniform approximation by Bernstein-type operators, *Indag. Math.* **46** (1984), 87-93.

[16] Z. Walczak, On the rate of convergence for modified Baskakov operators, *Liet. matem. rink.* **44**(1)(2004), 124-130

[17] Z. Walczak, On the rate of convergence for some linear operators, *Hiroshima Math. J.* **35**(1)(2005), 115-124.

[18] Z. Walczak, Approximation by some linear positive operators in polynomial weighted spaces, *Publ. Math. Debrecen* **64** (2004)(3-4), 353-367.

Institute of Mathematics,
Poznań University of Technology Piotrowo 3A
60-965 Poznań, Poland
email:rewal@icpnet.pl