On the Center Problem for $p : -q$ Resonant Polynomial Vector Fields

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Abstract

We define the center variety for families of $p : -q$ resonant polynomial vector fields and prove the correctness of the definition. We also derive an algorithm for computing the focus quantities of such vector fields.

1 Introduction

Consider a real analytic system of ordinary differential equations on $\mathbb{R}^2$ with an isolated equilibrium at the origin, at which the eigenvalues of the linear part are non-zero pure imaginary numbers. By an analytic change of coordinates the system has the form $\dot{x} = \omega y + \cdots$, $\dot{y} = -\omega x + \cdots$. The classical Poincaré-Lyapunov Center Theorem states that the origin is a center if and only if the system admits an analytic first integral of the form $\Phi(x, y) = x^2+y^2+\cdots$. If the system is complexified in a natural way, there arises an analytic system of ordinary differential equations on $\mathbb{C}^2$ of the form $\dot{z} = i\omega z + \cdots$, $\dot{w} = -i\omega w + \cdots$. Existence of the integral Φ is equivalent to existence of an analytic first integral of the complexification of the form $\Psi(z, w) = zw + \cdots$. It is known that, on both $\mathbb{R}^2$ and $\mathbb{C}^2$, and in the slightly more general setting of an isolated equilibrium at the origin at which the eigenvalues λ_1, λ_2 of the linear part are non-zero, all nonlinear terms in the system are non-resonant, hence all through any specified order can be eliminated by a single analytic coordinate change (and *all* of them by a formal coordinate change), unless the eigenvalues λ_1 and λ_2 are rationally related ($\lambda_1/\lambda_2 \in \mathbb{Q}$). These facts motivate the generalization as in Definition 1 below of the concept of a (real) center to certain classes of systems of ordinary differential equations on $\mathbb{C}^2$. In this paper we study

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one such class, the $p:-q$ resonant polynomial vector fields, namely, systems on $\mathbb{C}^2$ of the form

$$\begin{aligned} \dot{x} &= \quad px - \sum_{(i,j)\in S} a_{ij}x^{i+1}y^j = P(x,y), \\ \dot{y} &= -qy + \sum_{(i,j)\in S} b_{ji}x^j y^{i+1} = Q(x,y), \end{aligned} \tag{1}$$

where $p,q\in\mathbb{N}$, $GCD(p,q)=1$, and where S is the set

$$S=\{(u_k,v_k): u_k+v_k\geq 1, k=1,\dots,\ell\}\subset \mathbb{N}_{-1}\times\mathbb{N}_0.$$

Here $\mathbb{N}$ denotes the set of natural numbers and for any nonnegative integer n, $\mathbb{N}_{-n}=\{-n,\dots,-1,0\}\cup\mathbb{N}$. The notation (1) simply emphasizes that we take into account only non-zero coefficients of the polynomials of interest, and will simplify formulas that occur later. We denote by $(a,b)=(a_{u_1,v_1},a_{u_2,v_2},\dots,b_{u_1,v_1})$ the ordered vector of the coefficients of system (1), by $E(a,b)=\mathbb{C}^{2\ell}$ the parameter space of (1), and by $\mathbb{C}[a,b]$ the polynomial ring in the variables a_{ij}, b_{ji}. We will study the problem of how to determine all systems within the family (1) that have a center at the origin in the sense of Definition 1 below, which means a determination of all such systems that have an analytic first integral in a neighborhood of the origin.

Note that if we are presented with a system of the form (1) for which p and q are not relatively prime, then we can always rescale the system by a suitably chosen positive real number so that for the new system the condition $GCD(p,q)=1$ holds (even if originally $p=q>1$).

It is easy to see from condition (3) below that if system (1) has a first integral represented by the formal power series

$$\Psi(x,y)=\sum_{\substack{i+j>0\\ i,j\in\mathbb{N}_0}} u_{i,j}x^iy^j$$

that begins with terms of order at most $p+q$, then up to rescaling by a non-zero constant Ψ must be of the form

$$\Psi(x,y)=x^qy^p+\sum_{\substack{i+j>p+q\\ i,j\in\mathbb{N}_0}} v_{i-q,j-p}x^iy^j, \tag{2}$$

where the indexing has been chosen so as to simplify formulas that we will obtain below.

Definition 1. A system of the form (1) is said to have a *center* at the origin if it admits a formal first integral of the form (2).

The condition that a function $\Psi(x,y)$ be a first integral of (1) is the identity

$$D(\Psi)\overset{\text{def}}{=}\frac{\partial\Psi}{\partial x}P(x,y)+\frac{\partial\Psi}{\partial y}Q(x,y)\equiv 0\,, \tag{3}$$

which for functions of the form (2) is

$$\begin{aligned} &\Bigg(qx^{q-1}y^p+\sum_{i+j>p+q} iv_{i-q,j-p}x^{i-1}y^j\Bigg)\Bigg(px-\sum_{(m,n)\in S}a_{mn}x^{m+1}y^n\Bigg) \\ &\quad+\Bigg(px^qy^{p-1}+\sum_{i+j>p+q} jv_{i-q,j-p}x^iy^{j-1}\Bigg)\Bigg(-qy+\sum_{(m,n)\in S}b_{nm}x^ny^{m+1}\Bigg)\equiv 0\,. \end{aligned} \tag{4}$$

We augment the set of coefficients in (2) with the collection

$$J = \{v_{-q+s,q-s} : s = 0, \dots, p+q\}, \tag{5}$$

where in agreement with formula (2) we set $v_{00} = 1$ and $v_{mn} = 0$ for all other elements of J, so that elements of J are the coefficients of the terms of degree $p+q$ in $\Psi(x, y)$. We also set $a_{mn} = b_{nm} = 0$ for $(m, n) \notin S$. With these conventions, for $(k_1, k_2) \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}$, the coefficient g_{k_1,k_2} of $x^{k_1+q}y^{k_2+p}$ in (4) is zero for $k_1 + k_2 \leq 0$ and for $k_1 + k_2 \geq 1$ is

$$g_{k_1,k_2} = (pk_1 - qk_2)v_{k_1,k_2} - \sum_{\substack{s_1+s_2=0 \\ s_1 \geq -q,\ s_2 \geq -p}}^{k_1+k_2-1} [(s_1+q)a_{k_1-s_1,k_2-s_2} - (s_2+p)b_{k_1-s_1,k_2-s_2}]\, v_{s_1,s_2}. \tag{6}$$

This formula can be used recursively to construct a formal first integral Ψ for system (1), at the first stage finding all v_{k_1,k_2} for which $k_1 + k_2 = 1$, at the second all v_{k_1,k_2} for which $k_1 + k_2 = 2$, and so on. For any pair k_1 and k_2, if

$$qk_1 \neq pk_2\,, \tag{7}$$

and if all coefficients v_{ℓ_1,ℓ_2} are already known for $\ell_1 + \ell_2 < k_1 + k_2$, then v_{k_1,k_2} is uniquely determined by (6) and the condition that g_{k_1,k_2} be zero. But at each of the stages $k_1 + k_2 = k(p+q)$, $k \in \mathbb{N}$ (but only at these stages, since $GCD(p, q) = 1$) there occurs the one "resonance" pair $(k_1, k_2) = (kq, kp)$ for which (7) does not hold, hence for which (6) becomes

$$g_{kq,kp} = - \sum_{\substack{s_1+s_2=0 \\ s_1 \geq -q, s_2 \geq -p}}^{kq+kp-1} [(s_1+q)a_{k_1-s_1,k_2-s_2} - (s_2+p)b_{k_1-s_1,k_2-s_2}]\, v_{s_1,s_2}\,, \tag{8}$$

so that the process of constructing a first integral Ψ succeeds at this step only if the expression on the right hand side of (8) is zero. In this case the value of $v_{k_1,k_2} = v_{kq,kp}$ is not determined by equation (6) and may be assigned arbitrarily.

It is evident from (6) that for all indices $(k_1, k_2) \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}$, v_{k_1,k_2} is a polynomial function of the coefficients of (1), that is, is an element of the set that we have denoted $\mathbb{C}[a, b]$, hence by (8) so are the expressions $g_{kq,kp}$ for all k. We would like to regard the polynomial $g_{kq,kp}$ as the kth "obstruction" to the existence of the integral (2). It is certainly true that if at a point (a^*, b^*) of our parameter space $E(a, b)$, $g_{kq,kp}(a^*, b^*) \neq 0$, then the construction process fails at that step. However, although $g_{q,p}$ is uniquely determined, for $k > 1$ $g_{kq,kp}$ is not, since for $\ell < k$ $v_{\ell q,\ell p}$ was arbitrary. Thus although it is true that the vanishing of $g_{kq,kp}(a^*, b^*)$ for all $k \in \mathbb{N}$ is sufficient for the existence of a formal first integral of the form (2), it is not clear *a priori* that it is necessary. As our first main result we will prove in the next section that the condition is indeed necessary, independently of the choices of the $v_{\ell q,\ell p}$. Consequently, the variety in $E(a, b) = \mathbb{C}^{2\ell}$ determined by the collection $\{g_{kq,kp} : k \in \mathbb{N}\}$ is the same for all choices of these polynomials, so that the variety identified in the following definition is well-defined.

Definition 2. Fix a set S. The polynomial $g_{kq,kp}$ is called the *kth focus quantity* of the family (1). The ideal of focus quantities, $\mathcal{B} = \langle g_{q,p}, g_{2q,2p}, \dots, g_{jq,jp}, \dots \rangle \subset \mathbb{C}[a, b]$, is called the *Bautin ideal*. The variety of the Bautin ideal, $V_{\mathcal{C}} = \mathbf{V}(\langle g_{q,p}, g_{2q,2p}, \ldots, g_{jq,jp}, \ldots\rangle) = \{(a,b) : g_{jq,jp}(a,b) = 0 \text{ for all } j \in \mathbb{N}\}$ is called the *center variety* of the family (1).

The center variety therefore corresponds exactly to those systems of the form (1) for which there is a center at the origin of $\mathbb{C}^2$, in the sense of Definition 1. In Section 3 we will present an efficient algorithm for computing the focus quantities. A result along the same lines for $1 : -1$ resonance was announced in [15], but the proof is given here for the first time.

In the case that (1) is a system on $\mathbb{R}^2$ rather than $\mathbb{C}^2$, the polynomials $g_{jq,jp}$ are still defined, but in that context are called the *saddle quantities* of the system. The algorithm presented in the appendix applies equally well in this situation as an efficient method for computing them.

We remark finally that even though it is not generally true that an integral of the form (2) exists, the construction process described above always yields a formal series of the form (2) for which $D(\Psi) = \Psi_x P + \Psi_y Q$ reduces to

$$D(\Psi) = g_{q,p}(x^q y^p)^2 + g_{2q,2p}(x^q y^p)^3 + g_{3q,3p}(x^q y^p)^4 + \cdots . \tag{9}$$

2 Normal forms and the center variety

It is well known (see e.g. [3] for details) that by means of a change of variables of the form

$$\begin{aligned} x &= x_1 + \sum_{k_1+k_2>1} h_1^{(k_1,k_2)} x_1^{k_1} y_1^{k_2} = x_1 + h_1(x_1,y_1) \\ y &= y_1 + \sum_{k_1+k_2>1} h_2^{(k_1,k_2)} x_1^{k_1} y_1^{k_2} = y_1 + h_2(x_1,y_1) \end{aligned} \tag{10}$$

system (1) can be transformed into a system of the form

$$\begin{aligned} \dot{x}_1 &= px_1 + x_1 \sum_{j=1}^{\infty} X^{(jq+1,jp)} (x_1^q y_1^p)^j = px_1 + x_1 X(x_1^q y_1^p) \\ \dot{y}_1 &= -qy_1 + y_1 \sum_{j=1}^{\infty} Y^{(jq,jp+1)} (x_1^q y_1^p)^j = -qy_1 + y_1 Y(x_1^q y_1^p), \end{aligned} \tag{11}$$

a *normal form* of (1). The normal form (11) of system (1) is not unique, since the so-called resonant coefficients $h_1^{(kq+1,kp)}, h_2^{(kq,kp+1)}$, $k \in \mathbb{N}$, may be selected arbitrarily. A change of variables (10) transforming (1) to a normal form is called *distinguished* if all the resonant coefficients $h_1^{(kq+1,kp)}, h_2^{(kq,kp+1)}$, $k \in \mathbb{N}$, are zero, and is uniquely defined. If the transformation (10) is distinguished then the normal form (11) is also termed *distinguished*, and is also uniquely defined. The following theorem is a particular case of Theorem 3.2 in [3, p.18].

Theorem 1. *Let $\kappa = (p, -q)$, $e_1 = (1,0)$, and $e_2 = (0,1)$. Suppose there exists a positive constant d such that for all α and β in $\mathbb{N}_0^2$ for which $\beta \le \alpha + e_m$, $m \in \{1,2\}$, the following inequality holds:*

$$\left|\beta_1 X^{(\alpha-\beta+e_1)} + \beta_2 Y^{(\alpha-\beta+e_2)}\right| \le d|(\beta,\kappa)| \left(\left|X^{(\alpha-\beta+e_1)}\right| + \left|Y^{(\alpha-\beta+e_2)}\right|\right) . \tag{12}$$

Then the distinguished normalizing transformation (10) *is analytic, that is, each of the functions $h_1(x_1, x_1)$ and $h_2(x_1, y_1)$ is given by a convergent power series, so that system* (1) *is analytically equivalent to its normal form* (11).

An immediate consequence of Theorem 1 is the following more convenient criterion for the convergence of the normal form (11) of system (1).

Proposition 1. *Consider a system of the form* (1), *where $p, q \in \mathbb{N}$, $GCD(p, q) = 1$. Let* (10) *be the distinguished normalizing transformation, producing the unique distinguished normal form* (11). *Let $w = x_1^q y_1^p$. If $qX(w) + pY(w) \equiv 0$ then the distinguished normalizing transformation* (10) *is convergent.*

We let G be the function defined by

$$G = qX + pY. \tag{13}$$

We now show that the existence of a formal first integral is enough to guarantee that $G(w) = qX(w) + pY(w) \equiv 0$, hence that the normalizing transformation that transforms (1) to (11) is convergent. Theorems of this sort, an analogue of the Poincaré-Lyapunov theorem, are known, but the proof given here, along lines similar to [3], differs from others in the literature (see [4] and [14]).

Theorem 2. *Consider a system of the form* (1), *where $p, q \in \mathbb{N}$, $GCD(p, q) = 1$. Let G be the function computed from the normal form* (11) *of* (1), *defined by* (13).

1. *If system* (1) *has a formal integral of the form* (2), *then $G \equiv 0$, hence the normalizing transformation is convergent.*
2. *Conversely, if $G \equiv 0$ then* (1) *has an analytic first integral of the form* (2).

Proof. Suppose (1) has a formal integral of the form (2), that is, $\Psi(x, y) = x^q y^p + \sum_{i+j>p+q} v_{i-q,j-p} x^i y^j$. If $\mathbf{H}$ is the normalizing transformation (10) that converts (1) into its normal form (11), then $F = \Psi \circ \mathbf{H}$ is a formal integral for the normal form, hence

$$\frac{\partial F}{\partial x_1}(x_1, y_1)\Big[px_1 + x_1 X(x_1^q y_1^p)\Big] + \frac{\partial F}{\partial y_1}(x_1, y_1)\Big[-qy_1 + y_1 Y(x_1^q y_1^p)\Big] \equiv 0\,, \tag{14}$$

which we rearrange as

$$\begin{aligned} px_1 \frac{\partial F}{\partial x_1}&(x_1, y_1) - qy_1 \frac{\partial F}{\partial y_1}(x_1, y_1) \\ &= -x_1 \frac{\partial F}{\partial x_1}(x_1, y_1) X(x_1^q y_1^p) - y_1 \frac{\partial F}{\partial y_1}(x_1, y_1) Y(x_1^q y_1^p)\,. \end{aligned} \tag{15}$$

Recalling from (10) the form of the normalizing transformation, we see that $F(x_1, y_1)$ has the form

$$F(x_1, y_1) = x_1^q y_1^p + \sum_{\alpha_1+\alpha_2>p+q} F^{(\alpha_1,\alpha_2)} x_1^{\alpha_1} y_1^{\alpha_2}\,. \tag{16}$$

A simple computation on the left hand side of (15), and insertion of (11) into the right, yields

$$\begin{aligned}\sum_{\alpha_1+\alpha_2>p+q}&(\alpha_1 p-\alpha_2 q)F^{(\alpha_1,\alpha_2)}x_1^{\alpha_1}y_1^{\alpha_2}\\ &=-\left[qx_1^q y_1^p+\sum_{\alpha_1+\alpha_2>p+q}\alpha_1 F^{(\alpha_1,\alpha_2)}x_1^{\alpha_1}y_1^{\alpha_2}\right]\left[\sum_{j=1}^{\infty}X^{(jq+1,jp)}(x_1^q y_1^p)^j\right]\\ &\quad-\left[px_1^q y_1^p+\sum_{\alpha_1+\alpha_2>p+q}\alpha_2 F^{(\alpha_1,\alpha_2)}x_1^{\alpha_1}y_1^{\alpha_2}\right]\left[\sum_{j=1}^{\infty}Y^{(jq,jp+1)}(x_1^q y_1^p)^j\right].\end{aligned}\tag{17}$$

We claim that $F(x_1,y_1)$ is a function of $x_1^q y_1^p$, alone, that is, that

$$F(x_1,y_1)=f(x_1^q y_1^p)=x_1^q y_1^p+f_2\,(x_1^q y_1^p)^2+f_3\,(x_1^q y_1^p)^3+\cdots.\tag{18}$$

The claim is precisely the statement that for any term $F^{(\alpha_1,\alpha_2)}x_1^{\alpha_1}y_1^{\alpha_2}$ of F,

$$p\alpha_1-q\alpha_2\neq 0\quad\Rightarrow\quad F^{(\alpha_1,\alpha_2)}=0\,.\tag{19}$$

Equation (16) shows that (19) holds for $|(\alpha_1,\alpha_2)|=\alpha_1+\alpha_2\le p+q$. This implies that the right hand side of (17) has the form $c_2(x_1^q y_1^p)^2+\cdots$ for some c_2, hence by (17) implication (19) holds for $\alpha_1+\alpha_2\le 2(p+q)$. But if that is the case, then the right hand side of (17) must have the form $c_2(x_1^q y_1^p)^2+c_3(x_1^q y_1^p)^3+\cdots$ for some c_3, hence by (17) implication (19) must hold for $\alpha_1+\alpha_2\le 3(p+q)$. Clearly by mathematical induction (19) must hold in general, establishing the claim.

But if $F(x_1,y_1)=f(x_1^q y_1^p)$ then

$$x_1\frac{\partial F}{\partial x_1}(x_1,y_1)=qx_1^q y_1^p f'(x_1^q y_1^p)\quad\text{and}\quad y_1\frac{\partial F}{\partial y_1}(x_1,y_1)=px_1^q y_1^p f'(x_1^q y_1^p)$$

so that, letting $w=x_1^q y_1^p$, (15) becomes

$$0\equiv -qwf'(w)X(w)-pwf'(w)Y(w)\,.$$

But because F is of the form (18), we see that $wf'(w)=w+\cdots$. So we immediately obtain $qX(w)+pY(w)\equiv 0$, proving part (1).

Direct calculations show that if $G\equiv 0$ then $\widehat{\Psi}(x_1,y_1)=x_1^q y_1^p$ is a first integral of (11), and by Proposition 1 the transformation to the normal form (11) is convergent. Therefore system (1) admits an analytic first integral of the form (2), $\Psi(x,y)=x^q y^p+\cdots$. ∎

We now prove the correctness of the definition of the center variety, that is, we show that the variety $V_{\mathcal{C}}$ is the same for all choices of the polynomials $v_{jq,jp}$, $j\in\mathbb{N}$, that determine $g_{kq,kp}$, and thus that the center variety $V_{\mathcal{C}}$ is well-defined.

Theorem 3. *Consider a family of systems of the form* (1)*, with parameter space* $E(a,b)=\mathbb{C}^{2\ell}$*, where* $p,q\in\mathbb{N}$*,* $GCD(p,q)=1$*.*

1. *Let* Ψ *be a formal series of the form* (2) *and let* $g_{q,p}(a,b),g_{2q,2p}(a,b),\ldots$ *be polynomials satisfying* (9) *with respect to the system* (1)*. Then the system in*

family (1) *corresponding to the choice of coefficients* $(a^*, b^*) \in E(a,b)$ *has a center at the origin if and only if* $g_{kq,kp}(a^*, b^*) = 0$ *for all* $k \in \mathbb{N}$.

2. *Let* Ψ *and* $g_{qk,pk}$ *be as above and suppose there exists another function* Ψ' *of the form* (2) *and polynomials* $g'_{q,p}(a,b), g'_{2q,2p}(a,b), \ldots$ *satisfying* (9) *with respect to the family* (1). *Then* $V_{\mathcal{C}} = V'_{\mathcal{C}}$, *where* $V_{\mathcal{C}} = \mathbf{V}(\langle g_{q,p}(a,b), g_{2q,2p}(a,b), \ldots\rangle)$ *and* $V'_{\mathcal{C}} = \mathbf{V}(\langle g'_{q,p}(a,b), g'_{2q,2p}(a,b), \ldots\rangle)$.

Proof. 1) Suppose that family (1) is as in the statement of the theorem. Let Ψ be a formal series of the form (2) and let $\{g_{kq,kp}(a,b) : k \in \mathbb{N}\}$ be polynomials in (a,b) that satisfy (9).

If, for $(a^*, b^*) \in E(a,b)$, $g_{kq,kp}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$ then Ψ is a formal first integral for the corresponding family in (1), so by Definition 1 the system has a center at the origin of $\mathbb{C}^2$.

To prove the converse, we first make the following observations. Suppose that there exists a $k \in \mathbb{N}$ and a choice (a^*, b^*) of the parameters such that $g_{jq,jp}(a^*, b^*) = 0$ for $1 \le j \le k-1$ but $g_{kq,kp}(a^*, b^*) \ne 0$. Let $\mathbf{H}(x_1, y_1)$ be the distinguished normalizing transformation (10), producing the distinguished normal form (11), and consider the function $F = \Psi \circ \mathbf{H}$. By construction

$$\begin{aligned}
\Big[px_1 + x_1 X(x_1^q y_1^p)\Big]\frac{\partial F}{\partial x_1}(x_1, y_1) + \Big[-qy_1 + y_1 Y(x_1^q y_1^p)\Big]\frac{\partial F}{\partial y_1}(x_1, y_1) \\
= g_{kq,kp}(a^*, b^*)[x_1 + h_1(x_1,y_1)]^{kq}[y_1 + h_2(x_1,y_1)]^{kp} + \cdots \\
= g_{kq,kp}(a^*, b^*) x_1^{kq} y_1^{kp} + \cdots .
\end{aligned} \tag{20}$$

Through order $k(p+q)-1$ this is precisely equation (14), so that if we repeat verbatim the argument that follows (14), we obtain the identity (19) through order $k(p+q)$; that is,

$$F(x_1, y_1) = x_1^q y_1^p + f_2\,(x_1^q y_1^p)^2 + \cdots + f_k\,(x_1^q y_1^p)^k + U(x_1, y_1) = f(x_1^q y_1^p) + U(x_1, y_1)$$

where $U(x_1, y_1)$ begins with terms of order at least $k(p+q)+1$. Thus

$$x_1 \frac{\partial F}{\partial x_1} = q x_1^q y_1^p f'(x_1^q y_1^p) + \alpha(x_1, y_1) \qquad \text{and} \qquad y_1 \frac{\partial F}{\partial y_1} = p x_1^q y_1^p f'(x_1^q y_1^p) + \beta(x_1, y_1)$$

where $\alpha(x_1, y_1)$ and $\beta(x_1, y_1)$ begin with terms of order at least $k(p+q)+1$, and so the left hand side of (20) is

$$\begin{aligned}
p\,\alpha(x_1, y_1) - q\,\beta(x_1, y_1) \\
+ (qX(x_1^q y_1^p) + pY(x_1^q y_1^p))\, x_1^q y_1^p\, f'(x_1^q y_1^p) \\
+ X(x_1^q y_1^p)\,\alpha(x_1, y_1) + Y(x_1^q y_1^p)\,\beta(x_1, y_1)\,.
\end{aligned}$$

Hence if we subtract

$$p\,\alpha(x_1, y_1) - q\,\beta(x_1, y_1) + X(x_1^q y_1^p)\,\alpha(x_1, y_1) + Y(x_1^q y_1^p)\,\beta(x_1, y_1)\,,$$

which begins with terms of order at least $k(p+q)+1$, from each side of (20) we obtain

$$G(x_1^q y_1^p)\, x_1^q y_1^p\, f'(x_1^q y_1^p) = g_{kq,kp}(a^*, b^*)(x_1^q y_1^p)^k + \cdots\,, \tag{21}$$

where G is the function of (13).

Now suppose, contrary to what we wish to show, that system (1) for the choice $(a,b) = (a^*, b^*)$ has a center at the origin of $\mathbb{C}^2$, so that it admits a first integral $\Phi(x,y) = x^q y^p + \cdots$. Then by part (1) of Theorem 2 the function G vanishes identically, hence the left hand side of (21) is identically zero, whereas the right hand side is not, a contradiction.

2) If $V_{\mathcal{C}} \neq V'_{\mathcal{C}}$, then there exists (a^*, b^*) that belongs to one of the varieties $V_{\mathcal{C}}$ and $V'_{\mathcal{C}}$ but not to the other, say $(a^*, b^*) \in V_{\mathcal{C}}$ but $(a^*, b^*) \notin V'_{\mathcal{C}}$. The inclusion $(a^*, b^*) \in V_{\mathcal{C}}$ means that the system corresponding to (a^*, b^*) has a center at the origin. Therefore by part (1) $g'_{kq,kp}(a^*, b^*) = 0$ for all $k \in \mathbb{N}$. This contradicts our assumption that $(a^*, b^*) \notin V'_{\mathcal{C}}$. ∎

3 Properties of focus quantities and their computation

The focus quantities $g_{kq,kp}$ are difficult to compute for large k if we use only formulas (6) and (8), because the number of terms in these polynomials grows so fast. In this section we identify structure in the focus quantities $g_{kq,kp}$ for systems of the form (1) and give an efficient algorithm for computing the focus quantities. The algorithm is a further development of the methods of [6, 11, 13, 16, 17].

To illustrate the ideas that are the basis of the algorithm we consider as an example the family

$$\begin{aligned} \dot{x} &= \quad\ x - a_{20}x^3 - a_{-13}y^3, \\ \dot{y} &= -3y + b_{3,-1}x^3 + b_{02}y^3. \end{aligned} \tag{22}$$

Of course for $k_1 + k_2 = 0$, $v_{k_1,k_2} = 0$ if $(k_1,k_2) \neq (0,0)$, and $v_{00} = 1$. Computing using formula (6) we find that: for $k_1 + k_2 = 1$: $v_{k_1,k_2} = 0$ for all pairs (k_1,k_2); for $k_1 + k_2 = 2$: $v_{3,-1} = -\frac{1}{6}b_{3,-1}$, $v_{20} = \frac{3}{2}a_{20}$, $v_{11} = 0$, $v_{02} = \frac{1}{6}b_{02}$, $v_{-13} = -\frac{3}{10}a_{-13}$, $v_{-24} = v_{-35} = 0$; for $k_1 + k_2 = 3$: $v_{k_1,k_2} = 0$ for all pairs (k_1,k_2); and for $k_1 + k_2 = 4$: $v_{-37} = 0$, $v_{-26} = \frac{3}{100}a_{-13}^2$, $v_{2,2} = \frac{1}{20}(5a_{20}b_{02} - a_{-13}b_{3,-1})$, and similarly for the remaining six coefficients at this level.

We observe that all these polynomials have the following property: for any monomial that appears in v_{k_1,k_2}, the sum of the product of the index of each term (as an element of $\mathbb{N}_{-q} \times \mathbb{N}_{-p}$) and its exponent is the index (k_1,k_2) of v_{k_1,k_2}. For example, for v_{-26}:

$$a_{-13}^2 : 2 \cdot (-1,3) = (-2,6)$$

and for v_{22}:

$$\begin{aligned} a_{02}b_{20} &: 1 \cdot (2,0) + 1 \cdot (0,2) = (2,2) \\ a_{-13}b_{3,-1} &: 1 \cdot (-1,3) + 1 \cdot (3,-1) = (2,2)\,. \end{aligned}$$

To express this fact in general, we introduce the following notation. We order S in some manner, say by degree lexicographic order, writing $S = \{(p_1,q_1),\ldots,(p_\ell,q_\ell)\}$ and ordering the parameters accordingly as $(a_{p_1,q_1},\cdots,a_{p_\ell,q_\ell},b_{q_\ell,p_\ell},\cdots,b_{q_1,p_1})$. Any monomial appearing in v_{ij} has the form $a_{p_1,q_1}^{\nu_1}\cdots a_{p_\ell,q_\ell}^{\nu_\ell} b_{q_\ell,p_\ell}^{\nu_{\ell+1}}\cdots b_{q_1,p_1}^{\nu_{2\ell}}$ for some $\nu = (\nu_1,\ldots,\nu_{2\ell})$, which for simplicity we write as

$$[\nu] \stackrel{\text{def}}{=} a_{p_1,q_1}^{\nu_1}\cdots a_{p_\ell,q_\ell}^{\nu_\ell} b_{q_\ell,p_\ell}^{\nu_{\ell+1}}\cdots b_{q_1,p_1}^{\nu_{2\ell}}\,.$$

We write $\mathbb{Q}[a,b]$ for $\mathbb{Q}[a_{p_1,q_1},\dots,a_{p_\ell,q_\ell},b_{q_\ell,p_\ell},\dots,b_{q_1,p_1}]$. For $f\in\mathbb{Q}[a,b]$ we write $f=\sum_{\nu\in\mathrm{Supp}(f)}f^{(\nu)}[\nu]$, where $\mathrm{Supp}(f)$ denotes those $\nu\in\mathbb{N}_0^{2\ell}$ such that the coefficient of $[\nu]$ in the polynomial f is non–zero.

Let $L:\mathbb{N}_0^{2\ell}\to\mathbb{Z}^2$ be the linear map defined by

$$\begin{aligned}
L(\nu) &= (L_1(\nu),L_2(\nu))\\
&= \nu_1(p_1,q_1)+\cdots+\nu_\ell(p_\ell,q_\ell)+\nu_{\ell+1}(q_\ell,p_\ell)+\cdots+\nu_{2\ell}(q_1,p_1)\\
&= (p_1\nu_1+\cdots+p_\ell\nu_\ell+q_\ell\nu_{\ell+1}+\cdots+q_1\nu_{2\ell},\\
&\qquad q_1\nu_1+\cdots+q_\ell\nu_\ell+p_\ell\nu_{\ell+1}+\cdots+p_1\nu_{2\ell}).
\end{aligned}\tag{23}$$

It is the formal expression for the sums displayed in the example above. The fact that we have observed about the v_{ij} is that for each monomial $[\nu]$ appearing in v_{ij}, $L(\nu)=(i,j)$. This motivates the following definition.

Definition 3. For $(i,j)\in\mathbb{N}_{-q}\times\mathbb{N}_{-p}$, a polynomial $f\in\mathbb{Q}[a,b]$, $f=\sum_{\nu\in\mathrm{Supp}(f)}f^{(\nu)}[\nu]$, is an (i,j)*–polynomial* if for every $\nu\in\mathrm{Supp}(f)$, $L(\nu)=(i,j)$.

Theorem 4. *Consider a family of systems of the form* (1)*, with parameter space $E(a,b)=\mathbb{C}^{2\ell}$, where $p,q\in\mathbb{N}$, $GCD(p,q)=1$. There exists a formal series $\Psi(x,y)$ of the form* (2) *and polynomials $g_{q,p}$, $g_{2q,2p}$, $g_{3p,3q}$, … in $\mathbb{C}[a,b]$ such that*
(a) equation (9) *holds;*
(b) for every pair $(i,j)\in\mathbb{N}_{-q}\times\mathbb{N}_{-p}$ such that $i+j\geq 0$, $v_{ij}\in\mathbb{Q}[a,b]$, and v_{ij} is an (i,j)–polynomial;
(c) for every $k\geq 1$, $v_{kq,kp}=0$; and
(d) for every $k\geq 1$, $g_{kq,kp}\in\mathbb{Q}[a,b]$, and $g_{kq,kp}$ is a (kq,kp)–polynomial.

Proof. Let J be the set (5). The discussion leading from equation (2) to equation (9) shows that if we set $v_{00}=1$ and $v_{ij}=0$ for the other elements of the set J, then for $k_1\geq -q$, $k_2\geq -p$, if v_{k_1,k_2} are defined recursively by

$$v_{k_1,k_2}=\begin{cases}\dfrac{1}{k_1p-k_2q}\displaystyle\sum_{\substack{s_1+s_2=0\\ s_1\geq -q,s_2\geq -p}}^{k_1+k_2-1}[(s_1+q)a_{k_1-s_1,k_2-s_2}-(s_2+p)b_{k_1-s_1,k_2-s_2}]v_{s_1,s_2} & \text{if } k_1p\neq k_2q\\[2ex] 0 & \text{if } k_1p=k_2q,\end{cases}\tag{24}$$

where the recursion is on k_1+k_2 (i.e., find all v_{k_1,k_2} for which $k_1+k_2=1$, then find all v_{k_1,k_2} for which $k_1+k_2=2$, and so on), and once all v_{k_1,k_2} are known for $k_1+k_2\leq kq+kp-1$, $g_{kq,kp}$ is defined by

$$g_{kq,kp}=-\left[\sum_{\substack{s_1+s_2=0\\ s_1\geq -q,s_2\geq -p}}^{k_1+k_2-1}[(s_1+q)a_{k_1-s_1,k_2-s_2}-(s_2+p)b_{k_1-s_1,k_2-s_2}]\,v_{s_1,s_2}\right]\tag{25}$$

then for every pair (i,j), $v_{ij}\in\mathbb{Q}[a,b]$, for every k, $g_{kq,kp}\in\mathbb{Q}[a,b]$, and equation (9) holds. (An assumption that should be recalled is that in (24) and (25) $a_{k_1-s_1,k_2-s_2}$ and $b_{k_1-s_1,k_2-s_2}$ are replaced by zero when $(k_1-s_1,k_2-s_2)\notin S$.)

By definition (c) holds. To show that v_{ij} is an (i,j)–polynomial we proceed by induction on $i+j$.

Basis step. For $k_1 + k_2 = 0$, the corresponding polynomials are those from the set J defined in (5). Since $\mathrm{Supp}(v_{ij}) = \varnothing$ if $v_{ij} \in J$ and $v_{ij} \neq v_{00}$, the condition in Definition 3 is vacuous for such v_{ij}. For v_{00}, $\mathrm{Supp}(v_{00}) = (0, \dots, 0)$, and $L(0, \dots, 0) = (0, 0)$.

Inductive step. Suppose that v_{ij} is an (i, j)–polynomial for all (i, j) satisfying $i + j \leq m$, and that $k_1 + k_2 = m + 1$. Consider a term $v_{s_1,s_2} a_{k_1-s_1,k_2-s_2}$ in the sum in (24). If $(k_1 - s_1, k_2 - s_2) \notin S$ then $a_{k_1-s_1,k_2-s_2} = 0$, by convention, and the term does not appear. If $(k_1 - s_1, k_2 - s_2) = \bar{\imath}_c \in S$, then

$$v_{s_1,s_2} a_{k_1-s_1,k_2-s_2} = \left(\sum_{\nu \in \mathrm{Supp}(v_{s_1,s_2})} v^{(\nu)}_{s_1,s_2} [\nu] \right) [\mu] = \sum_{\nu \in \mathrm{Supp}(v_{s_1,s_2})} v^{(\nu)}_{s_1,s_2} [\nu + \mu] \,, \tag{26}$$

where $\mu = (0, \dots, 0, 1, 0, \dots, 0)$, with the one in the cth position, counting from the left. Clearly $L(\mu) = (k_1 - s_1, k_2 - s_2)$, hence by the inductive hypothesis and additivity of L, every term in (26) satisfies

$$L(\nu + \mu) = L(\nu) + L(\mu) = (s_1, s_2) + (k_1 - s_1, k_2 - s_2) = (k_1, k_2) \,.$$

Similarly for any term $v_{s_1,s_2} b_{k_1-s_1,k_2-s_2}$ such that $(k_1 - s_1, k_2 - s_2) = \bar{\jmath}_c$, i.e., such that $(k_2 - s_2, k_1 - s_1) = \bar{\imath}_c \in S$, we obtain an expression just like (26), except that now the 1 in μ is in the cth position counting from the *right*, and we easily compute that every term in this expression satisfies $L(\nu + \mu) = (k_1, k_2)$. Thus point (b) is fully established.

It is clear that the same argument as in the inductive step shows that $g_{kq,kp}$ is a (kq, kp)–polynomial, completing the proof of point (d). ■

The next theorem shows how to compute the coefficients $v^{(\nu)}_{k_1,k_2}$. The following definition and lemma will be needed.

Let a family (1) for some set S of indices be fixed, and for any $\nu \in \mathbb{N}_0^{2\ell}$ define $V(\nu) \in \mathbb{Q}$ recursively, with respect to $|\nu| = \nu_1 + \cdots + \nu_{2\ell}$, as follows:

$$V((0, \dots, 0)) = 1; \tag{27a}$$

for $\nu \neq (0, \dots, 0)$

$$V(\nu) = 0 \qquad \text{if} \qquad L_1(\nu) p = L_2(\nu) q; \tag{27b}$$

and when $L_1(\nu) p \neq L_2(\nu) q$,

$$\begin{aligned} V(\nu) = &\frac{1}{L_1(\nu)p - L_2(\nu)q} \times \\ &\left[\sum_{j=1}^{\ell} \widetilde{V}(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell})(L_1(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell}) + q) \right. \\ &\left. - \sum_{j=\ell+1}^{2\ell} \widetilde{V}(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell})(L_2(\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell}) + p) \right] \end{aligned} \tag{27c}$$

where

$$\widetilde{V}(\eta) = \begin{cases} V(\eta) & \text{if } \ \eta \in \mathbb{N}_0^{2\ell} \\ 0 & \text{if } \ \eta \in (\mathbb{N}_{-1})^{2\ell} \setminus \mathbb{N}_0^{2\ell} \end{cases}$$

and $L(\nu)$ is defined by (23).

Lemma 1. *Suppose $\nu \in \mathbb{N}_0^{2\ell}$ is such that either $L_1(\nu) < -q$ or $L_2(\nu) < -p$. Then $V(\nu) = 0$.*

Proof. Since $V(\nu) = 0$ is automatic if $L_1(\zeta)p = L_2(\zeta)q$, we may implicitly assume that $L_1(\zeta)p \neq L_2(\zeta)q$ throughout the proof. Since the coordinate change that exchanges x and y reverses the roles of p and q, without loss of generality we assume that $q \leq p$. The proof is by induction on $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_{2\ell}$.

Basis step. By definition of L,

$$L_1(\nu) \geq -(\nu_1 + \cdots + \nu_\ell) \qquad \text{and} \qquad L_2(\nu) \geq -(\nu_\ell + \cdots + \nu_{2\ell})\,. \tag{28}$$

Thus if $0 \leq |\nu| \leq q$ then both $L_1(\nu) \geq -q$ and $L_2(\nu) \geq -p$, so the basis step is $|\nu| = q+1$. Suppose $\nu = \zeta$ is such that $|\zeta| = q+1$, and that either (i) $L_1(\zeta) < -q$ or (ii) $L_2(\zeta) < -p$.

Case (i). By the first inequality in (28) $\zeta = (\zeta_1, \ldots, \zeta_\ell, 0, \ldots, 0)$ and we must have $u_k = -1$ for all k, $1 \leq k \leq \ell$, for which $\zeta_k \neq 0$. Thus $L_1(\zeta) = -q-1$. By the definition of $\widetilde{V}$, if the jth term ζ_j of ζ is zero, then the corresponding summand in (27c) is zero; in particular, every summand in the second sum in (27c) is zero. If ζ_j is non-zero then the corresponding summand in (27c) is a constant times

$$L_1(\zeta_1, \ldots, \zeta_j - 1, \ldots, \zeta_\ell, 0, \ldots 0) + q = L_1(\zeta) - u_j + q = -q - 1 + 1 + q = 0\,.$$

Case (ii). The proof is similar. We remark that now $\zeta = (0, \ldots, 0, \zeta_{\ell+1}, \ldots, \zeta_{2\ell})$, and that this case arises only if $q = p$ (recall our standing assumption that $q \leq p$).

Inductive step. Assume the statement of the lemma is true for all ν satisfying $|\nu| \leq m$ and let $\nu = \zeta$ be such that $|\zeta| = m+1$ and either $L_1(\zeta) < -q$ or $L_2(\zeta) < -p$. For any term in either sum in (27c), the argument $\mu = (\zeta_1, \ldots, \zeta_j - 1, \ldots, \zeta_{2\ell})$ of $\widetilde{V}$ satisfies $|\mu| = m$.

Suppose $L_1(\zeta) < -q$. For any summand in the first sum in (27c), $j \leq \ell$, hence $L_1(\mu) = L_1(\zeta) - u_j$. If $u_j \geq 0$, then $L_1(\mu) < -q$, so by the induction hypothesis $\widetilde{V}(\mu) = 0$, and the summand is zero. Suppose then that $u_j = -1$. If in fact $L_1(\zeta) < -q-1$, then $L_1(\mu) < -q$, so $\widetilde{V}(\mu) = 0$ and the summand is zero for the same reason. If $L_1(\zeta) = -q-1$, then $L_1(\mu) = -q$, hence $L_1(\mu) + q = 0$, and the summand is zero because the second factor is zero.

For any summand in the second sum in (27c), we do not examine $L_2(\mu)$, but instead still examine $L_1(\mu)$. Since $j \geq \ell+1$, $L_1(\mu) = L_1(\zeta) - v_{2\ell-j+1}$. Since $v_{2\ell-j+1} \geq 0$, $L_1(\mu) < -q$, so by the induction hypothesis $\widetilde{V}(\mu) = 0$, and the summand is zero.

The argument for the case $L_2(\zeta) < -p$ is completely analogous. ∎

Theorem 5. *For a family of systems of the form* (1)*, with parameter space $E(a,b) = \mathbb{C}^{2\ell}$, where $p, q \in \mathbb{N}$, $GCD(p,q) = 1$, let $\Psi(x,y)$ be the formal series of the form* (2) *and let $\{g_{kq,kp} : k \in \mathbb{N}\}$ be the polynomials in $\mathbb{C}[a,b]$ given by Theorem 4. Then*
(a) for $\nu \in Supp(v_{k_1,k_2})$, the coefficient $v^{(\nu)}_{k_1,k_2}$ of $[\nu]$ in v_{k_1,k_2} is $V(\nu)$;
(b) for $\nu \in Supp(g_{kq,kp})$, the coefficient $g^{(\nu)}_{kq,kp}$ of $[\nu]$ in $g_{kq,kp}$ is

$$\begin{aligned} W(\nu) = -\Bigg[&\sum_{j=1}^{\ell} \widetilde{V}(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell})(L_1(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell}) + q) \\ &- \sum_{j=\ell+1}^{2\ell} \widetilde{V}(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell})(L_2(\nu_1, \ldots, \nu_j - 1, \ldots, \nu_{2\ell}) + p) \Bigg]\,. \end{aligned} \tag{29}$$

Proof. Without loss of generality we assume that $q \le p$.

The proof of part (a) is by induction on $k_1 + k_2$.

Basis step. For $k_1 + k_2 = 0$, the corresponding polynomials form the set J of (5). The only polynomial in J with non-empty support is v_{00}: $\mathrm{Supp}(v_{00}) = (0, \dots, 0)$, and $v_{00} = 1 \cdot a^0_{\bar{\imath}_1} \cdots a^0_{\bar{\imath}_\ell} b^0_{\bar{\jmath}_\ell} \cdots b^0_{\bar{\jmath}_1} = V((0, \dots, 0)) \cdot a^0_{\bar{\imath}_1} \cdots a^0_{\bar{\imath}_\ell} b^0_{\bar{\jmath}_\ell} \cdots b^0_{\bar{\jmath}_1}$, as required. Thus statement (a) holds vacuously.

Inductive step. Suppose statement (a) holds for v_{k_1,k_2} for $k_1 + k_2 \le m$, and let k_1 and k_2 be such that $k_1 + k_2 = m+1$. If $k_1 p = k_2 q$ then because p and q are relatively prime $v_{k_1,k_2} = 0$ by Theorem 4(c). By Theorem 4(b), for any $\nu \in \mathrm{Supp}(v_{k_1,k_2})$, $L(\nu) = (k_1, k_2)$, so $L_1(\nu)p = L_2(\nu)q$ and by definition $V(\nu) = 0$, as required. If $k_1 p \ne k_2 q$, then by (24)

$$\begin{aligned}
&(k_1 p - k_2 q)\, v_{k_1,k_2} \\
&\quad = \sum_{\substack{s_1+s_2=0 \\ s_1 \ge -q, s_2 \ge -p}}^{k_1+k_2-1} [(s_1+q) a_{k_1-s_1,k_2-s_2} - (s_2+p) b_{k_1-s_1,k_2-s_2}] v_{s_1,s_2} \\
&\quad = \sum_{\substack{s_1+s_2=0 \\ s_1 \ge -q, s_2 \ge -p}}^{k_1+k_2-1} \Bigg[(s_1+q) \Bigg(\sum_{\mu \in \mathrm{Supp}(v_{s_1,s_2})} v^{(\mu)}_{s_1,s_2} [\mu] a_{k_1-s_1,k_2-s_2} \Bigg) \\
&\qquad\qquad - (s_2+p) \Bigg(\sum_{\mu \in \mathrm{Supp}(v_{s_1,s_2})} v^{(\mu)}_{s_1,s_2} [\mu] b_{k_1-s_1,k_2-s_2}] \Bigg) \Bigg] \\
&\quad = \sum_{\substack{s_1+s_2=0 \\ s_1 \ge -q, s_2 \ge -p}}^{k_1+k_2-1} \Bigg[(s_1+q) \sum_{\mu \in \mathrm{Supp}(v_{s_1,s_2})} v^{(\mu)}_{s_1,s_2} a^{\mu_1}_{\bar{\imath}_1} \cdots a^{\mu_c+1}_{\bar{\imath}_c} \cdots a^{\mu_\ell}_{\bar{\imath}_\ell} b^{\mu_{\ell+1}}_{\bar{\jmath}_\ell} \cdots b^{\mu_{2\ell}}_{\bar{\jmath}_1} \\
&\qquad\qquad - (s_2+p) \sum_{\mu \in \mathrm{Supp}(v_{s_1,s_2})} v^{(\mu)}_{s_1,s_2} a^{\mu_1}_{\bar{\imath}_1} \cdots a^{\mu_\ell}_{\bar{\imath}_\ell} b^{\mu_{\ell+1}}_{\bar{\jmath}_\ell} \cdots b^{\mu_{2\ell-d+1}+1}_{\bar{\jmath}_d} \cdots b^{\mu_{2\ell}}_{\bar{\jmath}_1} \Bigg]
\end{aligned} \tag{30}$$

where $\bar{\imath}_c = (k_1 - s_1, k_2 - s_2)$ (provided $(k_1 - s_1, k_2 - s_2) \in S$, else by convention the product is zero) and $\bar{\jmath}_d = (k_1 - s_1, k_2 - s_2)$ (provided $(k_2 - s_2, k_1 - s_1) \in S$, else by convention the product is zero).

Fix $\nu \in \mathbb{N}_0^{2\ell}$ for which $L(\nu) = (k_1, k_2)$. We wish to find the coefficient $v^{(\nu)}_{k_1,k_2}$ of $[\nu]$ in v_{k_1,k_2}. For a fixed $j \in \{1, \dots, \ell\}$, we first ask which pairs (s_1, s_2) are such that $\bar{\imath}_c = (k_1 - s_1, k_2 - s_2) = \bar{\imath}_j = (u_j, v_j)$. There is at most one such pair: $s_1 = k_1 - u_j$ and $s_2 = k_2 - v_j$; it exists if and only if $k_1 - u_j \ge -q$ and $k_2 - v_j \ge -p$. For that pair, we then ask which $\mu \in \mathbb{N}_0^{2\ell}$ are such that $(\mu_1, \dots, \mu_j + 1, \dots, \mu_{2\ell}) = (\nu_1, \dots, \nu_{2\ell})$. There is at most one such multi–index: $(\mu_1, \mu_2, \dots, \mu_{2\ell}) = (\nu_1, \dots, \nu_j - 1, \dots, \nu_{2\ell})$; it exists if and only if $\nu_j \ge 1$. For this μ, $L(\mu) = \nu_1 \bar{\imath}_1 + \cdots + \nu_{2\ell} \bar{\jmath}_1 - (u_j, v_j) = (k_1 - u_j, k_2 - v_j) = (s_1, s_2)$, although $\mu \notin \mathrm{Supp}(v_{s_1,s_2})$ is possible. Applying the same considerations to the cases $\bar{\jmath}_d = \bar{\jmath}_{2\ell-j+1}$ for $j = \ell+1, \dots, 2\ell$, we see that for any term $v^{(\nu)}_{k_1,k_2}[\nu]$ appearing in v_{k_1,k_2} there is at most one term on the right hand side of (30) for which the value of c is 1, at most one for which the value of c is 2, and so on through $c = \ell$, and similarly at most one term for which the value of d is ℓ, at most one for which the value of d is $\ell - 1$, and so on through $d = 1$. Thus the coefficient of $[\nu]$ in (24) is (recalling that v_{k_1,k_2} is a (k_1, k_2)–polynomial so that

$(k_1,k_2)=(L_1(\nu),L_2(\nu))$, and similarly for v_{s_1,s_2}):

$$\begin{aligned} v^{(\nu)}_{k_1,k_2} = & \frac{1}{pL_1(\nu)-qL_2(\nu)} \times \\ & \Bigg[\sum_{j=1}^{\ell}{}'(L_1(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})+q)v^{(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})}_{k_1-u_j,k_2-v_j} \\ & - \sum_{j=\ell+1}^{2\ell}{}'(L_2(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})+p)v^{(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})}_{k_1-v_{2\ell-j+1},k_2-u_{2\ell-j+1}} \Bigg], \end{aligned} \tag{31}$$

where the prime on the first summation symbol indicates that if (i) $\nu_j-1<0$, or if (ii) $k_1-u_j<-q$, or if (iii) $k_2-v_j<-p$, or if (iv) $\nu_j-1\geq 0$, $k_1-u_j\geq -q$, and $k_2-v_j\geq -p$, but $(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})\notin \operatorname{Supp}(v_{k_1-u_j,k_2-v_j})$, then the corresponding term does not appear in the sum, and the prime on the second summation symbol has a similar meaning.

If in either sum j is such that $\nu_j-1<0$, then since the corresponding term does not appear, the sum is unchanged if we replace $v^{(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})}_{k_1-u_j,k_2-v_j}$ by $\widetilde{V}(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})$, since the latter is zero in this situation.

In the first sum, suppose j is such that $\nu_j-1\geq 0$. If both $k_1-u_j\geq -q$ and $k_2-v_j\geq -p$, then there are two subcases. If the corresponding term appears in the sum, then because $|\nu_1+\cdots+(\nu_j-1)+\cdots+\nu_{2\ell}|\leq m$, the induction hypothesis applies and $v^{(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})}_{k_1-u_j,k_2-v_j}=V(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})$. Since in this situation $\widetilde{V}(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})=V(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})$, in the corresponding term we may replace $v^{(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})}_{k_1-u_j,k_2-v_j}$ by $\widetilde{V}(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})$ and the sum is unchanged. The second subcase is that in which the corresponding term does not appear, meaning that $(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})\notin\operatorname{Supp}(v_{k_1-u_j,k_2-v_j})$. But again the induction hypothesis applies, and now yields $V(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})=0$, so again the sum is unchanged by the same replacement.

Finally, suppose that in the first sum j is such that $\nu_j-1\geq 0$ but either $k_1-u_j<-q$ or $k_2-v_j<-p$, so the corresponding term is not present in the sum. Then because $L(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})=(k_1-u_j,k_2-v_j)$, Lemma 1 applies, by which we can make the same replacement as above, and thus the first sum in (31) is the same as the first sum in (27c). The second sum in (31) is treated similarly. This proves point (a). The same argument as in the inductive step gives point (b). ∎

The results of this section yield the Focus Quantity Algorithm for computation of the focus quantities for family (1) given at the end of this paper.

As an example, we now resolve the center problem for system (22).

Theorem 6. *System (22) on $\mathbb{C}^2$ has a center at the origin, that is, has a local first integral of the form (2) if and only if at least one of the following conditions holds:*
(1) $a_{-13}=b_{3,-1}=0$,
(2) $a_{20}=b_{3,-1}=0$,
(3) $a_{-13}=b_{02}=0$.

Proof. Using the Focus Quantity Algorithm we compute the first five focus quantities; for example, $g_{3,1}=\frac{1}{2}b_{3,-1}b_{02}$ and $g_{6,2}=\frac{3}{8}a_{20}^2a_{-13}b_{3,-1}+\frac{5}{36}b_{3,-1}^2b_{02}^2$; $g_{9,3}$,

$g_{12,4}$, and $g_{15,5}$ have degrees 6, 8, and 10 and contain five, seven, and eleven terms, respectively.

Let $V_1 = \mathbf{V}(\langle g_{3,1}\rangle)$, the variety of the ideal in $\mathbb{C}[a,b]$ generated by $g_{3,1}$, let $V_2 = \mathbf{V}(\langle g_{3,1}, g_{6,2}\rangle)$, and so on. Using the Radical Membership Test [7], which states that a polynomial $f \in \mathbb{C}[x_1,\dots,x_n]$ vanishes on the variety $\mathbf{V}(I)$ of an ideal $I \subset \mathbb{C}[x_1,\dots,x_n]$ if and only if the reduced Gröbner basis of $\langle I, 1-wf\rangle \subset \mathbb{C}[w,x_1,\dots,x_n]$ is $\{1\}$, we find that $V_3 \neq V_4$, but $V_4 = V_5$. It follows from the Hilbert Basis Theorem that the chain of varieties $V_1 \supset V_2 \supset V_3 \supset \dots$ stabilizes; the computation thus far suggests that it stabilizes on the fourth step, that is, that $V_{\mathcal{C}} = \mathbf{V}(\mathcal{B}) = V_4$. Solving the system $g_{3,1} = g_{6,2} = g_{9,3} = g_{12,4} = 0$ we find that the variety V_4 is the union of the three components defined in the statement of the theorem. Hence the three conditions stated in the theorem are necessary conditions for the existence of an integral of the form (2). To prove that they are also sufficient we construct a first integral in each case.

The first case is immediate: direct integration of the equation

$$\frac{dy}{dx} = \frac{b_{02}y^3 - 3y}{x - a_{20}x^3}$$

yields an integral of the form $\Psi(x,y) = x^3y + \cdots$, as in (2).

To treat the remaining cases we use the Darboux method of integration. Recall that a polynomial $f(x,y)$ defines an algebraic invariant curve $f(x,y) = 0$ of system (1) if and only if there exists a polynomial $k(x,y)$, the cofactor of f, such that $D(f) = k \cdot f$. It is easily verified that if $f_1,\dots,f_m$ are algebraic invariant curves of (1) with respective cofactors $k_1,\dots,k_m$, and if there exist constants $\alpha_1,\dots,\alpha_m$ such that

$$\sum_{i=1}^{m} \alpha_i k_i(x,y) + \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y) = 0 \tag{32}$$

then $\mu = f_1^{\alpha_1}\cdots f_m^{\alpha_m}$ is an integrating factor for system (1).

In the second case of the theorem system (22) has the form

$$\dot{x} = x - a_{-13}\, y^3, \quad \dot{y} = -3y + b_{02}\, y^3. \tag{33}$$

If $b_{02} = 0$ then a simple integration after multiplying by the integrating factor $y^{-2/3}$ gives the first integral $\Phi(x,y) = 3xy^{1/3} - \frac{3a}{10}y^{10/3}$. Then $\Psi := \Phi^3/27$ is a first integral of the form (2). When $b_{02} \neq 0$ then factoring the right hand side of the $\dot{y}$ equation in (33) yields the three invariant lines $f_i(x,y) = 0$, $i = 1, 2, 3$, where

$$f_1(x,y) = y, \quad f_2(x,y) = 1 + \sqrt{\frac{b_{02}}{3}}\, y, \quad f_3(x,y) = 1 - \sqrt{\frac{b_{02}}{3}}\, y,$$

with their respective cofactors

$$k_1(x,y) = -3 + b_{02}\, y^2, \quad k_2(x,y) = -\sqrt{3b_{02}}\, y + b_{02}\, y^2, \quad k_3(x,y) = \sqrt{3b_{02}}\, y + b_{02}\, y^2.$$

Solving (32) for α_1, α_2, and α_3 we find that $\mu = y^{-2/3}(f_2f_3)^{-7/6} = y^{-2/3}(1-\frac{b_{02}}{3}y^2)^{-7/6}$ is an integrating factor for (33), with corresponding first integral

$$\Phi(x,y) = 3xy^{1/3}(1 - \tfrac{b}{3}y^2)^{-1/6} - \tfrac{3}{10}\, a_{-13}\, y^{10/3}\, {}_2F_1(\tfrac{7}{6}, \tfrac{5}{3}; \tfrac{8}{3}; \tfrac{b_{02}}{3}y^2)\,,$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function. Then $\Psi := \Phi^3/27$ is a first integral of the form (2).

In the third case of the theorem system (22) has the form

$$\dot{x} = x - a_{20}x^3, \quad \dot{y} = -3y + b_{3,-1}x^3 \,. \tag{34}$$

If $a_{20} = 0$ then a simple integration after multiplying by the integrating factor x^2 gives the first integral $\Psi(x,y) = x^3\,y - \frac{b_{3,-1}\,x^6}{6}$. When $a_{20} \neq 0$ we factor the $\dot{x}$ equation to obtain three invariant lines, find the cofactors, and solve equation (32) to obtain the integrating factor $\mu = x^2/(1-a_{20}\,x^2)^{5/2}$, which yields the first integral

$$\Phi(x,y) = \frac{b_{3,-1}(8 - 12a_{20}\,x^2 + 3a_{20}^2\,x^4) + 3a_{20}^3\,x^3y}{3a_{20}^3\,(1 - a_{20}\,x^2)^{3/2}}\,.$$

Then $\Psi(x,y) = \Phi(x,y) - \frac{8b_{3,-1}}{3a_{20}^3}$ is a first integral of the form (2). ∎

4 Appendix: The Focus Quantity Algorithm

Combining Theorem 5 with parts (b) and (d) of Theorem 4 we obtain the following efficient algorithm for computing the focus quantities for family (1). For $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.

Input:

$K \in \mathbb{N}$

Ordered set $S = \{(u_1,v_1),\ldots,(u_\ell,v_\ell)\} \subset (\{-1\} \times \mathbb{N}_0)^2$

$p, q \in \mathbb{N}$, $GCD(p,q) = 1$

Output:

Focus quantities $g_{kq,kp}$, $1 \le k \le K$, for family (1)

Procedure:

$w := \min\{u_1+v_1,\ldots,u_\ell+v_\ell\}$
$M := \lfloor \frac{K(p+q)}{w} \rfloor$
$g_{qp} := 0;\ \ldots,\ g_{Kq,Kp} := 0;$
$V(0,\ldots,0) := 1;$
FOR $k = 1$ TO M DO
  FOR $\nu \in \mathbb{N}_0^{2\ell}$ such that $|\nu| = k$
  Compute $L(\nu)$ using (23)
  Compute $V(\nu)$ using (27)
  IF
  $L_1(\nu)\,p = L_2(\nu)\,q$
  THEN
  Compute $W(\nu)$ using (29)
  $g_{L(\nu)} := g_{L(\nu)} + W(\nu)\,[\nu]$

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