Finite groups determined by an inequality of the orders of their subgroups

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Abstract

In this article we introduce and study two classes of finite groups for which the orders of their subgroups satisfy a certain inequality. These are closely connected to some well-known arithmetic classes of natural numbers.

1 Preliminaries

Let n be a natural number and $\sigma(n)$ be the sum of all divisors of n. We say that n is a *deficient number* if $\sigma(n) < 2n$ and a *perfect number* if $\sigma(n) = 2n$ (for more details on these numbers, see [1]). Thus, the set consisting of both the deficient numbers and the perfect numbers can be characterized by the inequality

$$\sum_{d \in L_n} d \leq 2n,$$

where $L_n = \{d \in \mathbb{N} \mid d|n\}$.

Now let G be a finite group. Then the set $L(G)$ of all subgroups of G forms a complete lattice with respect to set inclusion, called the *subgroup lattice* of G. A remarkable subposet of $L(G)$ is constituted by all cyclic subgroups of G. It is called the *poset of cyclic subgroups* of G and will be denoted by $C(G)$. Clearly, if the group G is cyclic of order n, then $L(G) = C(G)$ and they are isomorphic to the lattice L_n. So, n is deficient or perfect if and only if

$$\sum_{H \in L(G)} |H| \leq 2|G|, \tag{1}$$

or equivalently

$$\sum_{H \in C(G)} |H| \le 2|G| \,. \tag{2}$$

This fact suggests us to consider the classes $\mathcal{C}_1$ and $\mathcal{C}_2$ consisting of *all* finite groups G which satisfy the inequalities (1) and (2), respectively. Remark that $\mathcal{C}_1$ is properly contained in $\mathcal{C}_2$ (an example of a finite group in $\mathcal{C}_2$ but not in $\mathcal{C}_1$ is the symmetric group S_3). As we have seen above, finite cyclic groups of deficient or perfect order are contained in both $\mathcal{C}_1$ and $\mathcal{C}_2$, but at first sight it is difficult to characterize finite groups in these classes. Therefore they must be investigated more carefully. Their study is the main goal of our paper.

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [3]. For subgroup lattice concepts we refer the reader to [2] and [4].

2   Main results

For a finite group G let us denote

$$\sigma_1(G) = \sum_{H \in L(G)} \frac{|H|}{|G|} = \sum_{H \in L(G)} \frac{1}{|G : H|}$$

and

$$\sigma_2(G) = \sum_{H \in C(G)} \frac{|H|}{|G|} = \sum_{H \in C(G)} \frac{1}{|G : H|} \,.$$

In this way, $\mathcal{C}_i$ is the class of all finite groups G for which $\sigma_i(G) \le 2, i = 1, 2$. Observe that we have $\sigma_2(G) \le \sigma_1(G)$, for all finite groups G, with equality on the class of finite cyclic groups. A common property of these functions is that they are multiplicative, that is if G and G' are two finite groups satisfying $\gcd(|G|, |G'|) = 1$, then

$$\sigma_i(G \times G') = \sigma_i(G)\sigma_i(G'), \ i = 1, 2 \,;$$

it then follows that if G_j, $j = 1, \ldots, m$, are finite groups of coprime orders, then

$$\sigma_i\Big(\underset{j=1}{\overset{m}{\times}} G_j\Big) = \prod_{j=1}^{m} \sigma_i(G_j), \ i = 1, 2 \,.$$

In the rest of this paper, we will investigate the classes $\mathcal{C}_i$, $i = 1, 2$. First of all, we focus on $\mathcal{C}_1$. Obviously, it contains the finite cyclic groups of prime order. On the other hand, we easily obtain $\sigma_1(\mathbb{Z}_2 \times \mathbb{Z}_2) = 11/4$ and $\sigma_1(S_3) = 8/3 \cdot$ These equalities show that the class $\mathcal{C}_1$ is not closed under direct products or extensions. If N is a normal subgroup of G, then we get

$$\sigma_1(G/N) = \sum_{\substack{H \in L(G) \\ N \subseteq H}} \frac{1}{|G : H|} \quad \text{and} \quad \sigma_1(N) = |G : N| \sum_{\substack{H \in L(G) \\ H \subseteq N}} \frac{1}{|G : H|} \,.$$

Thus, the function σ_1 satisfies the inequality

$$\sigma_1(G) \ge \sigma_1(G/N) + \frac{1}{|G : N|} \left(\sigma_1(N) - 1 \right) \,.$$

In particular, it follows that $\sigma_1(G) \geq \sigma_1(G/N)$, which shows that $\mathcal{C}_1$ is closed under homomorphic images. Note that for the moment we are not able to decide whether $\mathcal{C}_1$ is closed under subgroups, i.e. whether all subgroups of a group in $\mathcal{C}_1$ also belong to $\mathcal{C}_1$.

A precise characterization of the groups in $\mathcal{C}_1$ is given by the following theorem. It shows that the finite cyclic groups of deficient or perfect order are in fact the only finite groups contained in $\mathcal{C}_1$.

Theorem 1. *Let G be a finite group of order n. Then G is contained in $\mathcal{C}_1$ if and only if it is cyclic and n is a deficient or perfect number.*

Proof. Suppose that G belongs to $\mathcal{C}_1$ and let $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$ be the decomposition of n as a product of prime factors. If M is a nonnormal maximal subgroup of G, then M coincides with its normalizer in G and so it has exactly $r = |G : M|$ conjugates, say $M_1, M_2, \ldots, M_r$. Hence

$$\sigma_1(G) \geq 1 + \frac{1}{n} + \sum_{i=1}^{r} \frac{1}{|G : M_i|} = 1 + \frac{1}{n} + \sum_{i=1}^{r} \frac{1}{r} = 2 + \frac{1}{n} > 2,$$

a contradiction. Therefore all maximal subgroups of G are normal, which implies that G is nilpotent.

Let $G = \bigtimes_{j=1}^{m} G_j$, where $|G_j| = p_j^{n_j}$ for every $j \in \{1, \ldots, m\}$, and take an arbitrary index $s \in \{1, \ldots, m\}$. If the Sylow p_s-subgroup G_s of G is not cyclic, then it contains at least $p_s + 1$ maximal subgroups (of order $p_s^{n_s-1}$): $G_{s,1}, G_{s,2}, \ldots, G_{s,p_s+1}$. It is clear that for all $u \neq v \in \{1, 2, \ldots, p_s + 1\}$ the subgroups $G_{s,u} \bigtimes_{\substack{j=1 \\ j \neq s}}^{m} G_j$ and $G_{s,v} \bigtimes_{\substack{j=1 \\ j \neq s}}^{m} G_j$ are distinct. In this way, G has at least $p_s + 1$ subgroups of order n/p_s. It follows that

$$\sigma_1(G) \geq \frac{1 + (p_s + 1)\, n/p_s + n}{n} > 2\,,$$

again contradicting the hypothesis. Hence all Sylow subgroups of G are cyclic and therefore G itself is cyclic of deficient or perfect order.

Since the converse is already shown, our proof is finished. ∎

It follows from Theorem 1 that the restriction of σ_1 to $\mathcal{C}_1$ is connected to the classical function σ by the equality $\sigma_1(G) = \dfrac{\sigma(|G|)}{|G|}$, for any G in $\mathcal{C}_1$. Because all divisors d of n satisfy $\dfrac{\sigma(d)}{d} \leq \dfrac{\sigma(n)}{n}$, we also infer that the class $\mathcal{C}_1$ is closed under subgroups.

We will now study the larger class $\mathcal{C}_2$. We will obtain a general formula for σ_2 for arbitrary finite groups, but unfortunately, it seems difficult to compute it explicitly in general. For p-groups, however, the formula simplifies, and this will allow us to compute $\sigma_2(G)$ for arbitrary finite *nilpotent* groups G.

Theorem 2. *Let G be an arbitrary finite group. Then*

$$\sigma_2(G) = |G|^{-1} \cdot \sum_{h \in G} \frac{|h|}{\phi(|h|)},$$

where ϕ is the Euler totient function.

Proof. The claim is equivalent to the statement

$$\sum_{H \in C(G)} |H| = \sum_{h \in G} \frac{|h|}{\phi(|h|)}. \tag{$*$}$$

The left hand side of this expression is equal to the cardinality of the set

$$\begin{aligned} S &= \{(H, g) \mid H \in C(G), g \in H\} \\ &= \{(\langle h \rangle, g) \mid h \in G, g \in \langle h \rangle\}. \end{aligned}$$

Every element $h \in G$ gives rise to $|h|$ such pairs $(\langle h \rangle, g)$ with $g \in \langle h \rangle$, but then every such pair has been counted $\phi(|h|)$ times, since the number of generators of the cyclic group $\langle h \rangle$ is precisely $\phi(|h|)$. Hence $|S|$ is equal to the expression on the right hand side of equation $(*)$. ∎

Theorem 3. *Let G be a finite p-group of order p^t for some integer $t \geq 1$. Then*

$$\sigma_2(G) = \frac{p^{1+t} - 1}{p^t(p-1)}.$$

In particular, we have

$$1 + \frac{1}{p} \leq \sigma_2(G) < 1 + \frac{1}{p-1}.$$

Proof. If $n = p^i$ for some $i \geq 1$, then $n/\phi(n) = p^i/(p^i - p^{i-1}) = p/(p-1)$, and this is clearly independent of i. (For $i = 0$, we get $n = 1$ and hence $n/\phi(n) = 1$.) It now follows from Theorem 2 that

$$\begin{aligned} \sigma_2(G) &= |G|^{-1} \cdot \left(1 + \sum_{h \in G \setminus \{1\}} \frac{p}{p-1}\right) \\ &= p^{-t} \cdot \left(1 + (p^t - 1) \cdot \frac{p}{p-1}\right) = \frac{p^{1+t} - 1}{p^t(p-1)}. \end{aligned}$$ ∎

Corollary 4. *All finite p-groups are contained in the class $\mathcal{C}_2$.*

Proof. Since for any prime p we have $1 + \dfrac{1}{p-1} \leq 2$, this follows immediately from Theorem 3. ∎

Using the multiplicity of σ_2, the explicit formula found in Theorem 3 can be extended to the general case of arbitrary finite nilpotent groups, in the following way.

Theorem 5. *Let G be a finite nilpotent group of order $n = \prod_{j=1}^{m} p_j^{n_j}$, and let $\prod_{j=1}^{m} G_j$ be the direct decomposition of G as a product of its Sylow p-subgroups, where $|G_j| = p_j^{n_j}$, for all $j \in \{1, \ldots, m\}$. Then*

$$\sigma_2(G) = \prod_{j=1}^{m} \frac{p_j^{n_j+1} - 1}{p_j^{n_j}(p_j - 1)} = \frac{\sigma(n)}{n}.$$

In particular, we have

$$\prod_{j=1}^{m} \left(1 + \frac{1}{p_j}\right) \leq \sigma_2(G) < \prod_{j=1}^{m} \left(1 + \frac{1}{p_j - 1}\right).$$

Proof. This follows from Theorem 3 and the fact that σ_2 is multiplicative.   ∎

Corollary 6. *A finite nilpotent group is contained in $\mathcal{C}_2$ if and only if its order is a deficient or perfect number.*

Proof. This is an immediate consequence of Theorem 5.   ∎

Observe that the above equivalence fails without the supplementary hypothesis that G is nilpotent. Indeed, consider the dihedral group D_{12} of order 12; a simple exercise shows that $\sigma_2(D_{12}) = 2$ (therefore D_{12} belongs to the class $\mathcal{C}_2$), but $\sigma(12)/12 = 7/3 > 2$ (therefore $|D_{12}|$ is not a deficient or perfect number).

It is clear that the Klein's group $\mathbb{Z}_2 \times \mathbb{Z}_2$ belongs to $\mathcal{C}_2$. The same thing can be also said about the cyclic group $\mathbb{Z}_{15}$, but not about the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{15}$ (see Corollary 6). This remark shows that the class $\mathcal{C}_2$ is not closed under direct products.

The class $\mathcal{C}_2$ is not closed under subgroups either, not even under normal subgroups. The smallest counterexample is given by the non-abelian group $G = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_9$ of order 36. Indeed, G itself is in $\mathcal{C}_2$, but it has a normal subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, which is not in $\mathcal{C}_2$.

It turns out, however, that the class $\mathcal{C}_2$ is closed under homomorphic images. More precisely, we have the following result.

Theorem 7. *Let G be a finite group and N a normal subgroup of G. Then $\sigma_2(G/N) \leq \sigma_2(G)$. In particular, the class $\mathcal{C}_2$ is closed under homomorphic images.*

Proof. We first observe that if n is a natural number and $d \mid n$, then $d/\phi(d) \leq n/\phi(n)$. It now follows from Theorem 2 that

$$\sigma_2(G/N) = \frac{|N|}{|G|} \cdot \sum_{h \in G/N} \frac{|h|}{\phi(|h|)} = \frac{1}{|G|} \cdot \sum_{h \in G} \frac{|hN|}{\phi(|hN|)},$$

since every coset hN is obtained by $|N|$ different elements $h \in G$. (Note that $|hN|$ denotes the order of the element hN in the group G/N.) But for every $h \in G$, the order of hN in G/N is a divisor of the order of h in G, and hence

$$\sigma_2(G/N) \leq \frac{1}{|G|} \cdot \sum_{h \in G} \frac{|h|}{\phi(|h|)} = \sigma_2(G).$$ ∎

Inspired by Theorem 5, we came up with the following conjecture, which we have verified by computer for all groups of order ≤ 512.

Conjecture 8. *Let G be a finite group of order n. Then $\sigma_2(G) \leq \sigma(n)/n$, and we have equality if and only if G is nilpotent.*

Finally, we indicate another way to extend our initial conditions (1) and (2). Another interesting condition in the same spirit is given by

$$\sum_{H \in N(G)} |H| \leq 2|G|\,, \tag{3}$$

where $N(G)$ denotes the normal subgroup lattice of G. Hence, in the same manner as above, one can introduce and study the class $\mathcal{C}_3$ consisting of all finite groups G which satisfy this inequality.

References

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