Weighted integral representations of entire functions of several complex variables

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Abstract

In the paper we consider the spaces of entire functions $f(z), z \in C^n$, satisfying the condition

$$\int_{R^n} \left(\int_{R^n} |f(x+iy)|^p dx \right)^s |y|^\alpha e^{-\sigma |y|^\rho} dy < +\infty.$$

For these classes the following integral representation is obtained:

$$f(z) = \int_{C^n} f(u+iv) \Phi(z, u+iv) |v|^\alpha e^{-\sigma |v|^\rho} du dv, \quad z \in C^n,$$

where the reproducing kernel $\Phi(z, u+iv)$ is written in an explicit form as a Fourier type integral. Also, an estimate for Φ is obtained.

Introduction

0.1. It is well known that the Hardy class $H^2(Rez > 0)$ is defined as the set of all holomorphic functions $f(z), Rez > 0$, satisfying the condition

$$\sup_{x>0} \left(\int_{-\infty}^{+\infty} |f(x+iy)|^2 dy \right) < +\infty. \tag{0.1}$$

The following result was established by R. Paley and N. Wiener [1]:

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Theorem 0.1. The class $H^2(Rez>0)$ admits an integral representation of the form

$$f(z)=\int_0^{+\infty}F(t)e^{-zt}dt, \quad Rez>0, \tag{0.2}$$

where $F(t)\in L^2(0;+\infty)$.

Theorem 0.1 initiated numerous investigations, where this classical result was generalized in various directions.

M.M. Djrbashian and A.E. Avetisian [2] (see also [3, Chapter 7]) introduced Hardy type weighted classes in arbitrary angular domains and established Paley-Wiener type integral representations for these classes by means of Mittag-Leffler type kernels

$$E_\rho(z;\mu)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k/\rho)}. \tag{0.3}$$

S. Bochner [4](see also [5]) established an analogue of Theorem 0.1 for multidimensional Hardy classes H^2 over radial tube domains in C^n. Later on S.G. Gindikin [6] extended Bochner's result in the case of Siegel domains of type two, which are much more general than radial tube domains. Moreover, in [6, §5] a somewhat different problem was set and solved: to obtain Paley-Wiener type integral representations for classes of functions holomorphic in Siegel domains of type two and belonging to L^2 over the whole domain. On the basis of these integral representations reproducing kernels for the classes were constructed in an explicit form.

In order to give a brief survey of works where S.G. Gindikin's investigation was continued and developed, we need some notations.

0.2. For arbitrary $z=(z_1,\dots,z_n)\in C^n$ and $w=(w_1,\dots,w_n)\in C^n$ set

$$<z,w>=\sum_{k=1}^{n}z_k\overline{w_k}. \tag{0.4}$$

Suppose that $B\subset R^n$ is a domain and $\gamma(y)>0, y\in B$, is an arbitrary continuous function. We put

$$\gamma^*(t)=\int_B e^{-<y,t>}\gamma(y)dy, \quad t\in R^n. \tag{0.5}$$

Further, for $p,s\in(0;+\infty)$ we denote by $H^p_{s,\gamma}(T_B)$ the set of all functions $f(z)\equiv f(x+iy)$ holomorphic in the tube domain

$$T_B=\{z=x+iy\in C^n : x\in R^n, y\in B\} \tag{0.6}$$

and satisfying the condition

$$M^p_{s,\gamma}(f)\equiv\int_B\left(\int_{R^n}|f(x+iy)|^p dx\right)^s\gamma(y)dy<+\infty. \tag{0.7}$$

Note that for $s=1$ the space $H^p_{s,\gamma}(T_B)=H^p_{1,\gamma}(T_B)$ consists of those functions holomorphic in T_B, which belong to $L^p\{T_B;\gamma(y)dxdy\}$. Besides, when $B=R^n$, i.e. $T_B=C^n$, the corresponding spaces are denoted by $H^p_{s,\gamma}(C^n)$.

The following theorem is valid:

Theorem 0.2.

1. Assume that $2 \leq p < +\infty, 1/p + 1/q = 1, 0 < s < +\infty$ and a measurable function $F(t), t \in R^n$, satisfies the condition

$$\int_B \left(\int_{R^n} |F(t)|^q e^{-q<y,t>} dt \right)^{s(p-1)} \gamma(y) dy < +\infty. \tag{0.8}$$

Then the function

$$f(z) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} F(t) e^{i<z,t>} dt, \quad z \in T_B, \tag{0.9}$$

belongs to the space $H^p_{s,\gamma}(T_B)$ and

$$M^p_{s,\gamma}(f) \leq \frac{1}{(2\pi)^{\frac{n}{2}s(p-2)}} \int_B \left(\int_{R^n} |F(t)|^q e^{-q<y,t>} dt \right)^{s(p-1)} \gamma(y) dy < +\infty. \tag{0.10}$$

2. Assume that $1 \leq p \leq 2$ and $0 < s < +\infty$. Then each function $f \in H^p_{s,\gamma}(T_B)$ admits an integral representation of the form (0.9), where:

— if $p = 1$, $F(t), t \in R^n$, is continuous and satisfies the condition

$$\sup_{t \in R^n} \{ |F(t)| \gamma^*(st) \} \leq \frac{M^1_{s,\gamma}(f)}{(2\pi)^{\frac{n}{2}s}} < +\infty; \tag{0.11}$$

— if $1 < p \leq 2$, $F(t), t \in R^n$, is measurable and satisfies the condition

$$\int_B \left(\int_{R^n} |F(t)|^q e^{-q<y,t>} dt \right)^{s(p-1)} \gamma(y) dy \leq \frac{M^p_{s,\gamma}(f)}{(2\pi)^{\frac{n}{2}s(2-p)}} < +\infty \quad (1/p + 1/q = 1). \tag{0.12}$$

Also, for a.e. $y \in B$ we have

$$\hat{f}_y(t) = F(t) e^{-<y,t>}, \quad t \in R^n, \tag{0.13}$$

where $\hat{f}_y$ is the Fourier transform of

$$f_y(x) \equiv f(x + iy), \quad x \in R^n.$$

3. For $p = 2$ and $0 < s < +\infty$ the formula (0.9) gives an integral representation of the whole class $H^p_{s,\gamma}(T_B)$, i.e. $H^2_{s,\gamma}(T_B)$ coincides with the set of all functions $f(z)$ representable in the form (0.9) with a measurable function $F(t), t \in R^n$, satisfying the condition

$$\int_B \left(\int_{R^n} |F(t)|^2 e^{-2<y,t>} dt \right)^s \gamma(y) dy < +\infty. \tag{0.14}$$

Moreover, the Parseval identity holds:

$$M^2_{s,\gamma}(f) = \int_B \left(\int_{R^n} |F(t)|^2 e^{-2<y,t>} dt \right)^s \gamma(y) dy. \tag{0.15}$$

For $n \geq 1, \gamma(y) \equiv 1 (y \in B)$ and $p = 2, s = 1$ Theorem 0.2 follows from S.G. Gindikin's results [6] for Siegel domains.

For $n \geq 1, \gamma(y) \equiv 1 (y \in B)$ and under the assumptions $1 \leq p \leq 2, s = 1$ or $1 < p < 2, s = 1/(p-1)$ the formulated assertions were established by T.G. Genchev [7-9].

For $n = 1, \gamma(y) \equiv 1 (y \in B)$ the theorem follows from more general results by M.M. Djrbashian and V.M. Martirosian [10].

Finally, in the form formulated above, Theorem 0.2 was established in [11, Theorems 2.3-2.5] and [12, Theorem 2].

It turns out that the supports of the functions $F(t)$ in Theorem 0.2 (2,3) become significantly narrower if the domain $B \subset R^n$ and the weight function $\gamma(y), (y \in B)$ satisfy certain conditions. We mean the following cases:

(a) (see [13]) $B = V$ is a sharp (or an acute) open convex cone in R^n and

$$\underline{\lim}_{|y| \to +\infty} \frac{\ln \gamma(y)}{|y|} \geq 0 \quad (y \in V);$$

(b) (see [14]) $B = V$ is a sharp open convex cone in R^n and $\gamma(y)$ is of a form $\varphi(\rho_V(y))$, where $\rho_V(y) = dist(y, \partial V), y \in V$ and $\varphi(\tau) > 0, \tau \in (0; +\infty)$, is a continuous function such that

$$\underline{\lim}_{\tau \to +\infty} \frac{\ln \varphi(\tau)}{\tau} \geq 0;$$

(particularly, $\varphi(\tau) = \tau^\alpha$) ;

(c) (see [15]) $B = V$ is an affine-homogeneous sharp open convex cone (see [6]) in R^n and $\gamma(y)$ is defined in accordance with the inner structure of V.

In all these cases we have $supp\, F(t) \subset V^*$, where V^* is a cone conjugate to V. Furthermore, in [13-15] for $1 \leq p \leq 2$ and under additional conditions on $\gamma(y), y \in V$ and on the parameter s, a reproducing kernel for the class $H^p_{s,\gamma}(T_V)$ was constructed. In other words, a kernel $\Phi(z, w), z, w \in T_V$ was constructed, such that for all $f \in H^p_{s,\gamma}(T_V)$

$$f(z) = \frac{1}{(2\pi)^n} \int_{T_V} f(w) \Phi(z, w) \gamma(v) du dv, \quad z \in T_V \quad (w = u + iv). \tag{0.16}$$

S. Saitoh [16-18] considered and solved, in a sense, a converse problem: given a domain $D \subset R^n$, integrals of the form

$$f(z) = \frac{1}{(2\pi)^{n/2}} \int_D F(t) e^{i<z,t>} dt, \tag{0.17}$$

were considered for, in general, arbitrary functions $F(t), t \in D$. Then, requiring for holomorphic functions $f(z)$ to belong to Hardy type spaces (including the condition of square integrability over the domain), S. Saitoh determined the corresponding conditions on functions $F(t)$ and the maximal tube domain in C^n where the functions (0.17) can be extended. For the considered spaces reproducing kernels were constructed.

It should be mentioned that for the case $p = 2, s = 1, \gamma(y) \equiv 1$ the results of [11-15] are close to [16-18].

R.A. Zalik and T. Abuabara Saad [19] considered a case of Theorem 0.2 when $n = 1, s = 1/(p-1), B = R$ and $\gamma(y) \equiv |y|^\alpha e^{-\sigma|y|^\rho}, y \in R(\alpha > -1)$.

Due to this explicit form of $\gamma(y)$ they obtained an explicit asymptotic behaviour of $\gamma^*(t)$ when $|t| \to +\infty$. As a consequence, the conditions (0.8), (0.11), (0.12) simplified. Roughly speaking (see [19]), $F(t)$ belongs to a space of type $L^q\left\{R; e^{a|t|^{\frac{\rho}{\rho-1}}}dt\right\}$ for some $a > 0$.

A.M. Sedletskii [20] extended the result of [19] on the case of entire functions satisfying the condition of type

$$\int_R \left(\int_R |f(x+iy)|^p |x|^s dx\right)^{\frac{1}{p-1}} |y|^\alpha e^{-\sigma|y|^\rho} dy < +\infty. \tag{0.18}$$

In the present paper for the spaces $H^p_{s,\gamma}(C^n)(1 \le p \le 2)$, where $\gamma(y) \equiv |y|^\alpha e^{-\sigma|y|^\rho}, y \in R^n(\alpha > -n)$, reproducing kernels (in the sense of (0.16)) are constructed and applications are given.

1 Main integral representations

1.1. From now on we suppose that $n > 1, 1 \le p \le 2 \quad (1/p + 1/q = 1), 0 < s < \infty$ and $\alpha > -n, 0 < \sigma < \infty, 1 < \rho < \infty$. For these parameters put

$$\gamma(y) \equiv |y|^\alpha e^{-\sigma|y|^\rho}, \quad y \in R^n. \tag{1.1}$$

According to (0.5) we have

$$\gamma^*(t) = \int_{R^n} e^{-<y,t>} |y|^\alpha e^{-\sigma|y|^\rho} dy, \quad t \in R^n. \tag{1.2}$$

Obviously, γ^* is continuous in $t \in R^n$ and $\gamma^*(t) > 0, t \in R^n$.

Further, we shall consider the space $H^p_{s,\gamma}(C^n)$, i.e. the space of all entire functions $f(z), z \in C^n$, satisfying the condition

$$M^p_{s,\gamma}(f) = \int_{R^n} \left(\int_{R^n} |f(x+iy)|^p dx\right)^s |y|^\alpha e^{-\sigma|y|^\rho} dy < +\infty. \tag{1.3}$$

Below we shall construct reproducing kernels (in the sense of (0.16)) for these spaces. To this end we establish some auxiliary results.

1.2. First of all, let us obtain an asymptotic behaviour of $\gamma^*(t), t \in R^n$.

We put

$$c_\rho = max_{[0;1]} \left\{x^{\frac{1}{\rho-1}}(1-x)\right\} = (\frac{1}{\rho})^{\frac{1}{\rho-1}}(1-\frac{1}{\rho}). \tag{1.4}$$

It is easy to verify that $0 < c_\rho < 1$.

The following estimates for $\gamma^*(t)$ are valid:

Lemma 1.1.

1. When $|t| \geq 1$, then

$$\gamma^*(t) \leq const(n, \alpha, \rho, \sigma) \cdot |t|^{\frac{n+\alpha}{\rho-1}} e^{\sigma^{-1/(\rho-1)} c_\rho |t|^{\rho/(\rho-1)}}. \tag{1.5}$$

2. For an arbitrary small $\varepsilon > 0$, when $t \in R^n$, then

$$\gamma^*(t) \leq const(n, \alpha, \rho, \sigma, \varepsilon) \cdot e^{\sigma^{-1/(\rho-1)} c_\rho (1+\varepsilon)|t|^{\rho/(\rho-1)}}. \tag{1.5'}$$

3. For an arbitrary small $\varepsilon > 0$, when $t \in R^n$, then

$$\gamma^*(t) \geq const(n, \alpha, \rho, \sigma, \varepsilon) \cdot |t|^{\frac{n+\alpha}{\rho-1}} e^{\sigma^{-1/(\rho-1)} c_\rho (1-\varepsilon)|t|^{\rho/(\rho-1)}}. \tag{1.6}$$

4. For an arbitrary small $\varepsilon > 0$, when $t \in R^n$, then

$$\gamma^*(t) \geq const(n, \alpha, \rho, \sigma, \varepsilon) \cdot e^{\sigma^{-1/(\rho-1)} c_\rho (1-\varepsilon)|t|^{\rho/(\rho-1)}}. \tag{1.6'}$$

Proof. Let us write (1.2) in polar coordinates:

$$\gamma^*(t) = \int_0^{+\infty} r^{n+\alpha-1} e^{-\sigma r^\rho} \int_{S_n} e^{-r<\zeta,t>} d\sigma_n(\zeta) dr, \quad t \in R^n, \tag{1.7}$$

where S_n is the unit sphere in R^n and σ_n is the surface measure on S_n. Then we have:

$$\gamma^*(t) \leq const \int_0^{+\infty} r^{n+\alpha-1} e^{r(|t|-\sigma r^{\rho-1})} dr. \tag{1.8}$$

The change of variable $\sigma r^{\rho-1} = x$ in the right-hand side of (1.8) yields

$$\gamma^*(t) \leq const \int_0^{+\infty} x^{\frac{n+\alpha+1-\rho}{\rho-1}} e^{(\frac{x}{\sigma})^{1/(\rho-1)}(|t|-x)} dx =$$

$$= const \cdot \left\{ \int_0^{|t|} + \int_{|t|}^{+\infty} \right\} \equiv const \cdot \{\gamma_1^*(t) + \gamma_2^*(t)\}. \tag{1.9}$$

Further, making a change of variable $x \to |t|x \quad (t \neq 0)$ in both the integrals γ_1^* and γ_2^*, we obtain:

$$\gamma_1^*(t) = |t|^{\frac{n+\alpha}{\rho-1}} \int_0^1 x^{\frac{n+\alpha+1-\rho}{\rho-1}} e^{\sigma^{-1/(\rho-1)} |t|^{\rho/(\rho-1)} x^{1/(\rho-1)}(1-x)} dx \leq$$

$$\leq const \cdot |t|^{\frac{n+\alpha}{\rho-1}} e^{\sigma^{-1/(\rho-1)} c_\rho |t|^{\rho/(\rho-1)}}, \tag{1.10}$$

$$\gamma_2^*(t) = |t|^{\frac{n+\alpha}{\rho-1}} \int_1^{\infty} x^{\frac{n+\alpha+1-\rho}{\rho-1}} e^{-\sigma^{-1/(\rho-1)} |t|^{\rho/(\rho-1)} x^{1/(\rho-1)}(x-1)} dx.$$

If $|t| \geq 1$, then $|t|^{\rho/(\rho-1)} \geq 1$, consequently

$$\gamma_2^*(t) \leq |t|^{\frac{n+\alpha}{\rho-1}} \int_1^{\infty} x^{\frac{n+\alpha+1-\rho}{\rho-1}} e^{-\sigma^{-1/(\rho-1)} x^{1/(\rho-1)}(x-1)} dx \leq const \cdot |t|^{\frac{n+\alpha}{\rho-1}}. \tag{1.11}$$

Combining (1.9) - (1.11), we obtain (1.5) when $|t| \geq 1$.

The inequality (1.5′) is a direct consequence of (1.5).

To establish (1.6), fix a sufficiently small $\varepsilon > 0$. Then find a sufficiently small $\eta \in (0; \frac{1}{2})$ such that $(1-\eta)^{\frac{2\rho-1}{\rho-1}} > 1-\varepsilon$. Further, note that

$$-<-\zeta_0, \zeta_0>= 1, \quad \forall \zeta_0 \in S_n.$$

Hence for an arbitrary $\zeta_0 \in S_n$ there exists a neighbourhood $G(-\zeta_0) \subset S_n$ of the point $-\zeta_0$ such that

$$-<\zeta, \zeta_0>\geq 1-\eta, \quad \forall \zeta \in G(-\zeta_0). \tag{1.12}$$

Of course, $G(-\zeta_0)$ depends on ζ_0, but

$$\sigma_n\left(G(-\zeta_0)\right) \equiv const > 0, \quad \forall \zeta_0 \in S_n. \tag{1.13}$$

Combining (1.12) and (1.13), we obtain the inequality

$$\int_{S_n} e^{-<\zeta,t>} d\sigma_n(\zeta) \geq const \cdot e^{(1-\eta)|t|}, \quad t \in R^n, \tag{1.14}$$

where the constant does not depend on t .

Consequently, (1.7) and (1.14) give

$$\gamma^*(t) \geq const \int_0^{+\infty} r^{n+\alpha-1} e^{-\sigma r^\rho} e^{(1-\eta)r|t|} dr = const \int_0^{+\infty} r^{n+\alpha-1} e^{r[(1-\eta)|t|-\sigma r^{\rho-1}]} dr.$$

After the change of variable $\sigma r^{\rho-1} = x$ we have

$$\gamma^*(t) \geq const \int_0^{+\infty} x^{\frac{n+\alpha+1-\rho}{\rho-1}} e^{(\frac{x}{\sigma})^{1/(\rho-1)}[(1-\eta)|t|-x]} dx \geq$$
$$\geq const \int_0^{(1-\eta)|t|} x^{\frac{n+\alpha+1-\rho}{\rho-1}} e^{(\frac{x}{\sigma})^{1/(\rho-1)}[(1-\eta)|t|-x]} dx.$$

Finally, after another change of variable $x \to (1-\eta)|t|x \quad (t \neq 0)$ we obtain

$$\gamma^*(t) \geq const \cdot [(1-\eta)|t|]^{\frac{n+\alpha}{\rho-1}} \int_0^1 x^{\frac{n+\alpha+1-\rho}{\rho-1}} e^{\sigma^{-1/(\rho-1)}(1-\eta)^{\rho/(\rho-1)}|t|^{\rho/(\rho-1)} x^{1/(\rho-1)}(1-x)} dx.$$

In view of (1.4)

$$x^{\frac{1}{\rho-1}}(1-x) \geq (1-\eta)c_\rho, \quad x \in (\frac{1}{\rho}-\delta; \frac{1}{\rho}+\delta) \subset (0;1), \quad \delta = \delta(\eta) > 0.$$

Also,

$$[(1-\eta)|t|]^{\frac{n+\alpha}{\rho-1}} > (\frac{1}{2})^{\frac{n+\alpha}{\rho-1}} \cdot |t|^{\frac{n+\alpha}{\rho-1}}.$$

Hence

$$\gamma^*(t) \geq const \cdot |t|^{\frac{n+\alpha}{\rho-1}} e^{\sigma^{-1/(\rho-1)}(1-\eta)^{\frac{2\rho-1}{\rho-1}} c_\rho |t|^{\rho/(\rho-1)}} \geq$$

$$\geq const \cdot |t|^{\frac{n+\alpha}{\rho-1}} e^{\sigma^{-1/(\rho-1)}(1-\varepsilon)c_\rho|t|^{\rho/(\rho-1)}}.$$

The inequality (1.6′) immediately follows from (1.6). Thus the lemma is proved.

Remark 1.1.
In [19] for $n = 1$ a more explicit (compared to (1.5)-(1.6)) asymptotic estimate was established:

$$\gamma^*(t) \approx const \cdot |t|^{\frac{\alpha+1}{\rho-1}-\frac{\rho}{2(\rho-1)}} e^{\sigma^{-1/(\rho-1)}c_\rho|t|^{\rho/(\rho-1)}}, \quad |t| \to +\infty. \tag{*}$$

We think that for $n > 1$ "arbitrary small $\varepsilon > 0$ " can not be omitted.

Remark 1.2.
When $\rho = 2, \alpha = 0$, we have

$$\gamma^*(t) = \int_{R^n} e^{-<y,t>} e^{-\sigma|y|^2} dy \equiv const \cdot e^{\frac{|t|^2}{4\sigma}}, \quad t \in R^n. \tag{1.15}$$

This slightly differs from (1.5)-(1.6)(for $\rho = 2, \alpha = 0$) and coincides with (*)(for $n = 1$). Consequently, estimates (1.5)-(1.6) are not explicit. However, this is sufficient for our purposes.

Remark 1.3.
In view of Lemma 1.1 we can give a new interpretation of conditions (0.11) and (0.12) of Theorem 0.2.
The condition (0.11) is close to the condition of the form

$$|F(t)| \leq const \cdot e^{-const \cdot |t|^{\rho/(\rho-1)}}, |t| \to +\infty.$$

For $1 < p \leq 2, s = \frac{1}{p-1}$ we can change the order of integrations in the left-hand side of (0.12). As a result we obtain

$$\int_{R^n} |F(t)|^q \gamma^*(qt) dt < +\infty.$$

In view of (1.6), this implies the condition

$$\int_{R^n} |F(t)|^q |t|^{\frac{n+\alpha}{\rho-1}} e^{const \cdot |t|^{\rho/(\rho-1)}} dt < +\infty.$$

Further, recall Minkowski generalized integral inequality ($q \geq 1$):

$$\left\{ \int_Y \left\{ \int_X f(x,y) d\mu(x) \right\}^q d\nu(y) \right\}^{1/q} \leq \int_X \left\{ \int_Y \{f(x,y)\}^q d\nu(y) \right\}^{1/q} d\mu(x),$$

where $(X;\mu)$ and $(Y;\nu)$ are spaces with positive measures and f is an $X \times Y$-measurable nonnegative function.

For $1 < p \leq 2, s = \frac{1}{p}$ (when $\frac{1}{q} = s(p-1)$) the application of this inequality to (0.12) (with $B = R^n$)implies

$$+\infty > \int_{R^n} \left(\int_{R^n} |F(t)|^q e^{-q<y,t>} dt \right)^{s(p-1)} \gamma(y) dy \geq$$

$$\geq \left(\int_{R^n}\left(\int_{R^n}|F(t)|e^{-<y,t>}dy\right)^q dt\right)^{1/q} = \left(\int_{R^n}|F(t)|^q[\gamma^*(t)]^q dt\right)^{1/q}.$$

In other words,

$$\int_{R^n}|F(t)|^q[\gamma^*(t)]^q dt < +\infty,$$

which, in view of (1.6), implies the condition of the form

$$\int_{R^n}|F(t)|^q|t|^{q\frac{n+\alpha}{\rho-1}}e^{const\cdot|t|^{\rho/(\rho-1)}}dt < +\infty.$$

1.3. For arbitrary $z = x + iy \in C^n$ and $v \in R^n$ set

$$R_{z,v}(t) = (2\pi)^{n/2}\frac{e^{i<z+iv,t>}}{\gamma^*(2t)}, t \in R^n. \tag{1.16}$$

In view of Lemma 1.1(4) $R_{z,v}(t) \in L^p(R^n)$ for all $0 < p \leq \infty$. Further, set

$$R_z(v) = \|R_{z,v}(t)\|_p = (2\pi)^{n/2}\left(\int_{R^n}\frac{e^{-p<y+v,t>}}{[\gamma^*(2t)]^p}dt\right)^{1/p}, 0 < p < \infty. \tag{1.17}$$

Note that $R_z(v)$ is continuous in $v \in R^n$ and $R_z(v) > 0, v \in R^n$.

Lemma 1.2. For an arbitrary small $\varepsilon > 0$

$$R_z(v) \leq const(p, n, \alpha, \rho, \sigma, \varepsilon)\cdot e^{\frac{\sigma}{2^\rho}(1+\varepsilon)|y+v|^\rho}, \quad z \in C^n, \quad v \in R^n, \tag{1.18}$$

$$R_z(v) \leq const(p, n, \alpha, \rho, \sigma, \varepsilon, y)\cdot e^{\frac{\sigma}{2^\rho}(1+\varepsilon)|v|^\rho}, \quad v \in R^n. \tag{1.18'}$$

Proof. In what follows we suppose that $\varepsilon \in (0; 3)$ is arbitrary small. Then find a sufficiently small $\varepsilon_1 > 0$ such that $(1-\varepsilon_1)^{-(\rho-1)} < 1 + \frac{\varepsilon}{3}$. As it follows from (1.6′),

$$\gamma^*(2t) \geq const\cdot e^{\sigma^{-1/(\rho-1)}c_\rho(1-\varepsilon_1)|2t|^{\rho/(\rho-1)}}, \quad t \in R^n.$$

Hence

$$[R_z(v)]^p \leq const\int_{R^n}e^{p|y+v||t|-p2^{\rho/(\rho-1)}\sigma^{-1/(\rho-1)}c_\rho(1-\varepsilon_1)|t|^{\rho/(\rho-1)}}dt =$$

$$= const\int_0^{+\infty}r^{n-1}e^{r\{p|y+v|-p2^{\rho/(\rho-1)}\sigma^{-1/(\rho-1)}c_\rho(1-\varepsilon_1)r^{\rho/(\rho-1)-1}\}}dr. \tag{1.19}$$

Then note that the last integral is of type (1.8) with parameters $\alpha_1 = 0$ instead of α, $\rho_1 = \rho/(\rho-1)$ instead of ρ, $\sigma_1 = p2^{\rho/(\rho-1)}\sigma^{-1/(\rho-1)}c_\rho(1-\varepsilon_1)$ instead of σ and $t_1 = p(y+v)$ instead of t. And since the integral at the right-hand side of (1.8) has been already estimated (see (1.5′)), we have

$$[R_z(v)]^p \leq const\cdot e^{\sigma_1^{-1/(\rho_1-1)}c_{\rho_1}(1+\varepsilon/3)|t_1|^{\rho_1/(\rho_1-1)}}, \quad t_1 \in R^n. \tag{1.20}$$

Now we need some calculations.

First of all, $|t_1| = p|y+v|$. Further,

$$1/(\rho_1-1) = \rho-1, \rho_1/(\rho_1-1) = \rho,$$

$$\sigma_1^{-1/(\rho_1-1)} = \sigma_1^{-(\rho-1)} = p^{-(\rho-1)}2^{-\rho}\sigma c_\rho^{-(\rho-1)}(1-\varepsilon_1)^{-(\rho-1)} =$$

$$= p^{-(\rho-1)}2^{-\rho}\rho(\frac{\rho}{\rho-1})^{\rho-1}\sigma(1-\varepsilon_1)^{-(\rho-1)};$$

$$c_{\rho_1} = (\frac{1}{\rho_1})^{1/(\rho_1-1)}(1-\frac{1}{\rho_1}) = (\frac{\rho-1}{\rho})^{\rho-1}\frac{1}{\rho};$$

$$|t_1|^{\rho_1/(\rho_1-1)} = p^\rho|y+v|^\rho.$$

Thus for $z \in C^n, v \in R^n$, (1.20) implies

$$[R_z(v)]^p \leq const \cdot e^{\frac{p\sigma}{2^\rho}(1-\varepsilon_1)^{-(\rho-1)}(1+\varepsilon/3)|y+v|^\rho} <$$
$$< const \cdot e^{\frac{p\sigma}{2^\rho}(1+\varepsilon/3)(1+\varepsilon/3)|y+v|^\rho} <$$
$$< const \cdot e^{\frac{p\sigma}{2^\rho}(1+\varepsilon)|y+v|^\rho}. \tag{1.21}$$

It remains to note that (1.21) implies (1.18). As to (1.18′), it directly follows from (1.18). Lemma 1.2 is therefore proved.

1.4. For arbitrary $z, w \in C^n$ set

$$\Phi(z,w) = \int_{R^n} \frac{e^{i<z-\overline{w},t>}}{\gamma^*(2t)}dt. \tag{1.22}$$

Lemma 1.3.
1. The kernel $\Phi(z,w)$ is holomorphic in $z \in C^n$ and antiholomorphic in $w \in C^n$.
2. For fixed $z \in C^n$ and $v \in R^n$ the kernel $\Phi(z,u+iv)$, as a function of $u \in R^n$, is the Fourier transform of the function $R_{z,v}(t)$.
3. For an arbitrary small $\varepsilon > 0$ we have ($z = x+iy \in C^n, w = u+iv \in C^n$):

$$|\Phi(z,w)| \leq const(n,\alpha,\rho,\sigma,\varepsilon) \cdot e^{\frac{\sigma}{2^\rho}(1+\varepsilon)|y+v|^\rho}.$$

Proof. The estimate (1.6) immediately implies the assertions 1 and 2. As for 3, it easily follows from (1.21) with $p = 1$.

Theorem 1.1. Each function $f \in H^p_{s,\gamma}(C^n)$ has the integral representation

$$f(z) = \frac{1}{(2\pi)^n}\int_{C^n} f(w)\Phi(z,w)\gamma(v)dudv, \quad z \in C^n \quad (w = u+iv), \tag{1.23}$$

if the parameter s satisfies the conditions

$$\frac{1}{p} < s, \quad \frac{ps}{ps-1} < 2^\rho. \tag{1.24}$$

Proof. Let f be a function from $H^p_{s,\gamma}(C^n)$. According to Theorem 0.2(2)

$$f(z) = \frac{1}{(2\pi)^{n/2}}\int_{R^n} F(t)e^{i<z,t>}dt, \quad z \in C^n, \tag{1.25}$$

where $F(t), t \in R^n$, is continuous and satisfies (0.11) for $p = 1$, and $F(t), t \in R^n$, is measurable and satisfies (0.12) for $1 < p \leq 2$. Moreover, for a.e. $v \in R^n$

$$\hat{f}_v(t) = F(t)e^{-<v,t>}, \quad t \in R^n, \tag{1.26}$$

where $f_v(u) \equiv f(u + iv), u \in R^n$.

Now fix an arbitrary $z = x + iy \in C^n$ and set

$$I(z) = \frac{1}{(2\pi)^n} \int_{C^n} f(w)\Phi(z, w)\gamma(v)dudv. \tag{1.27}$$

Assuming absolute convergence of the integral (1.27), we establish the equality $I(z) = f(z)$ in the following way:

$$I(z) = \frac{1}{(2\pi)^n} \int_{R^n} \gamma(v) \int_{R^n} f(u + iv)\Phi(z, u + iv)dudv =$$

$$= \frac{1}{(2\pi)^n} \int_{R^n} \gamma(v) \int_{R^n} f_v(u)(\hat{R_{z,v}})(u)dudv =$$

$$= \frac{1}{(2\pi)^n} \int_{R^n} \gamma(v) \int_{R^n} \hat{f}_v(t)R_{z,v}(t)dtdv =$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{R^n} \gamma(v) \int_{R^n} F(t)e^{-<v,t>} \frac{e^{i<z+iv,t>}}{\gamma^*(2t)} dtdv =$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{R^n} F(t) \frac{e^{i<z,t>}}{\gamma^*(2t)} \int_{R^n} \gamma(v)e^{-2<v,t>}dvdt =$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{R^n} F(t)e^{i<z,t>}dt = f(z).$$

So it remains to show that the integral $I(z)$ converges absolutely. To this end set

$$\tilde{I}(z) = \int_{R^n} \gamma(v) \int_{R^n} |f_v(u)||\Phi(z, u + iv)|dudv. \tag{1.28}$$

An application of the Hőlder integral inequality and Lemma 1.3(2) gives:

$$\int_{R^n} |f_v(u)||\Phi(z, u + iv)|du \leq \tilde{f}(v) \cdot \left(\int_{R^n} |\Phi(z, u + iv)|^q du\right)^{1/q} =$$

$$= \tilde{f}(v) \cdot \left(\int_{R^n} |\hat{R_{z,v}}(t)|^q dt\right)^{1/q}, \quad (1/p + 1/q = 1),$$

where

$$\tilde{f}(v) = \left(\int_{R^n} |f_v(u)|^p du\right)^{1/p}, v \in R^n. \tag{1.29}$$

Further, the condition $1 \leq p \leq 2$ makes it possible to apply the Hausdorff-Young theorem (see [21,p.247]):

$$\int_{R^n} |f_v(u)||\Phi(z, u + iv)|du \leq const \cdot \tilde{f}(v) \cdot \left(\int_{R^n} |R_{z,v}(t)|^p dt\right)^{1/p} \equiv const \cdot \tilde{f}(v) \cdot R_z(v),$$

where $R_z(v), v \in R^n$, is defined by (1.17). Hence (1.28) implies:

$$\tilde{I}(z) \le const \int_{R^n} \tilde{f}(v) R_z(v)\gamma(v)dv. \qquad (1.30)$$

Note that the condition $f \in H^p_{s,\gamma}(C^n)$ implies

$$\int_{R^n} [\tilde{f}(v)]^{ps}\gamma(v)dv < +\infty. \qquad (1.31)$$

Therefore we set $r = \frac{ps}{ps-1}$ and apply the Hőlder integral inequality to (1.30):

$$\tilde{I}(z) \le const \left(\int_{R^n} [\tilde{f}(v)]^{ps}\gamma(v)dv\right)^{\frac{1}{ps}} \times$$

$$\times \left(\int_{R^n} [R_z(v)]^r \gamma(v)dv\right)^{\frac{1}{r}}.$$

In view of (1.31) it suffices to show that

$$J(z) \equiv \int_{R^n} [R_z(v)]^r \gamma(v)dv < +\infty. \qquad (1.32)$$

Using the estimate (1.18′) and (1.1), we have

$$J(z) \le const \int_{R^n} e^{\sigma|v|^\rho \frac{r}{2^\rho}(1+\varepsilon)} |v|^\alpha e^{-\sigma|v|^\rho} dv. \qquad (1.33)$$

Due to condition (1.24), $\frac{r}{2^\rho} < 1$, therefore the integral at the right-hand side of (1.33) converges for sufficiently small $\varepsilon > 0$. Thus $J(z) < +\infty$, i.e. (1.32) holds. This completes the proof of the theorem.

Remark 1.4. In fact, the conditions (1.24) are not complicated. For example, the requirement $s \ge 2/p$ ensures (1.24).

2 Examples

In this section we discuss the particular case $\alpha = 0, \rho = 2$. Remember (see(1.15)) that in this case

$$\gamma^*(t) = const \cdot e^{\frac{|t|^2}{4\sigma}}, \quad t \in R^n.$$

Hence in view of (1.22) we have

$$\Phi(z,w) = const \int_{R^n} e^{i<z-\overline{w},t>} e^{-\frac{|t|^2}{\sigma}} dt, \quad z,w \in C^n. \qquad (2.1)$$

Of course, the constant in (2.1) can be easily computed. It depends only on n and σ .

To calculate the integral in (2.1) let us set $w = z = x + iy \in C^n$. Then

$$\Phi(z,z) = const \int_{R^n} e^{-2<y,t>} e^{-\frac{|t|^2}{\sigma}} dt =$$

$$= const \cdot e^{\sigma|y|^2} = const \cdot e^{\sigma \sum_{k=1}^n y_k^2} =$$

$$= const \cdot e^{-\frac{\sigma}{4}\sum_{k=1}^{n}(z_k-\overline{z}_k)^2}, \quad z \in C^n.$$

The last relation means that entire functions $\Phi(z,\overline{w}), (z,w) \in C^{2n}$, and $const \cdot e^{-\frac{\sigma}{4}\sum_{k=1}^{n}(z_k-w_k)^2}, (z,w) \in C^{2n}$, coincide on the set $w=\overline{z}$, which is "the uniqueness set" in C^{2n}. Hence

$$\Phi(z,w) = const \cdot e^{-\frac{\sigma}{4}\sum_{k=1}^{n}(z_k-\overline{w}_k)^2}, \quad z,w \in C^n. \tag{2.2}$$

Consequently, in our case Theorem 1.1 gives:

Theorem 2.1. Each entire function $f(z) \equiv f(x+iy)$ satisfying the condition

$$\int_{R^n}\left(\int_{R^n}|f(x+iy)|^p dx\right)^s e^{-\sigma|y|^2}dy < +\infty, \tag{2.3}$$

has the integral representation

$$f(z) = const\int_{C^n} f(w)e^{-\frac{\sigma}{4}\sum_{k=1}^{n}(z_k-\overline{w}_k)^2}e^{-\sigma|v|^2}dudv, \quad z \in C^n \quad (w=u+iv), \tag{2.4}$$

where the constant depends only on $n>1$ and $\sigma>0$ and where it is assumed that $1 \le p \le 2, 1/p < s, ps/(ps-1) < 4$.

This theorem has an interesting application:

Theorem 2.2.(see [22],[23],[24],[25],[26],[18],[27]) Each entire function $\varphi(z) \equiv \varphi(x+iy)$ satisfying the condition

$$\int_{C^n}|\varphi(z)|^2 e^{-\sigma|z|^2}dxdy < +\infty, \tag{2.5}$$

has the integral representation

$$\varphi(z) = const\int_{C^n}\varphi(w)e^{\sigma<z,w>}e^{-\sigma|w|^2}dudv, \quad z \in C^n \quad (w=u+iv), \tag{2.6}$$

where the constant depends only on $n>1$ and $\sigma>0$.

Proof. Consider an arbitrary entire function $\varphi(z), z=x+iy \in C^n$, satisfying the condition (2.5). Then set

$$f(z) = \varphi(z)e^{-\frac{\sigma}{2}\sum_{k=1}^{n}z_k^2}, \quad z=(z_1,\cdots,z_n) \in C^n. \tag{2.7}$$

Obviously, $f(z)$ is entire function and

$$|f(z)|^2 = |\varphi(z)|^2 \cdot e^{-\sigma|x|^2}e^{\sigma|y|^2} = |\varphi(z)|^2 \cdot e^{-\sigma|z|^2}e^{2\sigma|y|^2}.$$

Hence in view of (2.5)

$$\int_{C^n}|f(z)|^2 e^{-2\sigma|y|^2}dxdy < +\infty. \tag{2.8}$$

Consequently, f is an entire function satisfying (2.3) for $p=2, s=1$ and with 2σ instead of σ. Therefore (2.4) gives:

$$f(z) = const \int_{C^n} f(w)e^{-\frac{\sigma}{2}\sum_{k=1}^{n}(z_k-\overline{w}_k)^2} e^{-2\sigma|v|^2} dudv, \quad z \in C^n \quad (w = u + iv). \quad (2.9)$$

The substitution of (2.7) into (2.9)gives

$$\varphi(z)e^{-\frac{\sigma}{2}\sum_{k=1}^{n} z_k^2} = const \int_{C^n} \varphi(w)e^{-\frac{\sigma}{2}\sum_{k=1}^{n} w_k^2} e^{-\frac{\sigma}{2}\sum_{k=1}^{n}(z_k-\overline{w}_k)^2} e^{-2\sigma|v|^2} dudv. \quad (2.10)$$

It remains to note that after simplifications (2.10) coincides with (2.6).

Remark 2.1. It should be mentioned that in [28] for classes of entire functions $\varphi(z), z \in C^n$, satisfying a condition of type

$$\int_{C^n} |\varphi(z)|^p |z|^\gamma e^{-\sigma|z|^\rho} dxdy < +\infty (1 < p < \infty; \rho, \sigma > 0; \gamma > -2n),$$

integral representations of type (2.6) were established.

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