Cross-Ratios and 6-Figures in some Moufang-Klingenberg Planes*

Atilla Akpinar Basri Celik Süleyman Ciftci

Abstract

This paper deals with Moufang-Klingenberg planes $\boldsymbol{M}(\mathcal{A})$ defined over a local alternative ring $\mathcal{A}$ of dual numbers. The definition of cross-ratio is extended to $\boldsymbol{M}(\mathcal{A})$. Also, some properties of cross-ratios and 6-figures that are well-known for Desarguesian planes are investigated in $\boldsymbol{M}(\mathcal{A})$; so we obtain relations between algebraic properties of $\mathcal{A}$ and geometric properties of $\boldsymbol{M}(\mathcal{A})$. In particular, we show that pairwise non-neighbour four points of the line g are in harmonic position if and only if they are harmonic, and that μ is Menelaus or Ceva 6-figure if and only if $r(\mu) = -1$ or $r(\mu) = 1$, respectively.

1 Introduction

One of the important problems of projective geometry is to find the relationships between algebraic properties of the coordinatizing ring and the geometric properties of the associated plane. For example, a projective plane is a Pappian plane, a Desarguesian plane or a Moufang plane if and only if the coordinatizing ring is a field, a division ring (skew field) or an alternative field (alternative division ring), respectively [10, p. 154]. Besides, a geometric structure is a Moufang-Klingenberg (MK) plane if and only if the coordinatizing ring is a local alternative ring [1, Theorem 3.10 and Theorem 4.1].

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One of the objectives of this paper is to extend the definition of cross-ratio for a certain class (which we will denote by $\boldsymbol{M}(\mathcal{A})$) of MK-planes coordinatized by a local alternative ring $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ (an alternative field $\mathbf{A}$, $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [4] to the whole plane $\boldsymbol{M}(\mathcal{A})$. Some properties of cross-ratios and 6-figures that are well-known for Desarguesian or Moufang planes are investigated in $\boldsymbol{M}(\mathcal{A})$ and so some relations between algebraic properties of $\mathcal{A}$ and geometric properties of $\boldsymbol{M}(\mathcal{A})$ are obtained, which is the other objective of this paper. In our previous paper [6], we have shown that the collineation group of $\boldsymbol{M}(\mathcal{A})$ acts transitively on 4-gons. This ensures the coordinatization of $\boldsymbol{M}(\mathcal{A})$ is independent of the choice of the coordinatization base. Also, in [6], we have obtained some important results related to 6-figures. We will give these results in the last section and we will get some new results for 6-figures.

Section 2 includes some basic definitions and results from the literature.

In Section 3, the concept of cross-ratio which has a great importance as a projective invariant is mentioned. First, the definition of cross-ratio, which is given in [4] for the points on $g = [1, 0, 0]$, is extended to the whole plane $\boldsymbol{M}(\mathcal{A})$. Next, a simple way for the calculation of the cross-ratio of points on the line l, according to type of l, is given. Finally the relation between the harmonic position (which is a geometric property) and the harmonicity (which is an algebraic property) is established: Any four pairwise non-neighbour points on the line g are in harmonic position if and only if they are harmonic.

In Section 4, the ratio of a 6-figure in $\boldsymbol{M}(\mathcal{A})$ is defined and it is given that a property for the 6-figures which are called conjugate, descendant and codescendant of a 6-figure in [5]. The paper is concluded by showing the ratio of any Menelaus 6-figure or any Ceva 6-figure is -1 or 1, respectively.

2 Preliminaries

Let $\boldsymbol{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$, respectively. Then $\boldsymbol{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If $P,\ Q$ are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If $g,\ h$ are non-neighbour lines, then there is a unique point $g \cap h$ on both g and h.

(PK3) There is a projective plane $\boldsymbol{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : \boldsymbol{M} \to \boldsymbol{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q,\ \Psi(g) = \Psi(h) \iff g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line $h \sim g$ such that $P \in h$.

Let $h, k \in \mathbf{L}$, $C \in \mathbf{P}$, $C \nsim h, k$. Then the well-defined bijection $\sigma := \sigma_C(k, h) : \begin{cases} h \to k \\ X \to XC \cap k \end{cases}$ mapping h to k is called a *perspectivity* from h to k with center C. A product of a finite number of perspectivities is called a *projectivity.*

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of $\boldsymbol{M}$. The notion of a *centre* and an *axis of a collineation* is as ordinary projective planes.

A PK-plane $\boldsymbol{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is called *Moufang-Klingenberg plane* (MK-plane), if it is (C, a)-transitive for all $C \in \mathbf{P},\ a \in \mathbf{L}$ with $C \in a$, i.e. if all possible elations exist. For every MK-plane the canonical image $\boldsymbol{M}^*$ is a Moufang plane [4].

An *alternative ring* $\mathbf{R}$ is a not necessarily associative ring that satisfies the alternative laws $a\,(ab) = a^2b, (ba)\,a = ba^2,\ \forall a, b \in \mathbf{R}$. An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We are now ready to give consecutively two lemmas related to alternative rings, which are used implicitly in many calculations throughout this paper.

Lemma 1. *([12, Theorem 3.1]) The subring generated by any two elements of an alternative ring is associative.*

Lemma 2. *([11, p. 160]) The identities*

$$\begin{aligned} x\,(y\,(xz)) &= (xyx)\,z \\ ((yx)\,z)\,x &= y\,(xzx) \\ (xy)\,(zx) &= x\,(yz)\,x \end{aligned}$$

which are known as Moufang identities are satisfied in every alternative ring.

Now we give the definition of an n-gon, which is meaningful when $n \geq 3$: An n-tuple of pairwise non-neighbour points is called an (ordered) *n-gon* if no three of its elements are on neighbour lines [6].

We summarize some basic concepts about the coordinatization of MK-planes from [2].

Let $\mathbf{R}$ be a local alternative ring. Then $\mathbf{M(R)} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned} \mathbf{P} &= \{(x, y, 1)|\ x, y \in \mathbf{R}\} \cup \{(1, y, z)|\ y \in \mathbf{R},\ z \in \mathbf{I}\} \cup \{(w, 1, z)|\ w, z \in \mathbf{I}\}, \\ \mathbf{L} &= \{[m, 1, p]\ |\ m, p \in \mathbf{R}\} \cup \{[1, n, p]\ |\ p \in \mathbf{R},\ n \in \mathbf{I}\} \cup \{[q, n, 1]\ |\ q, n \in \mathbf{I}\}, \end{aligned}$$

$$\begin{aligned} [m, 1, p] &= \{\,(x, xm + p, 1)|\, x \in \mathbf{R}\} \cup \{\,(1, zp + m, z)|\, z \in \mathbf{I}\}\,, \\ [1, n, p] &= \{\,(yn + p, y, 1)|\, y \in \mathbf{R}\} \cup \{\,(zp + n, 1, z)|\, z \in \mathbf{I}\}\,, \\ [q, n, 1] &= \{\,(1, y, yn + q)|\, y \in \mathbf{R}\} \cup \{\,(w, 1, wq + n)|\, w \in \mathbf{I}\}\,. \end{aligned}$$

$$\begin{aligned} P &= (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \iff x_i - y_i \in \mathbf{I}\ \ (i = 1, 2, 3)), \forall P, Q \in \mathbf{P}; \\ g &= [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h \Leftrightarrow x_i - y_i \in \mathbf{I}\ \ (i = 1, 2, 3)), \forall g, h \in \mathbf{L}. \end{aligned}$$

Baker *et al.* [1] use $(O = (0, 0, 1)\,, U = (1, 0, 0)\,, V = (0, 1, 0)\,, E = (1, 1, 1))$ as a coordinatization 4-gon. We stick to this notation throughout this paper. For more detailed information about the coordinatization see [1] and [2].

Now it is time to give the following theorem from [1].

Theorem 3. $\mathbf{M}(\mathbf{R})$ *is an MK-plane, and each MK-plane is isomorphic to some* $\mathbf{M}(\mathbf{R})$.

Let $\mathbf{A}$ be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1 b_1 + (a_1 b_2 + a_2 b_1)\varepsilon, \quad (a_i, b_i \in \mathbf{A},\ i = 1, 2)$$

Then $\mathcal{A}$ is a local alternative ring with ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of $\mathcal{A}$ is denoted as $\mathbf{I}^{-1}$. Calculations with the elements of $\mathbf{I}^{-1}$ are defined as follows [3]:

$(a\varepsilon)^{-1} + t := (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1}$, $q(a\varepsilon)^{-1} := (aq^{-1}\varepsilon)^{-1}$, $(a\varepsilon)^{-1} q := (q^{-1}a\varepsilon)^{-1}$, $\left((a\varepsilon)^{-1}\right)^{-1} = a\varepsilon$ where $(a\varepsilon)^{-1} \in \mathbf{I}^{-1}$, $t \in \mathcal{A}$, $q \in \mathcal{A} \setminus \mathbf{I}$. (Other terms are not defined.). For more information about $\mathcal{A}$ and its relation to MK-planes, the reader is referred to the papers of Blunck [3] and [4]. In [4], the centre $\mathbf{Z}(\mathcal{A})$ is defined to be the (commutative, associative) subring of $\mathcal{A}$ which is commuting and associating with all elements of $\mathcal{A}$. It is $\mathbf{Z}(\mathcal{A}) := \mathbf{Z}(\varepsilon) = \mathbf{Z} + \mathbf{Z}\varepsilon$ where $\mathbf{Z} = \{z \in \mathbf{A} \mid za = az,\ \forall a \in \mathbf{A}\}$ is the centre of $\mathbf{A}$. If $\mathbf{A}$ is not associative, then $\mathbf{A}$ is a Cayley division algebra over its centre $\mathbf{Z}$ see [11] or [13]. Throughout we assume char $\mathbf{A} \neq \mathbf{2}$ and also we restrict ourselves to the MK-plane $\boldsymbol{M}(\mathcal{A})$.

Throughout this paper we denote the conjugacy relation by $\equiv$ ($a \equiv b :\Leftrightarrow \exists$ unit c, $a = c^{-1}bc$). It is known that $\equiv$ is an equivalence relation on every alternative field $\mathbf{A}$ [8]. Moreover, $\equiv$ is an equivalence relation also on $\mathcal{A}$ ([3], behind Theorem 1).

3 Cross-ratios in $\boldsymbol{M}(\mathcal{A})$

Ferrar [8] gives the following algebraic definition of the cross-ratio for the points on the line $[0, 0]$ in Moufang planes coordinated by the alternative field $\mathbf{A}$.

$$(A, B; C, D) = (a, b; c, d) := < \left((a - d)^{-1}(b - d)\right)\left((b - c)^{-1}(a - c)\right) >$$

where $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$, $D = (d, 0)$. Here, $< x >$ denotes the conjugacy class of x in $\mathbf{A}$, i.e. $< x > = \{y^{-1}xy \mid y \in \mathbf{A}\}$. In the definition of the cross-ratio, if any one of the points A, B, C, D is ∞, the factors involving ∞ are cancelled.

Let $A = (0, a, 1)$, $B = (0, b, 1)$, $C = (0, c, 1)$, $D = (0, d, 1)$, $Z = (0, 1, z) \in g := OV = [1, 0, 0]$ be pairwise non-neighbour points (Notice that, $z \in \mathbf{I}$). The cross-ratio for the points is defined as follows (see [4]):

$$\begin{aligned}
(A, B; C, D) &: = (a, b; c, d) = < \left((a - d)^{-1}(b - d)\right)\left((b - c)^{-1}(a - c)\right) > \\
(Z, B; C, D) &: = \left(z^{-1}, b; c, d\right) = < \left((1 - dz)^{-1}(b - d)\right)\left((b - c)^{-1}(1 - cz)\right) > \\
(A, Z; C, D) &: = \left(a, z^{-1}; c, d\right) = < \left((a - d)^{-1}(1 - dz)\right)\left((1 - cz)^{-1}(a - c)\right) > \\
(A, B; Z, D) &: = \left(a, b; z^{-1}, d\right) = < \left((a - d)^{-1}(b - d)\right)\left((1 - zb)^{-1}(1 - za)\right) > \\
(A, B; C, Z) &: = \left(a, b; c, z^{-1}\right) = < \left((1 - za)^{-1}(1 - zb)\right)\left((b - c)^{-1}(a - c)\right) >
\end{aligned}$$

Calculations with the elements of $\mathbf{I}^{-1}$ have been given in Section 2.

In [3, Theorem 2], it is shown that the transformations

$$\begin{aligned} t_u(x) &= x+u;\ u\in\mathcal{A} \\ r_u(x) &= xu;\ u\in\mathcal{A}\setminus\mathbf{I} \\ i(x) &= x^{-1} \\ l_u(x) &= ux=\left(ir_u^{-1}i\right)(x);\ u\in\mathcal{A}\setminus\mathbf{I} \end{aligned}$$

which are defined on the line g preserve cross-ratios. In [2, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by Λ, equals to the group of projectivities of a line in $\boldsymbol{M}(\mathcal{A})$.

The following lemma gives another statement of the definition of the cross-ratio.

Lemma 4. *([3]) Let $A=(0,a,1)$, $B=(0,b,1)$, $C=(0,c,1)$, $D=(0,d,1)\in g$ be pairwise non-neighbour points. Then*

$$(A,B;C,D)=<\left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)^{-1}-(a-c)^{-1}\right)^{-1}>.$$

The next theorem, analogous to Theorem 2 in [8] for Moufang planes, tells an important result about cross-ratio in $\mathcal{A}$.

Theorem 5. *([3]) Every cross-ratio consists only of elements of $\mathcal{A}\backslash(\{0,1\}+\mathbf{I})$. Conversely, the conjugacy class of any such element appears as a cross-ratio; Given three pairwise non-neighbour points A,B,C and an element $r\in\mathcal{A}\setminus(\{0,1\}+\mathbf{I})$, then there is a (unique if $r\in\mathbf{Z}(\varepsilon)$) point $D\nsim A$, B, C with $(A,B;C,D)=<r>$.*

For any $x\in\mathcal{A}$, $<x>^{-1}$ and $1-<x>$ are defined by obvious way as $<x^{-1}>$ and $<1-x>$ respectively. In this situation, we can give the following results (which were discovered by Möbius for the real projective plane [9, p. 152]) which are implicitly given in [3, Corollary 1 and Lemma 8]:

$$\begin{aligned} (A,B;C,D) &= (B,A;D,C)=(C,D;A,B)=(D,C;B,A)=<w> \qquad (1)\\ (B,A;C,D) &= (A,B;D,C)=(D,C;A,B)=(C,D;B,A)=<w>^{-1} \\ (A,C;B,D) &= (B,D;A,C)=(C,A;D,B)=(D,B;C,A)=1-<w> \\ (B,C;A,D) &= (A,D;B,C)=(D,A;C,B)=(C,B;D,A)=1-<w>^{-1} \\ (C,A;B,D) &= (D,B;A,C)=(A,C;D,B)=(B,D;C,A)=<1-w>^{-1} \\ (C,B;A,D) &= (D,A;B,C)=(A,D;C,B)=(B,C;D,A)=<1-w^{-1}>^{-1} \end{aligned}$$

where $w\in(A,B;C,D)$. Hence, there exist at most six different values of the cross-ratio depending on the order of the points.

Now, we can extend the definition of the cross-ratio to the whole plane $\boldsymbol{M}(\mathcal{A})$ by the following definition.

Definition 6. *Let $\{O,U,V,E\}$ be the basis of $\boldsymbol{M}(\mathcal{A})$. Then*

(a) *If $l=[m,1,p]$ and A, B, C, D are pairwise non-neighbour points of l, then*

(i) *If $l\nsim U$, then $m\notin\mathbf{I}$. In this case, the cross-ratio $(A,B;C,D):=(\sigma(A),\sigma(B);\sigma(C),\sigma(D))$ where $\sigma=\sigma_U(g,l)$.*

(ii) *If* $l \sim U$, *then* $m \in \mathbf{I}$. *In this case, the cross-ratio* $(A, B; C, D) := (\sigma(A), \sigma(B); \sigma(C), \sigma(D))$ *where* $\sigma = \sigma_{(1,1,0)}(g, l)$.

(b) *If* $l = [1, n, p]$, *then the cross-ratio* $(A, B; C, D) := (\sigma(A), \sigma(B); \sigma(C), \sigma(D))$ *where* $\sigma = \sigma_U(g, l)$.

(c) *If* $l = [q, n, 1]$, *then the cross-ratio* $(A, B; C, D) := (\sigma(A), \sigma(B); \sigma(C), \sigma(D))$ *where* $\sigma = \sigma_{(1,0,1)}(g, l)$.

Thus we have the following lemma.

Lemma 7. *Let* σ *be the perspectivity in Definition 6. Then,*

(a) *Let* $l = [m, 1, p]$ *and* $A = (a, am + p, 1) \nsim UV$, $K = (1, m + kp, k) \sim UV$ *be the points of* l,

(i) *If* $m \notin \mathbf{I}$ *then* $\sigma(A) = (0, am + p, 1)$, $\sigma(K) = (0, 1, m^{-1}k)$,

(ii) *If* $m \in \mathbf{I}$ *then* $\sigma(A) = (0, a(m-1) + p, 1)$ *and*

$\sigma(K) = \left(0, 1, (m-1)^{-1} k\right)$.

(b) *Let* $l = [1, n, p]$ *and* $A = (an + p, a, 1) \nsim V$, $K = (kp + n, 1, k) \sim V$ *be the points of* l. *Then* $\sigma(A) = (0, a, 1)$, $\sigma(K) = (0, 1, k)$.

(c) *Let* $l = [q, n, 1]$ *and* $A = (1, a, q + an) \nsim V$, $K = (k, 1, kq + n) \sim V$ *be the points of* l. *Then* $\sigma(A) = \left(0, -[1 - (q + an)]^{-1} a, 1\right)$, $\sigma(K) = (0, 1, k(q-1) + n)$.

Proof. We give a detailed proof for only case (i) since the method for the others is the same. Immediately,

$$\begin{aligned} \sigma(A) &= AU \cap g \\ &= (a, am + p, 1)(1, 0, 0) \cap [1, 0, 0] \\ &= [0, 1, am + p] \cap [1, 0, 0] \\ &= (0, am + p, 1) \end{aligned}$$

and

$$\begin{aligned} \sigma(K) &= KU \cap g \\ &= (1, m + kp, k)(1, 0, 0) \cap [1, 0, 0] \\ &= \left[0, m^{-1}k, 1\right] \cap [1, 0, 0] \\ &= \left(0, 1, m^{-1}k\right) \end{aligned}$$

∎

This lemma enables us to make easily some calculations in the proof of the next theorem.

The following theorem we will prove, states a simple way for the calculation of the cross-ratio of the points on a line l in $\boldsymbol{M}(\mathcal{A})$.

Theorem 8. *According to types of lines, the cross-ratio of the points on the line l can be calculated as follows:*

If A, B, C, D and S are the pairwise non-neighbour points

(a) *of the line $l = [m, 1, p]$ where $A = (a, am + p, 1)$, $B = (b, bm + p, 1)$, $C = (c, cm + p, 1)$, $D = (d, dm + p, 1)$ are not near the line UV and $S = (1, m + sp, s) \sim UV$,*

(b) *of the line $l = [1, n, p]$ where $A = (an + p, a, 1)$, $B = (bn + p, b, 1)$, $C = (cn + p, c, 1)$, $D = (dn + p, d, 1)$ are not neighbour to V and $S = (n+sp, 1, s) \sim V$,*

(c) *of the line $l = [q, n, 1]$ where $A = (1, a, q + an)$, $B = (1, b, q + bn)$, $C = (1, c, q + cn)$, $D = (1, d, q + dn)$ are not near to V and $S = (s, 1, sq + n) \sim V$,*

then

$$\begin{aligned} (A, B; C, D) &= (a, b; c, d) \\ (S, B; C, D) &= \left(s^{-1}, b; c, d\right) \\ (A, S; C, D) &= \left(a, s^{-1}; c, d\right) \\ (A, B; S, D) &= \left(a, b; s^{-1}, d\right) \\ (A, B; C, S) &= \left(a, b; c, s^{-1}\right) \end{aligned}$$

Proof. We separate the proof into three cases as given in the statement of the theorem.

Case (a). There are two cases where $m \in \mathbf{I}$ and $m \notin \mathbf{I}$.

(a.1). If $m \in \mathbf{I}$, then under the perspectivity $\sigma_{(1,1,0)}(g, [m, 1, p])$, the points A, B, C, D and S transform to $A' = (0, a(m-1) + p, 1)$, $B' = (0, b(m-1) + p, 1)$, $C' = (0, c(m-1) + p, 1)$, $D' = (0, d(m-1) + p, 1)$ and $S' = \left(0, 1, (m-1)^{-1} s\right)$, respectively. Therefore, with $\gamma = r_{(m-1)^{-1}} \circ t_{-p} \in \Lambda$, we get

$$(A, B; C, D) = (A', B'; C', D')$$

$$= (a(m-1) + p, b(m-1) + p; c(m-1) + p, d(m-1) + p)$$

$$= (\gamma(a(m-1) + p), \gamma(b(m-1) + p); \gamma(c(m-1) + p), \gamma(d(m-1) + p))$$

$$= (a, b; c, d)$$

and

$$(S, B; C, D) = (S', B'; C', D')$$

$$= (s^{-1}(m-1), b(m-1) + p; c(m-1) + p, c(m-1) + p)$$

$$= (\gamma(s^{-1}(m-1)), \gamma(b(m-1) + p); \gamma(c(m-1) + p), \gamma(c(m-1) + p))$$

$$= (s^{-1}, b; c, d)$$

and similarly

$$\begin{aligned}(A,S;C,D) &= (a,s^{-1};c,d)\\(A,B;S,D) &= (a,b;s^{-1},d)\\(A,B;C,S) &= (a,b;c,s^{-1}).\end{aligned}$$

(a.2). If $m \notin \mathbf{I}$, then under the perspectivity $\sigma_U(g,[m,1,p])$, the points A, B, C, D and S transform to $A' = (0, am+p, 1)$, $B' = (0, bm+p, 1)$, $C' = (0, cm+p, 1)$, $D' = (0, dm+p, 1)$ and $S' = (0, 1, m^{-1}s)$ respectively. Therefore, with $\gamma = r_{m^{-1}} \circ t_{-p} \in \Lambda$, we have

$$\begin{aligned}(A,B;C,D) &= (A',B';C',D') = (am+p, bm+p; cm+p, dm+p)\\&= (\gamma(am+p), \gamma(bm+p); \gamma(cm+p), \gamma(dm+p))\\&= (a,b;c,d)\end{aligned}$$

and

$$\begin{aligned}(S,B;C,D) &= (S',B';C',D') = \left(s^{-1}m, bm+p; cm+p, dm+p\right)\\&= \left(\gamma(s^{-1}m), \gamma(bm+p); \gamma(cm+p), \gamma(dm+p)\right)\\&= (s^{-1},b;c,d)\end{aligned}$$

also similarly

$$\begin{aligned}(A,S;C,D) &= (a,s^{-1};c,d)\\(A,B;S,D) &= (a,b;s^{-1},d)\\(A,B;C,S) &= (a,b;c,s^{-1}).\end{aligned}$$

Case (b). Let $A = (an+p, a, 1)$, $B = (bn+p, b, 1)$, $C = (cn+p, c, 1)$, $D = (dn+p, d, 1)$ and $S = (n+sp, 1, s)$ be pairwise non-neighbour points of $[1,n,p]$. Then under the perspectivity $\sigma_U(g,[1,n,p])$, the points A, B, C, D and S transform to $A' = (0,a,1)$, $B' = (0,b,1)$, $C' = (0,c,1)$, $D' = (0,d,1)$ and $S' = (1,0,s)$, respectively. Therefore,

$$\begin{aligned}(A,B;C,D) &= (A',B';C',D') = (a,b;c,d)\\(S,B;C,D) &= (S',B';C',D') = (s^{-1},b;c,d)\end{aligned}$$

and similarly

$$\begin{aligned}(A,S;C,D) &= (a,s^{-1};c,d)\\(A,B;S,D) &= (a,b;s^{-1},d)\\(A,B;C,S) &= (a,b;c,s^{-1}).\end{aligned}$$

Case (c). Let $A = (1, a, q+an)$, $B = (1, b, q+bn)$, $C = (1, c, q+cn)$, $D = (1, d, q+dn)$ and $S = (s, 1, sq+n)$ be pairwise non-neighbour points of $[q,n,1]$ where A, B, C, $D \nsim V$, $S \sim V$. Then under the perspectivity $\sigma_{(1,0,1)}(g,[q,n,1])$, the points A, B, C, D and S transform to $A' = (0, -(1-(q+an))^{-1}a, 1)$, $B' = (0, -(1-(q+bn))^{-1}b, 1)$, $C' = (0, -(1-(q+cn))^{-1}c, 1)$,

$D' = (0, -(1-(q+dn))^{-1}d, 1)$ and $S' = (0, 1, s\,(q-1)+n)$, respectively. Therefore, with $\gamma = i \circ r_{(1-q)^{-1}} \circ t_n \circ i \circ l_{-1} \in \Lambda$, we have

$$\begin{aligned}(A, B; C, D) &= (A', B'; C', D') \\ &= (-(1-(q+an))^{-1} a, -(1-(q+bn))^{-1} b; \\ &\quad -(1-(q+cn))^{-1} c, -(1-(q+dn))^{-1} d) \\ &= (\gamma(-(1-(q+an))^{-1} a), \gamma(-(1-(q+bn))^{-1} b); \\ &\quad \gamma(-(1-(q+cn))^{-1} c), \gamma(-(1-(q+dn))^{-1} d)) \\ &= (a, b; c, d)\end{aligned}$$

and

$$\begin{aligned}(S, B; C, D) &= (S', B'; C', D') \\ &= ((s\,(q-1)+n)^{-1}, -(1-(q+bn))^{-1} b; \\ &\quad -(1-(q+cn))^{-1} c, -(1-(q+dn))^{-1} d) \\ &= (\gamma((s\,(q-1)+n)^{-1}), \gamma(-(1-(q+bn))^{-1} b); \\ &\quad \gamma(-(1-(q+cn))^{-1} c), \gamma(-(1-(q+dn))^{-1} d)) \\ &= (s^{-1}, b; c, d)\end{aligned}$$

and similarly

$$\begin{aligned}(A, S; C, D)) &= (a, s^{-1}; c, d) \\ (A, B; S, D) &= (a, b; s^{-1}, d) \\ (A, B; C, S) &= (a, b; c, s^{-1}).\end{aligned}$$

■

As a result of this theorem, one can easily compute the cross-ratio of any four pairwise non-neighbour collinear points because of the following facts:

(i) By the results of Case (a), the cross-ratio of the points on the line $[m, 1, p]$ can be calculated by using the first coordinates of the points not near line UV and the last coordinate's inverse of the point near UV.

(ii) By the results of Case (b), the cross-ratio of the points on the line $[1, n, p]$ can be calculated by using the second coordinates of the points not neighbour V and the last coordinate's inverse of the point neighbour V.

(iii) By the results of Case (c), the cross-ratio of the points on the line $[q, n, 1]$ can be calculated by using the first coordinate's inverse of the point neighbour V and the second coordinates of the points not neighbour V.

Theorem 9. *In $\boldsymbol{M}(\mathcal{A})$, perspectivities preserve cross-ratios.*

Proof. Let A, B, C, D be pairwise non-neighbour points of a line l in $\boldsymbol{M}(\mathcal{A})$, $\sigma_M\,(g, l)$ be the perspectivity given in Definition 6 i.e.

$$(A, B; C, D) = (\sigma_M\,(A)\,, \sigma_M\,(B)\,; \sigma_M\,(C)\,, \sigma_M\,(D))$$

and $\sigma_N(g,l)$ be a perpectivity such that $N \not\sim M$, $N \not\sim l$, $N \not\sim g$. It is sufficient to show that $\sigma_N(g,l)$ preserves cross-ratio. Since $\sigma = \sigma_M \sigma_N^{-1}$ is a projectivity of g, it preserves cross-ratio [3]. Thus

$$\begin{aligned}(\sigma_N(A), \sigma_N(B); \sigma_N(C), \sigma_N(D)) &= (\sigma_M(A), \sigma_M(B); \sigma_M(C), \sigma_M(D)) \\ &= (A, B; C, D)\end{aligned}$$ ■

Now we can state the following corollary as a result of this theorem.

Corollary 10. *Cross-ratios are preserved by projectivities.*

Now we give a definition in $\boldsymbol{M}(\mathcal{A})$, well known from the case of Moufang planes [8].

Definition 11. *In $\boldsymbol{M}(\mathcal{A})$, any pairwise non-neighbour four points A, B, C, $D \in l$ are called as harmonic if $(A, B; C, D) =< -1 >$ and we let $h(A, B, C, D)$ represent the statement: A, B, C, D are harmonic. Let l be any line in $\boldsymbol{M}(\mathcal{A})$. Then the pairwise non-neighbour points A, B, C, D of the line l are called to be in harmonic position if there exists 4-gon (P_1, P_2, Q_1, Q_2) such that $P_1P_2 \cap Q_1Q_2 = A$, $P_1Q_2 \cap P_2Q_1 = B$, $P_1Q_1 \cap l = C$, $P_2Q_2 \cap l = D$ (see Figure 1). We let $H(A, B, C, D)$ represent the statement: A, B, C, D are in harmonic position.*

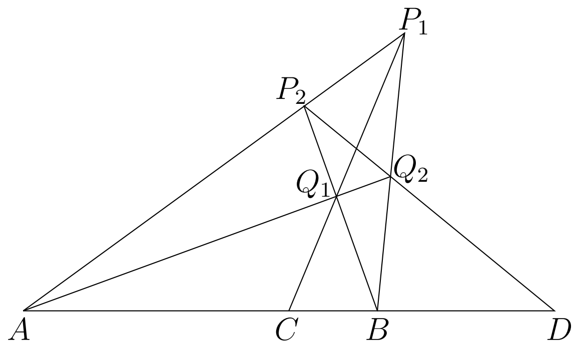

Figure 1

Now, we give two lemmas and a theorem which are necessary for the proof of Theorem 16.

Lemma 12. *In $\boldsymbol{M}(\mathcal{A})$ if f_1, f_2 are elations with center P and $f_1(Q) = f_2(Q) \neq Q$ for some points $Q \not\sim P$ then $f_1|_{PQ} = f_2|_{PQ}$.*

Proof. Suppose that f_1 is an elation with center P, axis l_1 and that $f_1 : Q, R \to A, B$ where $R \in PQ$. Suppose also that f_2 is an elation with center P, axis l_2 and that $f_2 : Q, R \to A, C$. Let $d := PQ$ then $P, Q, R, A, B, C \in d$. Let T_1 and T_2 be the points of l_1 and l_2, respectively such that $T_1 \not\sim P \not\sim T_2$ and $S := T_1T_2 \cap d$. Then there exists an elation h with center S, axis d, such that $h(T_1) = T_2$. Therefore $h(l_1) = l_2$ and h^{-1} is an elation with center S, axis d. It is easily verified that $j = h \circ f_1 \circ h^{-1}$ is an elation with center P, axis l_2 and it maps $Q \to A$, $R \to B$.One can easily show that an elation is completely determined by its center, its axis and the image of one point which is not neighbour to the axis. Therefore we have $j = f_2$ which means $B = C$. ■

Lemma 13. *In $\boldsymbol{M}(\mathcal{A})$ if $H(A,B,C,D)$ then there exist a line l and an elation f with center A, axis l, such that $f : C, B \to B, D$.*

Proof. Since $H(A,B,C,D)$, we have $A \nsim B$ and therefore there exists a unique line $d = AB$ and there is a 4-gon $(P_1,\ P_2,\ Q_1,\ Q_2)$ such that $P_1P_2 \cap Q_1Q_2 = A$, $P_1Q_2 \cap P_2Q_1 = B$, $P_1Q_1 \cap l = C$ and $P_2Q_2 \cap l = D$ (see Figure 1). Let $l = P_1P_2$ and f be an elation with center A, axis l such that $f : C \to B$. Then we have $f(CP_1) = f(C)\, f(P_1) = BP_1$ and therefore $f(Q_1) = f(Q_1Q_2 \cap CP_1) = f(Q_1Q_2) \cap f(CP_1) = Q_1Q_2 \cap BP_1 = Q_2$ and finally we have $f(B) = f(d \cap P_2Q_1) = f(d) \cap f(P_2Q_1) = d \cap P_2Q_2 = D$. ■

Theorem 14. *In $\boldsymbol{M}(\mathcal{A})$ if $H(A,B,C,D)$ and $H(A,B,C,D')$ then $D = D'$.*

Proof. If $H(A,B,C,D)$ and $H(A,B,C,D')$ then, by Lemma 13, there exist a line l and an elation f_1 with center A, axis l such that $f_1 : C, B \to B, D$ and also there exist a line d and an elation f_2 with center A, axis d such that $f_2 : C, B \to B, D'$. Then, by Lemma 12, $f_1|_{AB} = f_2|_{AB}$ and therefore we have $f_1(B) = f_2(B)$ i.e. $D = D'$. ■

Corollary 15. *If A, B, C are pairwise non-neighbour points of g and D is constructed from A, B, C, P_1, P_2 where P_1, $P_2 \nsim g$ and $P_1 \nsim P_2$ via the configuration in the Figure 1, the point D is uniquely determined by A, B, C. That is, the point D is independent of the choice of P_1 and P_2.*

Now we give a theorem for the points of the line g. This theorem will be valid for any line l if we can, in future, show that the collineations which are given in [6] and were used in showing the transitivity on 4-gons, can preserve the cross-ratio.

Theorem 16. *In $\boldsymbol{M}(\mathcal{A})$, $H(A,B,C,D)$ if and only if $h(A,B,C,D)$ where A, B, C, $D \in g$.*

Proof. Suppose that the points A, B, C, $D \in g$ are in harmonic position and also first none of them is near to V. Then we will show that $(A,B;C,D) =< -1 >$. Let $A = (0,a,1)$, $B = (0,b,1)$, $C = (0,c,1)$, $D = (0,d,1)$. Without loss of generality, by Corollary 15, we may assume that $P_1 = U = (1,0,0)$ and $P_2 = (1,a,1)$. Then $BP_2 = [a-b,1,b]$, $BP_2 \cap CP_1 = G = \left((c-b)(a-b)^{-1}, c, 1\right)$ and

$$\begin{aligned} BP_1 \cap AG &= F = \left((b-a)\left((c-a)^{-1}\left((c-b)(a-b)^{-1}\right)\right), b, 1\right) \\ &= \left((b-a)\left((c-a)^{-1}\left((c-a+a-b)(a-b)^{-1}\right)\right), b, 1\right) \\ &= \left(-1+(b-a)(c-a)^{-1}, b, 1\right) \end{aligned}$$

Since $P_2D = [a-d,1,d]$ and $F \in P_2D$, we have

$$\left((b-a)\left((c-a)^{-1}\left((c-b)(a-b)^{-1}\right)\right)\right)(a-d)+d = b$$

$$\Rightarrow (b-a)\left((c-a)^{-1}\left((c-b)(a-b)^{-1}\right)\right) = (b-d)(a-d)^{-1}$$

$$\Rightarrow (c-a)^{-1}\left((c-b)(a-b)^{-1}\right) = (b-a)^{-1}\left((b-d)(a-d)^{-1}\right)$$

$$\Rightarrow (c-a)^{-1}\left((c-a+a-b)(a-b)^{-1}\right) = (b-a)^{-1}\left((b-a+a-d)(a-d)^{-1}\right)$$

$$\Rightarrow (c-a)^{-1}\left((c-a)(a-b)^{-1}+1\right) = (b-a)^{-1}\left((b-a)(a-d)^{-1}+1\right)$$

$$\Rightarrow (a-b)^{-1}+(c-a)^{-1} = (a-d)^{-1}+(b-a)^{-1}$$

$$\Rightarrow (a-b)^{-1}-(a-c)^{-1} = (a-d)^{-1}-(a-b)^{-1}$$

$$\Rightarrow (a-b)^{-1}-(a-c)^{-1} = -\left((a-b)^{-1}-(a-d)^{-1}\right)$$

$$\Rightarrow \left((a-b)^{-1}-(a-d)^{-1}\right)\left((a-b)^{-1}-(a-c)^{-1}\right)^{-1} = -1$$

and the last equality (by Lemma 4) means that $h(A,B,C,D)$.

If $P_2F \cap g \sim V$ then $D = (0,1,z)$ and

$$P_2F = [1, -z, 1+az]$$

where $z = \left(-2(a-b)^{-1} - (c-a)^{-1}\right) \in \mathbf{I}$. In this case by the definition of the cross-ratio, we get

$$\begin{aligned}
(A,B;C,D) &= (a,b;c,z^{-1}) \\
&= <\left((1-za)^{-1}(1-zb)\right)\left((b-c)^{-1}(a-c)\right)> \\
&= <((1+za)(1-zb))\left((b-c)^{-1}(a-c)\right)> \\
&= <(1+z(a-b))\left((b-c)^{-1}(a-c)\right)> \\
&= <(1+z(a-b))\left((b-c)^{-1}(a-c)\right)>
\end{aligned}$$

and then by substituting z in the last equality we obtain

$$\begin{aligned}
&=<\left(1+\left(-2(a-b)^{-1}-(c-a)^{-1}\right)(a-b)\right)\left((b-c)^{-1}(a-c)\right)> \\
&=<-\left(1+(c-a)^{-1}(a-b)\right)\left((b-c)^{-1}(a-c)\right)> \\
&=<-\left(1+(c-a)^{-1}((a-c)+(c-b))\right)\left((b-c)^{-1}(a-c)\right)> \\
&=<-\left((c-a)^{-1}(c-b)\right)\left((b-c)^{-1}(a-c)\right)> \\
&=<-\left((c-a)^{-1}(b-c)\right)\left((b-c)^{-1}(c-a)\right)> \\
&=<-1>
\end{aligned}$$

This means that $h(A,B,C,D)$ even if D is near to V. If any one of the points A, B, C is near to V then the proof of this part follows from (1).

Conversely, let $h(A,B,C,D)$. Existence of the point D' such that $H(A,B,C,D')$ is obvious from Definition 11. Then $H(A,B,C,D')$ implies $h(A,B,C,D')$ (from the first part of the theorem). So, we have $h(A,B,C,D')$ and $h(A,B,C,D)$. Finally, by Theorem 5, we have $D = D'$ which gives $H(A,B,C,D)$. ∎

In the next section we will have a look at 6-figures in $\boldsymbol{M}(\mathcal{A})$ in view of the results of [5] and [7].

4 6-figures in $\boldsymbol{M}(\mathcal{A})$

In the final section, some properties of 6-figures that are well-known for Desarguesian planes or Moufang plane are investigated in $\boldsymbol{M}(\mathcal{A})$. We start with the definition of a 6-figure.

A *6-figure* is a sequence of six non-neighbour points $(ABC, A_1B_1C_1)$ such that (A, B, C) is a 3-gon, and $A_1 \in BC, B_1 \in CA, C_1 \in AB$. The points A, B, C, A_1, B_1, C_1 are called vertices of this 6-figure. The 6-figures $(ABC, A_1B_1C_1)$ and $(DGF, D_1G_1F_1)$ are *equivalent* if there exists a collineation of $\boldsymbol{M}(\mathcal{A})$ which transforms A, B, C, A_1, B_1, C_1 to D, G, F, D_1, G_1, F_1 respectively.

Now, we recall following two theorems from [6], which are very important for this section and are analogous to the theorems given in [5] for Desarguesian planes and in [7] for Moufang planes.

Theorem 17. *([6, Theorem 4]) Let $\mu = (ABC, A_1B_1C_1)$ be a 6-figure in $\boldsymbol{M}(\mathcal{A})$. Then, there is an $m \in \mathcal{A}\backslash\mathbf{I}$ such that μ is equivalent to $(UVO, (0,1,1)(1,0,1)(1,m,0))$ where $U = (1,0,0)$,$V = (0,1,0)$,$O = (0,0,1)$ are elements of the coordinatization basis of $\boldsymbol{M}(\mathcal{A})$.*

Theorem 18. *([6, Theorem 5]) The 6-figures $(ABC, A_1B_1C_1)$, $(BCA, B_1C_1A_1)$, $(CAB, C_1A_1B_1)$ are equivalent.*

Now we can obtain results related to ratios of 6-figures, which are analogous to results given in [5] for Desarguesian planes and in [7] for Moufang planes. So we give the definition of ratio of a 6-figure. But first we need another definition and a lemma.

Let $\mu = (ABC, A_1B_1C_1)$ be a 6-figure in $\boldsymbol{M}(\mathcal{A})$. Let $A^c = BC \cap B_1C_1$, $B^c = CA \cap C_1A_1$, $C^c = AB \cap A_1B_1$. The 6-figure $(ACB, A^cC^cB^c)$ is called the *first codescendant* of μ, written μ^c. μ is called a *first coancestor* of μ^c (see Figure 2).

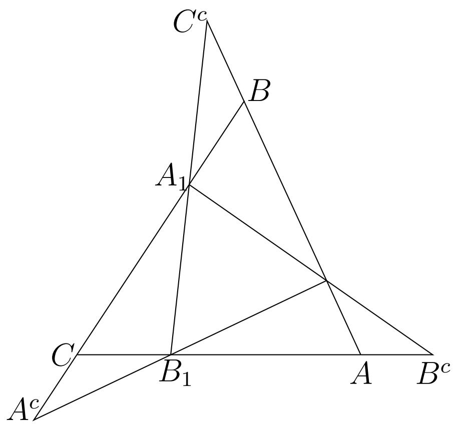

Figure 2

Lemma 19. *If $\mu = (ABC, A_1B_1C_1) = (UVO, (0,1,1)(1,0,1)(1,m,0))$, then*

$$(A, B; C_1, C^c) = (B, C; A_1, A^c) = (C, A; B_1, B^c) =< -m > .$$

Proof. The points A^c, B^c and C^c are respectively $(0,-m,1)$, $(-m^{-1},0,1)$ and $(1,-1,0)$. In this case the proof follows from Theorem 8. ■

We are now ready to state the definition of the ratio of a 6-figure. The conjugacy class $-(A,B;C_1,C^c)$ is called *the ratio of the 6-figure* $\mu=(ABC,A_1B_1C_1)$ and denoted by $r(\mu)$, that is, $r(\mu)=<m>$.

Now we give other definitions from [5] to introduce the remaining results of this section.

Let $\mu=(ABC,A_1B_1C_1)$ be a 6-figure in $\boldsymbol{M}(\mathcal{A})$. By Theorem 5, there exist unique points $A_2\in BC$, $B_2\in CA$, $C_2\in AB$ such that $h(A,B,C_1,C_2)$, $h(B,C,A_1,A_2)$, $h(C,A,B_1,B_2)$. The 6-figure $(ABC,A_2B_2C_2)$ is called the *conjugate* of μ, having symbol $-\mu$. Likewise μ is the conjugate of $-\mu$.

Let $C^d\in AB$ be the point such that C, C^d and $AA_1\cap BB_1$ are collinear. Let $A^d\in BC$ and $B^d\in CA$ be the points such that A, A^d and $BB_1\cap CC_1$ are collinear and B, B^d and $AA_1\cap CC_1$ are collinear. The 6-figure $(ACB,A^dC^dB^d)$ is called the *first descendant* of μ, written μ^d. μ is called a *first ancestor* of μ^d (see Figure 3).

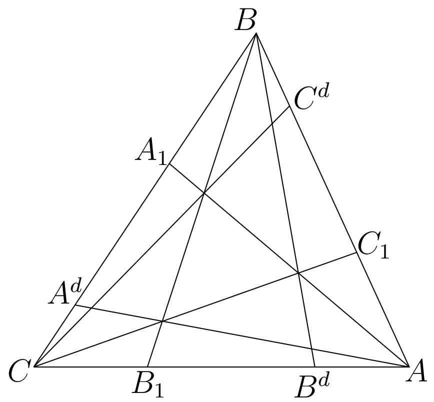

Figure 3

Using the definitions of $-\mu$, μ^c and μ^d the following lemma is obtained.

Lemma 20. *For any 6-figure μ we have*

(a) $(-\mu)^d=\mu^d$

(b) $\left(\mu^d\right)^c=(\mu^c)^c=(UVO,(0,-m^{-1},1)(-m,0,1)(1,-m^2,0))$

where $m\in\mathcal{A}\setminus\mathbf{I}$.

Proof. By Theorem 17, we may assume that $\mu=(UVO,(0,1,1)(1,0,1)(1,m,0))$ where $m\in\mathcal{A}\setminus\mathbf{I}$. Then we get

$$\begin{aligned}-\mu &= (UVO,(0,-1,1)(-1,0,1)(1,-m,0)),\\ \mu^d &= \left(UOV,(0,m,1)(1,1,0)(m^{-1},0,1)\right),\\ \mu^c &= \left(UOV,(0,-m,1)(1,-1,0)(-m^{-1},0,1)\right).\end{aligned}$$

(a) It is clear that $(-\mu)^d=\mu^d=(UOV,(0,m,1)(1,1,0)(m^{-1},0,1))$.

(b) We obtain that the first codescendant of μ^d i.e. $\left(\mu^d\right)^c$ is equal to the first codescendant of μ^c i.e. $(\mu^c)^c$. That is

$$\left(\mu^d\right)^c = (\mu^c)^c = \left(UVO, (0, -m^{-1}, 1)(-m, 0, 1)(1, -m^2, 0)\right). \qquad \blacksquare$$

$(ABC, A_1B_1C_1)$ is called a *Menelaus* 6-figure if A_1, B_1 and C_1 are collinear (see Figure 4); and $(ABC, A_1B_1C_1)$ is called a *Ceva* 6-figure if AA_1, BB_1 and CC_1 are concurrent (see Figure 5).

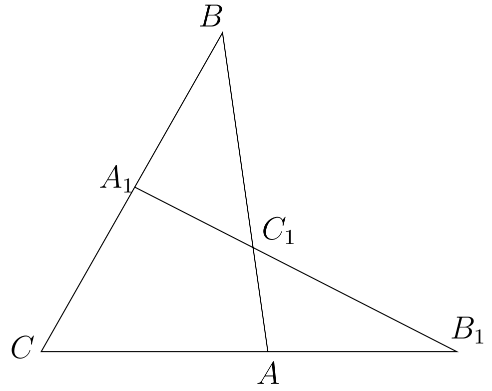

Figure 4

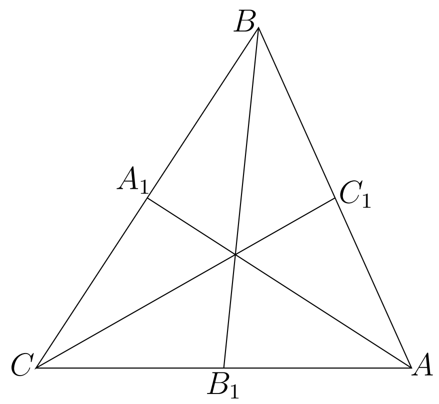

Figure 5

The following theorem is the analogue of the theorem given in [7] for Moufang planes.

Theorem 21. *μ is a Menelaus or Ceva 6-figure if and only if $r(\mu) = -1$ or $r(\mu) = 1$, respectively.*

Proof. By Theorem 17, we may assume that $\mu = (UVO, (0, 1, 1)(1, 0, 1)(1, m, 0))$ where $m \in \mathcal{A} \setminus \mathbf{I}$. If μ is Menelaus then the points $(0, 1, 1)$, $(1, 0, 1)$ and $(1, m, 0)$ are collinear, which means $m = -1$. Conversely, if $r(\mu) = -1$ for $\mu = (UVO, (0, 1, 1)(1, 0, 1)(1, m, 0))$ then $m = -1$. In this situation, the collinearity of $(0, 1, 1)$, $(1, 0, 1)$ and $(1, -1, 0)$ is obvious. If μ is Ceva then the lines $[0, 1, 1]$, $[1, 0, 1]$ and $[m, 1, 0]$ are concurrent, which means $m = 1$. Conversely, if $r(\mu) = 1$ for $\mu = (UVO, (0, 1, 1)(1, 0, 1)(1, m, 0))$ then $m = 1$. In this situation, the concurrence of $[0, 1, 1]$, $[1, 0, 1]$ and $[1, 1, 0]$ is clear. $\blacksquare$

Remark. There exist more open problems related to 6-figures. In [5] Cater gave a list of problems on the relations between geometric properties of Desarguesian planes and algebraic properties of their coordinatizing rings. In [7], Ciftci could carry over most of the results which are valid on Desarguesian planes to Moufang planes. In this paper we restrict ourselves to the results of [7]. So, more interesting and nice results can be obtained if one handles the other problems from Cater's list for Desarguesian, Moufang or MK-planes.

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Uludag University, Science and Art Faculty,
Department of Mathematics,
Gorukle, 16059, Bursa - TURKEY
E-mails:aakpinar@uludag.edu.tr, basri@uludag.edu.tr, sciftci@uludag.edu.tr