Asymptotic behavior of solutions to a perturbed ODE*

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Abstract

An existence result to infinite boundary-value problem (1) – (2) below is proved via Schauder-Tychonoff fixed point theorem.

1 Introduction

Last years, the boundary-value problems on infinite intervals have been treated especially for bounded or periodic solutions. In this field a different contribution is due to Jean Mawhin (see [8], [9], [10], [11]), who uses various topological methods (involving interesting applications of the topological degree theory). The reader can find in [1], [2], [3], [5], [8], [9], [10], [11], [12], [13] a rich bibliography in the study of the qualitative properties of the ODE of second order.

This Note is devoted to the existence of the solutions to the infinite boundary-value problem

$$x'' + 2f(t)\, x' + \beta(t)\, x + g(t, x) = 0, \qquad t \in \mathbb{R}_+, \tag{1}$$

$$x(\infty) = x'(\infty) = 0, \tag{2}$$

where f, $\beta : \mathbb{R}_+ \to \mathbb{R}$, and $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are three given functions, $\mathbb{R}_+ := [0, \infty)$, and

$$x(\infty) := \lim_{t\to\infty} x(t), \;\; x'(\infty) := \lim_{t\to\infty} x'(t).$$

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Equation (1) has been considered by different authors (see, e.g. [4], [6], [7], [14], [15], and the references therein). The most familiar interpretation of this equation is that it describes nonlinear oscillations (see [12], wherein the author presents a delightful history of the forced pendulum equation).

In [6], the authors have introduced a new method to study the stability of the null solution to equation (1), which is based on Schauder's fixed point theorem applied to an adequate operator H, built in the Banach space

$$C := \left\{ z : \mathbb{R}_+ \to \mathbb{R}^2,\ z \text{ continuous and bounded} \right\},$$

equipped with the usual norm $\|z\|_\infty := \sup_{t \in \mathbb{R}_+} \{\|z(t)\|\}$, where $\|\cdot\|$ represents a norm in $\mathbb{R}^2$.

In order to build the operator H one changes equation (1) to system

$$z' = A(t)\, z + G(t, z), \tag{3}$$

which is a perturbed system to

$$z' = A(t)\, z. \tag{4}$$

(Here A is a quadratic matrix 2×2, $z = (x, y)^\top$, and G is a function with values in $\mathbb{R}^2$; the expressions of A and G will be given in Section 3.)

In [14] we proved stability results for the null solution to (1), by using relatively classical arguments and in [15] we deduced the generalized exponential asymptotic stability of the trivial solution to the same equation, under more general assumptions, which required more sophisticated arguments (see Theorem 2.1 in [15]).

The purpose of the present paper is to answer to the following question: "How can we effectively use fixed point theory to prove that problem $(1) - (2)$ admits solutions ?" First we will show that for initial data small enough, equation (1) admits solutions defined on $\mathbb{R}_+$ and next we will prove that each such a solution fulfills boundary condition (2). Unlike [14] and [15], wherein the proof techniques are based on some Bernoulli type differential inequalities, we will apply, as in [4], Schauder-Tychonoff fixed point theorem in the Fréchet space

$$C_c := \left\{ z : \mathbb{R}_+ \to \mathbb{R}^2,\ z \text{ continuous} \right\},$$

endowed with a family of seminorms as chosen as to determine the convergence on compact subsets of $\mathbb{R}_+$. The proof is not too obvious because the fundamental matrix to system (4) can not be determined explicitly, as in the case when $\beta(t) = 1$, $\forall t \in \mathbb{R}_+$.

2 The main result

The following hypotheses will be required:

(i) $f \in C^1(\mathbb{R}_+)$ and $f(t) \geq 0$ for all $t \geq 0$;

(ii) $\int_0^\infty f(t)\, dt = \infty$;

(iii) there exists a constant $K \geq 0$, such that

$$\left| f'(t) + f^2(t) \right| \leq K f(t), \qquad \forall t \in \mathbb{R}_+; \tag{5}$$

(iv) $\beta \in C^1(\mathbb{R}_+)$, β is decreasing, and

$$\beta(t) \geq \beta_0 > K^2, \quad \forall t \in \mathbb{R}_+, \tag{6}$$

where β_0 is a constant;
(v) $g \in C(\mathbb{R}_+ \times \mathbb{R})$;
(vi) there exist two constants $M > 0$ and $\alpha > 1$, such that

$$|g(t,x)| \leq M f(t) |x|^{\alpha}, \qquad \forall x \in \mathbb{R}, \forall t \in \mathbb{R}_+.$$

These assumptions are inspired by those in [6]. Notice that (i) and (iii) imply that f is uniformly bounded (see [14], Remark 2.2).

The main result of this paper is the following theorem.

Theorem 2.1. *Suppose that hypotheses (i)-(vi) are fulfilled. Then, there exists an* $a > 0$ *such that every solution* x *to* (1) *with* $|x(0)| < a$ *is defined on* $\mathbb{R}_+$ *and satisfies condition* (2).

3 Proof of Theorem 2.1

As in [6], we write equation (1) as the following first order system

$$z' = A(t)z + B(t)z + F(t,z), \tag{7}$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(t) = \begin{pmatrix} -f(t) & 1 \\ -\beta(t) & -f(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 \\ f'(t) + f^2(t) & 0 \end{pmatrix},$$

$$F(t,z) = \begin{pmatrix} 0 \\ -g(t,x) \end{pmatrix}.$$

It is easily seen that our behavior question on the solutions to equation (1) at ∞ reduces to the behavior of the solutions to system (7) at ∞.

For $z_0 \in \mathbb{R}^2$, consider the initial condition

$$z(0) = z_0. \tag{8}$$

Let $Z(t)$, $t \geq 0$, be the fundamental matrix to linear system (4) which is equal to the identity matrix for $t = 0$.

Consider for $z = (x,y)^{\top} \in \mathbb{R}^2$ the norm $\|z\| := \sqrt{\beta_0 x^2 + y^2}$.

Then, as in [15], we have the following estimates

$$\|Z(t) z_0\| \leq \gamma \sqrt{1 + \beta(0)} e^{-\int_0^t f(u)du} \|z_0\|, \tag{9}$$

where $\gamma = \max\left\{1, 1/\sqrt{\beta_0}\right\}$ and

$$\left\| Z(t) Z(s)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| \leq e^{-\int_s^t f(u)du}, \qquad \forall t \geq s \geq 0. \tag{10}$$

Consider as fundamental the space

$$C_c := \left\{ z : \mathbb{R}_+ \to \mathbb{R}^2,\ z \text{ continuous} \right\}.$$

C_c is a Fréchet space (i.e. a complete, metrizable, and real linear space) with respect to the family of seminorms

$$\|z\|_n := \sup_{t \in [0,n]} \{\|z(t)\|\}, \qquad n \in \mathbb{N} \setminus \{0\}.$$

Notice that the topology defined by this family of seminorms is the topology of the convergence on compact subsets of $\mathbb{R}_+$; in addition, a family $\mathcal{A} \subset C_c$ is relatively compact if and only if it is equicontinuous and uniformly bounded on compacts subsets of $\mathbb{R}_+$ (Arzelá-Ascoli theorem).

Define in C_c the operator

$$(Hw)(t) := Z(t) z_0 + \int_0^t Z(t) Z^{-1}(s) \left[B(s) w(s) + F(s, w(s)) \right] ds, \tag{11}$$

for all $w \in C_c$, and for all $t \in \mathbb{R}_+$.

Remark 3.1. *It is obvious that the set of solutions to problem* $(7)-(8)$ *is identical the set of fixed points to* H.

Denote

$$B_\rho := \{ z \in C_c,\ \|z(t)\| \le \rho,\ \forall t \in \mathbb{R}_+ \},$$

where $\rho > 0$ is a fixed number; obviously, B_ρ is a nonempty, closed, bounded, and convex subset of C_c.

Lemma 3.1. *There exists a number* $h > 0$, *such that for every* $\rho \in (0, h)$, *there exists a number* $a > 0$ *with the property for every* z_0 *with* $\|z_0\| \in (0, a)$,

$$HB_\rho \subset B_\rho.$$

Proof. Let $z_0 \in \mathbb{R}^2$, $z_0 \ne 0$, $w \in B_\rho$, and $z := Hw$.

Then, by (11), for all $t \in \mathbb{R}_+$,

$$z(t) = Z(t) z_0 + \int_0^t Z(t) Z^{-1}(s) \left[B(s) w(s) + F(s, w(s)) \right] ds. \tag{12}$$

From hypotheses (iii), (iv), and (vi), we have the following estimates (see, e.g., [4], [14], [15]):

$$\|Z(t) z_0\| \le \gamma \sqrt{1 + \beta(0)}\, \|z_0\|\, e^{-\int_0^t f(s) ds},$$

$$\left\| \int_0^t Z(t) Z^{-1}(s) B(s) w(s)\, ds \right\| \le \frac{K}{\sqrt{\beta_0}} \int_0^t e^{-\int_s^t f(u) du} f(s) \|w(s)\|\, ds, \tag{13}$$

$$\left\| \int_0^t Z(t) Z^{-1}(s) F(s, w(s))\, ds \right\| \le \frac{M}{\left(\sqrt{\beta_0}\right)^\alpha} \int_0^t e^{-\int_s^t f(u) du} f(s) \|w(s)\|^\alpha\, ds. \tag{14}$$

By substituting the inequality $\|w(s)\| \leq \rho$, $\forall s \in \mathbb{R}_+$, in (13) and (14), from (12), and hypothesis (i), we get

$$\|z(t)\| \leq \gamma\sqrt{1+\beta(0)}\,\|z_0\| + \frac{K}{\sqrt{\beta_0}}\rho + \frac{M}{\left(\sqrt{\beta_0}\right)^{\alpha}}\rho^{\alpha}. \tag{15}$$

Let $h := \left(\frac{1-K/\sqrt{\beta_0}}{M/\left(\sqrt{\beta_0}\right)^{\alpha}}\right)^{\frac{1}{\alpha-1}}$ and consider $\rho \in (0,h)$ arbitrary. Set

$$a := \rho\left[1 - \left(\frac{K}{\sqrt{\beta_0}} + \frac{M}{\left(\sqrt{\beta_0}\right)^{\alpha}}\rho^{\alpha-1}\right)\right] / \left(\gamma\sqrt{1+\beta(0)}\right). \tag{16}$$

Obviously, $a > 0$; in addition, by (15) and (16), it follows that

$$(\|z_0\| < a) \Longrightarrow (\|(Hw)(t)\| \leq \rho,\ \forall t \in \mathbb{R}_+),$$

which ends the proof of Lemma 3.1. ∎

Lemma 3.2. *For $z_0 \in \mathbb{R}^2$, let z be a solution to problem $(7)-(8)$, defined on $\mathbb{R}_+$. Then for $\|z_0\|$ small enough, $z(\infty) = 0$.*

Proof. Let $z = (x,y)^{\top}$ be a solution to problem $(7)-(8)$ defined on $\mathbb{R}_+$, for $z_0 \in \mathbb{R}^2$. By (9), (10), and Remark 3.1 we infer that for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
\|z(t)\| &\leq \gamma\sqrt{1+\beta(0)}\,\|z_0\|\, e^{-\int_0^t f(s)ds} \\
&\quad + \int_0^t e^{-\int_s^t f(u)du}\left[Kf(s)|x(s)| + Mf(s)|x(s)|^{\alpha}\right]ds \\
&=: r(t).
\end{aligned} \tag{17}$$

Then, from (17), straightforward computations lead us to

$$\begin{cases} r'(t) \leq f(t)\left[\left(\frac{K}{\sqrt{\beta_0}} - 1\right) + \frac{M}{\left(\sqrt{\beta_0}\right)^{\alpha}} r(t)^{\alpha-1}\right] r(t), & \forall t \in \mathbb{R}_+ \\ r(0) = \gamma\sqrt{1+\beta(0)}\,\|z_0\|, & \end{cases}$$

and so

$$\begin{aligned}
\|z(t)\| &\leq r(t) \leq \left\{ e^{(\alpha-1)\left(1-\frac{K}{\sqrt{\beta_0}}\right)\int_0^t f(s)ds}\left[r(0)^{1-\alpha} - \frac{M/\left(\sqrt{\beta_0}\right)^{\alpha}}{1-K/\sqrt{\beta_0}}\right]\right. \\
&\quad \left. + \frac{M/\left(\sqrt{\beta_0}\right)^{\alpha}}{1-K/\sqrt{\beta_0}}\right\}^{\frac{1}{1-\alpha}},
\end{aligned} \tag{18}$$

for all $t \in \mathbb{R}_+$.

If

$$0 < \|z_0\| < \left(\frac{1-K/\sqrt{\beta_0}}{M/\left(\sqrt{\beta_0}\right)^{\alpha}}\right)^{\frac{1}{\alpha-1}} / \left(\gamma\sqrt{1+\beta(0)}\right),$$

then, from (18) and hypothesis (ii), it follows that $z(\infty) = 0$.

The proof of Lemma 3.2 is complete. ∎

In order to prove Theorem 2.1, it is enough to show that for $z_0 \in \mathbb{R}^2$ with $\|z_0\|$ small enough, problem $(7)-(8)$ admits solutions defined on $\mathbb{R}_+$. To this purpose, we will use Schauder-Tychonoff fixed point theorem, stated below (see, e.g., [16]).

Theorem 3.1. *Let E be a Fréchet space, $S \subset E$ a nonempty, closed, bounded, and convex subset of E, and $H : S \to S$ a continuous operator. If HS is relatively compact in E, then H admits fixed points.*

Setting $E = C_c$, H given by (11), and $S = B_\rho$ we have only to prove the continuity of H and the relative compactness of HS.

Let $w_n \in B_\rho$ such that $w_m \to w$ in C_c, as $m \to \infty$; that is, $\forall \varepsilon > 0$, $\exists m_0 = m_0(\varepsilon)$, $\forall m > m_0$, $\forall t \in [0,n]$, $\|w_m(t) - w(t)\| < \varepsilon$.

It is readily seen that there exist constants α_n and β_n, such that

$$\begin{aligned} \|(Hw)(t) - (Hw_m)(t)\| \leq\ & \alpha_n \int_0^n \|w(s) - w_m(s)\|\, ds \\ & + \beta_n \int_0^n \|F(s,w(s)) - F(s,w_m(s))\|\, ds. \end{aligned}$$

Since $F(t,z)$ is uniformly continuous for $t \in [0,n]$ and $\|z\| \leq \rho$, it follows that the sequence $F(t, w_m(t))$ converges uniformly on $[0,n]$ to $F(t,w(t))$, which finally proves the continuity of H.

Let us show that HB_ρ is relatively compact; from $HB_\rho \subset B_\rho$ it follows that HB_ρ is uniformly bounded in C_c.

Let $w \in B_\rho$ be arbitrary; since $z = Hw \in B_\rho$ and

$$z' = A(t)z + B(t)w + F(t,w)$$

there exist some constants γ_n and δ_n, such that

$$\|z'(t)\| \leq \gamma_n \rho + \delta_n, \qquad \forall t \in [0,n].$$

So, having the family of derivatives uniformly bounded, HB_ρ is equicontinuous on the compact subsets of $\mathbb{R}_+$. The proof of Theorem 2.1 is now complete. ∎

Remark 3.2. *While the classical transformation $(x := x,\ y := x')$ is useless when trying to obtain behavior results for the solutions to equation* (1) *at ∞, the transformation* (7)*, introduced in [6], is essential in deriving our estimates on the solution.*

Remark 3.3. *If $\beta(t) = 1$, $\forall t \in \mathbb{R}_+$, the fundamental matrix $Z(t)$ can be determined explicitly (see [4], [6], [14]),*

$$Z(t) = e^{-\int_0^t f(u)du} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

In general, this is not possible, so in our proof we had to get estimates without having an explicit form of $Z(t)$.

Example 3.1. *Some examples of typical functions $f,\ \beta,\ g$ fulfilling the assumptions $(i)-(vi)$ are:*

$$f(t) = \frac{1}{t+1},\ \beta(t) = 1 + e^{-t},\ g(t,x) = f(t)\,x^\alpha,\ \alpha > 1,\ \forall x \in \mathbb{R},\ \forall t \in \mathbb{R}_+.$$

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