Integral Characterizations For Exponential Stability Of Semigroups And Evolution Families On Banach Spaces

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Abstract

A proof of a sufficient condition for a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X to be uniformly exponentially stable is given. This result is a simplification of an earlier theorem by van Neerven, and concludes that a semigroup is uniformly exponentially stable provided $\sup_{||x||\leq 1} J(||T(\cdot)x||) < \infty$; here J is a certain nonlinear functional with certain natural properties. A non-autonomous version of this theorem for evolution families is also given. This implies the well-known Datko-Pazy and Rolewicz Theorems. This result is connected to the uniform asymptotic stability of the well-posed linear and non-autonomous abstract Cauchy problem

$$\begin{cases} \dot{u}(t) & = & A(t)u(t), \quad t \geq s \geq 0, \\ u(s) & = & x \in X. \end{cases}$$

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1 Introduction

Let X be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norm of vectors in X and operators in $\mathcal{L}(X)$ is denoted by $||\cdot||$. Let $\mathbf{T} := \{T(t)\}_{t\geq 0}$ be a semigroup of operators acting on X, that is, $T(t) \in \mathcal{L}(X)$ for every $t \geq 0$, $T(0) = I$ the identity operator in $\mathcal{L}(X)$ and $T(t+s) = T(t) \circ T(s)$ for every $t \geq 0$ and $s \geq 0$. The semigroup $\mathbf{T}$ is called strongly continuous if for each $x \in X$ the map $t \mapsto T(t)x : [0,\infty) \to X$ is continuous. Every strongly continuous semigroup is locally bounded, that is, there exist $h > 0$ and $M \geq 1$ such that $||T(t)|| \leq M$ for all $t \in [0, h]$. It is easy to see that every locally bounded semigroup is exponentially bounded, since there exist $\omega \in \mathbb{R}_+$ and $M \geq 1$ such that

$$||T(t)|| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

It is well-known that if $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on a Banach space X and there exists $p \in [1,\infty)$ such that for each $x \in X$,

$$\int_0^\infty ||T(t)x||^p dt = M(p,x) < \infty, \tag{1.1}$$

then $\mathbf{T}$ is uniformly exponentially stable, that is, its uniform growth bound

$$\omega_0(\mathbf{T}) := \inf_{t>0} \frac{\ln ||T(t)||}{t}$$

is negative. This result is usually referred to as the Datko-Pazy theorem, see [5, 11]. An important application of the Datko-Pazy theorem can be found in [15]. A quantitative version of this theorem states that if $M(p,x)$ from (1.1) is less than or equal to $C||x||^p$, where C is some positive constant, then $\omega_0(\mathbf{T}) < -\frac{1}{pC}$. See [9] Theorem 3.1.8 for details. An important generalization of the Datko-Pazy theorem was given by S. Rolewicz, [12]. In the autonomous case, the Rolewicz theorem reads as follows. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. If there exists a continuous non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ for each $t > 0$ and if*

$$\int_0^\infty \phi(||T(t)x||)dt := M_\phi(x) < \infty \text{ for each } x \in X, \tag{1.2}$$

then the semigroup $\mathbf{T}$ is uniformly exponentially stable. The same result was obtained independently by Littman [7]. In particular, from Rolewicz's theorem, it follows that the Datko-Pazy theorem remains valid for $p \in (0,1)$. The condition (1.1) indicates that for each $x \in X$ the map $t \mapsto ||T(t)x||$ belongs to $L^p(\mathbb{R}_+)$. Jan van Neerven has shown in [8] that a strongly continuous semigroup $\mathbf{T}$ on X is uniformly exponentially stable if there exists a Banach function space over $\mathbb{R}_+ := [0,\infty)$ with the property that

$$\lim_{t\to\infty} \left\| ||1_{[0,t]}|| \right\|_E = \infty, \tag{1.3}$$

such that

$$||T(\cdot)x|| \in E \text{ for every } x \in X. \tag{1.4}$$

He has also shown that the autonomous variant of the Rolewicz theorem can be derived from his result by taking for E a suitable Orlicz space over $\mathbb{R}_+$. In another paper, [10], Jan van Neerven has come to the same conclusion by replacing either (1.1), (1.2) or (1.4) by the hypothesis that the set of all $x \in X$ for which the following inequality holds

$$J(||T(\cdot)x||) < \infty,$$

is of the second category in X. Here J is a certain lower semi-continuous functional as defined in Theorem 2 from [10]. The proof of this latter result is based on a non-trivial result from operator theory given by V. Müler, see Lemma 1 from [10], for further details. We give here a surprisingly simple proof for a result of the same type, moreover, we do not require the lower semi-continuity of J.

In order to introduce some non-autonomous results of this type we recall the notion of an evolution family.

A family $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ of bounded linear operators on a Banach space X is a strongly continuous evolution family if

1. $U(t,t) = I$ and $U(t,s) = U(t,r)U(r,s)$ for $t \geq r \geq s \geq 0$.
2. The map $t \mapsto U(t,s)x : [s,\infty) \to X$ is continuous for every $s \geq 0$ and every $x \in X$.

The family $\mathcal{U}$ is exponentially bounded if there exist $\omega \in \mathbb{R}$ and $M_\omega \geq 0$ such that

$$||U(t,s)|| \leq M_\omega e^{\omega(t-s)} \text{ for } t \geq s \geq 0. \tag{1.5}$$

Then $\omega(\mathcal{U}) := \inf\{\omega \in \mathbb{R} : \text{there is } M_\omega \geq 0 \text{ such that (1.5) holds}\}$ is called the growth bound of $\mathcal{U}$. The family $\mathcal{U}$ is uniformly exponentially stable if its growth bound is negative.

In [1] it is proved that an exponentially bounded evolution family $\mathcal{U}$ is uniformly exponentially stable if there exists a Banach function space E satisfying (1.3) such that for each $s \geq 0$ and each $x \in X$ the map $||U(s+\cdot,s)x||$ belongs to E and

$$\sup_{s\geq 0} \Big\| ||U(s+\cdot,s)x|| \Big\|_E := K(x) < \infty.$$

The non-autonomous Datko theorem, [6], follows from this by taking $E = L^p(\mathbb{R}_+)$. The theorem of Rolewicz, [13], can be derived as well by taking for E a suitable Orlicz space over $\mathbb{R}_+$, see Theorem 2.10 from [1]. New guidelines about the proof of the Datko theorem can be found in [4] and [14]. In this paper we propose a more natural generalization of the theorems of Datko and Rolewicz which can also be extended to the general non-autonomous case. For some recently obtained autonomous or periodic versions of the above; see [3], [10].

2 A Generalization of the Datko-Pazy Theorem

We begin by stating and proving a lemma which is useful later.

Lemma 1. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a locally bounded semigroup such that for each* $x \in X$ *the map* $t \mapsto ||T(t)x||$ *is continuous on* $(0,\infty)$. *If there exist a positive* h *and* $0 < q < 1$ *such that for all* $x \in X$ *there exists* $t(x) \in (0, h]$ *with*

$$||T(t(x))x|| \leq q||x||, \tag{2.1}$$

then the semigroup $\mathbf{T}$ *is uniformly exponentially stable.*

Proof. Let $x \in X$ be fixed and $t_1 \in (0, h]$ such that $||T(t_1)x|| \leq q||x||$, then there exists $t_2 \in (0, h]$ such that

$$||T(t_2 + t_1)x|| \leq q||T(t_1)x|| \leq q^2||x||.$$

By mathematical induction it is easy to see that there exists a sequence (t_n), with $0 < t_n \leq h$ such that $||T(s_n)x|| \leq q^n||x||$, where $s_n := t_1 + t_2 + \cdots + t_n$.

If $s_n \to \infty$, then for each $t \in [s_n, s_{n+1}]$ we have that $t \leq (n + 1)h$ and

$$||T(t)x|| \leq ||T(t - s_n)||||T(s_n)x|| \leq Mq^n||x|| \leq Me^{-\ln(q)}e^{\frac{\ln(q)}{h}t}||x||;$$

here $M := \sup_{s\in[0,h]} ||T(s)||$.

If the sequence (s_n) is bounded, let $t(x)$ be the limit of (s_n). By the inequality $||T(s_n)x|| \leq q^n||x||$ and the assumption of continuity it follows that $T(t(x))x = 0$. This shows that the orbit $T(\cdot)x$ is eventually zero. Thus all orbits of the semigroup $\mathbf{T}$ are of negative exponential type and the desired result follows immediately. ∎

We can now state the main result of this section.

Theorem 1. *Let* $\mathcal{M}_{loc}([0,\infty))$ *be the space of all real valued locally bounded functions on* $\mathbb{R}_+ = [0,\infty)$ *endowed with the topology of uniform convergence on bounded sets and* $\mathcal{M}^+_{loc}(\mathbb{R}_+)$ *its positive cone.*

Let $J : \mathcal{M}^+_{loc}(\mathbb{R}_+) \to [0,\infty]$ *be a map with the following properties:*

1. *J is nondecreasing.*

2. *For each positive real number ρ,*

$$\lim_{t\to\infty} J(\rho \cdot 1_{[0,t]}) = \infty.$$

If $\mathbf{T}$ *is a locally bounded semigroup on a Banach space* X *such that for each* $x \in X$ *the map* $t \mapsto ||T(t)x||$ *is continuous on* $(0,\infty)$ *and if*

$$\sup_{||x||\leq 1} J(||T(\cdot)x||) < \infty, \tag{2.2}$$

then $\mathbf{T}$ *is uniformly exponentially stable.*

Proof. Suppose that $\mathbf{T}$ is not uniformly exponentially stable. For all $h > 0$ and all $0 < q < 1$ then there exists $x_0 \in X$ of norm one such that

$$||T(t)x_0|| > q \text{ for every } t \in [0, h],$$

as proved in Lemma 1. It follows then that

$$J(||T(\cdot)x_0||) \geq J(q \cdot 1_{[0,h]}),$$

which is a contradiction. ■

We remark here that the van Neerven theorem is an easy corollary of Theorem 1. Indeed, if J is lower-semicontinuous then the boundedness condition follows by a standard Baire category argument. We mention however that our second hypothesis about J is stronger than the similar one used by van Neerven in [10].

Corollary 1. *Let* $\mathbf{T} = \{T(t)\}_{t\geq 0}$ *be a locally bounded semigroup on a Banach space* X *such that for each* $x \in X$ *the map* $t \mapsto ||T(t)x||$ *is continuous on* $(0, \infty)$. *If there exists a non-decreasing function* $\phi : [0, \infty) \to [0, \infty)$ *such that* $\phi(t) > 0$ *for each* $t > 0$ *and (1.2) holds, then the semigroup* $\mathbf{T}$ *is uniformly exponentially stable.*

Proof. The natural proof uses the Fatou lemma in order to prove that the integral (1.2) defines a lower semi-continuous functional J. Application of van Neerven's version of Theorem 1 completes the proof. This argument is used in [10]. We mention here only the fact that it is possible to check directly that the boundedness condition (2.2) is satisfied and then apply our Theorem 1. ■

3 The Non-autonomous Case

We say that the evolution family $= \{U(t, s)\}_{t\geq s\geq 0}$ verifies the hypothesis (H) if it is exponentially bounded and for each $x \in X$ and each $s \geq 0$, the map $||U(s + \cdot, s)x||$ is continuous on $(0, \infty)$.

We state and prove a lemma that will be used in the sequel.

Lemma 2. *Let* $\mathcal{U} = \{U(t, s)\}_{t\geq s\geq 0}$ *be an evolution family on a Banach space* X *which verifies the hypothesis (H). If there exist positive real numbers* h *and* $q < 1$ *such that for every* $x \in X$ *there exists* $t(x) \in (0, h]$ *with the property that*

$$\sup_{s\geq 0} ||U(s + t(x), s)x|| \leq q||x||,$$

then the family $\mathcal{U}$ *is uniformly exponentially stable.*

Proof. Is similar to that of Lemma 1 and so we omit the details. ■

Theorem 2. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on a Banach space X verifying the hypothesis (H) and let J a functional satisfying the conditions **1.** and **2.** from Theorem 1. If there exists $r > 0$ such that*

$$\sup_{s\geq 0}\sup_{||x||\leq r} J(||U(s+\cdot,s)x||) < \infty, \tag{3.1}$$

then the evolution family $\mathcal{U}$ is uniformly exponentially stable.

Proof. Suppose that the family $\mathcal{U}$ is not uniformly exponentially stable. Under such circumstances as proved in Lemma 2, for every positive real number h and every $q \in (0,1)$ there exist $x_0 \in X$ of norm one and $s_0 \geq 0$ such that

$$||U(s_0+t,s_0)x_0|| > q \text{ for all } t \in [0,h].$$

Thus

$$J(||U(s_0+t,s_0)rx_0||) \geq J(rq \cdot 1_{[0,h]})$$

for each $h > 0$, which is a contradiction. ∎

Theorem 3. *We suppose, in addition, that J is lower semi-continuous and convex in the sense of Jensen (or sub-additive, that is, $J(f+g) \leq J(f) + J(g)$ for every f and g in $\mathcal{M}_{loc}(\mathbb{R}_+)$). Let $\mathcal{U}$ be an evolution family satisfying the hypothesis (H). If the set $\mathcal{X}$ of all $x \in X$ for which*

$$\sup_{s\geq 0} J(||U(s+\cdot,s)x||) < \infty$$

is of the second category in X, then the family $\mathcal{U}$ is uniformly exponentially stable.

Proof. Let $s \geq 0$ be fixed. The map $x \mapsto ||U(s+\cdot,s)x|| : X \to \mathcal{M}_{loc}(\mathbb{R}_+)$ is continuous. As a consequence, the map

$$x \mapsto \Phi_s(x) := J(||U(s+\cdot,s)x||) : X \to [0,\infty]$$

is lower semi-continuous as well. For each positive integer k, the set

$$X_k(s) := \{x \in X : J(||U(s+\cdot,s)x||) \leq k\}$$

is closed, because it is the reverse image of the real closed interval $[0,k]$ by the map Φ_s. It is clear that the set

$$X_k := \left\{x \in X : \sup_{s\geq 0} J(||U(s+\cdot,s)x||) \leq k\right\} = \cap_{s\geq 0} X_k(s)$$

is also closed and, moreover, that $\mathcal{X}$ is the union of all sets X_k. Because $\mathcal{X}$ is of the second category in X, there exists a set X_{k_0} whose interior is non empty. Let $x_0 \in X$ and $r_0 > 0$ such that $B(x_0,r_0)$ belongs to X_{k_0}. It is easy to see that $B\left(0,\frac{1}{2}r_0\right)$ belongs to X_{k_0}, that is,

$$\sup_{s\geq 0}\sup_{||x||\leq \frac{1}{2}r_0} J(||U(s+\cdot,s)x||) \leq k_0.$$

Indeed for every $x \in X$ with $||x|| \leq r_0$ we have:

$$\begin{aligned}
&J\left(\left\|U(s+\cdot,s)\left(\frac{1}{2}x\right)\right\|\right)\\
&= J\left(\frac{1}{2}||U(s+\cdot,s)[(x+x_0)-x_0]||\right)\\
&\leq J\left(\frac{1}{2}\left[||U(s+\cdot,s)(x+x_0)||+||U(s+\cdot,s)x_0||\right]\right)\\
&\leq \frac{1}{2}J(||U(s+\cdot,s)(x+x_0)||)+\frac{1}{2}J(||U(s+\cdot,s)x_0||)\\
&\leq k_0.
\end{aligned}$$

Application of Theorem 2 completes the proof. ∎

Corollary 2. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an exponentially bounded evolution family on a Banach space X such that for each $x \in X$ the map $t \mapsto ||U(s+t,s)x||$ is continuous on $(0,\infty)$ for every $s \geq 0$. Consider the following three inequalities:*

1. *There exists $p \in [1,\infty)$ such that*

$$\sup_{s\geq 0}\int_0^\infty ||U(s+t,s)x||^p dt < \infty$$

   for every $x \in X$.

2. *There exists a Banach function space E satisfying (1.3) such that for each $s \geq 0$ and each $x \in X$ the map $U(s+\cdot,s)x$ belongs to E and for every $x \in X$ we have*

$$\sup_{s\geq 0}|||U(s+\cdot,s)x|||_E < \infty.$$

3. *There exists a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(t) > 0$ for each $t > 0$ such that*

$$\sup_{s\geq 0}\int_0^\infty \phi(||U(s+t,s)x||)dt < \infty$$

   for every $x \in X$.

If any one of these statements is true then the family $\mathcal{U}$ is uniformly exponentially stable.

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