A remark on a functor of rational representations

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Abstract

Let k be a field of positive characteristic p. First we describe some specific subfunctors of the Burnside functor $k \otimes_{\mathbb{Z}} B$. We prove next that the restriction of the functor of rational representations $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$ to abelian finite p-groups, has a unique maximal filtration

$$k \otimes_{\mathbb{Z}} R_{\mathbb{Q}} = \overline{I_1} \supseteq \overline{I_2} \supseteq \overline{I_3} \supseteq \cdots$$

1 Introduction

The theory of Mackey functors for a finite group G over a ring k looks like an extension of the notion of kG-modules. So the usual notions of induction, restriction, inflation and deflation for modules, have their analogues for Mackey functors. This leads to the formalism of bisets, which gives a single natural framework involving restriction, inflation, induction and deflation. The classical properties of those constructions, such as the Mackey formula, become a single simple composition formula.

There are two kinds of Mackey functors, one kind defined only on the subgroups of a fixed group G, called by P. Webb *ordinary Mackey functors* (see [6]). The second kind defined on all finite groups, called *globally-defined Mackey functors*, or sometimes a subclass of finite groups. For example it could consist of all finite groups (see [1]), or just the identity group, or all nilpotent groups (see [2]) ore one of many other possibilities. In this work we consider the class of all abelian finite p-groups, over it some specific subfunctors of the Burnside functor will be described.

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The consequence of this description is the following :

Theorem : *The restriction of the functor of rational representations $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$ to abelian finite p-groups, has a unique maximal filtration*

$$k \otimes_{\mathbb{Z}} R_{\mathbb{Q}} = \overline{I_1} \supseteq \overline{I_2} \supseteq \overline{I_3} \supseteq \cdots$$

2 Specific subfunctors of $k \otimes_{\mathbb{Z}} B$

Let Q and P be groups. An P-set-Q is a set X with a left P-action and a right Q-action, which commute, i.e. if $g \in Q$, $h \in P$ and $x \in X$

$$h \cdot (x \cdot g) = (h \cdot x) \cdot g\,.$$

If X is an P-set-Q, and if Q and P are clear from context, we will also say that X is a biset.

As in [1], let k be a field of positive characteristic p, and $\mathcal{C}_k$ be the category whose objects are abelian finite p-groups, and morphisms are k-virtual bisets, i.e. linear combinations of bisets with coefficients in k.

If G and H are two objects of $\mathcal{C}_k$, then $\mathrm{Hom}_{\mathcal{C}_k}(H, G)$ is the tensor product by k of the Grothendieck group of the category of G-sets-H, the product of two morphisms is defined by k-linearity in the following way :
If L is a subgroup of $G \times H$ we denote by $(G \times H)/L$, the biset formed by the classes $(g, h)L$ for $(g, h) \in G \times H$, considered as G-set-H for the action

$$x \cdot (g, h)L \cdot y = (xg, y^{-1}h)L\,.$$

Let G' be another object of $\mathcal{C}_k$, E be a G-set-H and F be a H-set-G', we denote by $E \times_H F$ the set of orbits of H by its action over the product $E \times F$ given by $h \cdot (x, y) = (xh^{-1}, hy)$. It is a G-set-G': if $g \in G$ and $g' \in G'$, then by definition

$$g \cdot \overline{(x, y)} \cdot g' = \overline{(gx, yg')}\,,$$

where $\overline{(x, y)}$ is the image of (x, y) in $E \times_H F$.

Let H be a subgroup of G, the operation associated to the set $U = G$, viewed as a G-set-H, is called induction, and denoted by Ind_H^G :

$$\mathrm{Ind}_H^G = (G \times H)/\{(g, g) \mid g \in H\}\,.$$

Similarly, if G/N is a factor group of G, then the set $U = G/N$, viewed as a G-set-G/N, corresponds to inflation

$$\mathrm{Inf}_{G/N}^G = (G \times (G/N))/\{(g, gN) \mid g \in G\}\,.$$

When U is viewed as G/N-set-G, the associated operation is called deflation, and denoted by $\mathrm{Def}_{G/N}^G$:

$$\mathrm{Def}_{G/N}^G = ((G/N) \times G)/\{(gN, g) \mid g \in G\}\,.$$

Let φ be an isomorphism between an object G of $\mathcal{C}_k$ and another object G' of $\mathcal{C}_k$, the obvious associated operation of change of group is denoted by $\mathrm{Iso}_{G'}^{G}$, and corresponds to the set $U = G'$, viewed as a G'-set-G :

$$\mathrm{Iso}_{G'}^{G} = (G' \times G)/\triangle_{\varphi}(G), \text{ with } \triangle_{\varphi}(G) = \{(\varphi(g), g) \mid g \in G\} .$$

Let G and G' be two objects of $\mathcal{C}_k$ and L be a subgroup of $G \times G'$, we denote by $p_1(L)$ (resp. $p_2(L)$) the projection on L to G (resp. to G').
Let $k_1(L)$ and $k_2(L)$ denote

$$k_1(L) = \{g \in G \mid (g, 1) \in L\} \text{ and } k_2(L) = \{h \in G' \mid (1, h) \in L\} .$$

For every element y of $p_2(L)$, there exists x_y element of G such that $(x_y, y) \in L$. We associate to $yk_2(L)$ the element $x_y k_1(L)$, so we obtain a canonical isomorphism between $p_2(L)/k_2(L)$ and $p_1(L)/k_1(L)$.
If G'' is another object of $\mathcal{C}_k$, and M is a subgroup of $G' \times G''$, let $L * M$ denote

$$L * M = \{(g, g'') \in G \times G'' \mid \exists\, g' \in G',\ (g, g') \in L,\ (g', g'') \in M\} .$$

It is a subgroup of $G \times G''$.
Thus, we obtain the Mackey formula relating to bisets : (cf.[1], 3.2)

$$(G \times G'/L) \times_{G'} (G' \times G''/M) = \sum_{g \in p_2(L) \backslash G'/p_1(M)} (G \times G'')/(L *^{(g,1)}M) .$$

In the abelian case, this formula becomes

$$(G \times G'/L) \times_{G'} (G' \times G''/M) = |G'/(p_2(L) \cdot p_1(M))| \cdot (G \times G'')/(L * M) .$$

We denote by $\mathcal{F}_k$ the abelian category whose objects are the k-linear functors from $\mathcal{C}_k$ to the category of k-modules. Let k be a field of positive characteristic p, the standard operations on Grothendieck rings make $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$, after tensoring with k, into a functor in the category $\mathcal{F}_k$. Another example of object of $\mathcal{F}_k$ is $k \otimes_{\mathbb{Z}} B$, where B is Mackey functor which assigns to an abelian p-group G its Burnside ring $B(G)$. For more details we refer to [1]. The type of functors considered in the whole paper are objects of $\mathcal{F}_k$.

Let C_{p^n} be a cyclic group of order p^n. We consider the subfunctor I_n of $k \otimes_{\mathbb{Z}} B$ defined, for an object P of $\mathcal{C}_k$ by :

$$I_n(P) = \mathrm{Hom}_{\mathcal{C}_k}(C_{p^n}, P) \times_{C_{p^n}} \xi_n ,$$

where $\xi_n = C_{p^n}/1 - C_{p^n}/C$, with C is the unique subgroup of order p of C_{p^n}.
Lemma 1 : *For $n \geq 2$, we have $I_{n+1} \subseteq I_n$.*

Proof. We have

$$\xi_{n+1} = \mathrm{Ind}_{C_{p^n}}^{C_{p^{n+1}}} \xi_n \ \in I_n(C_{p^{n+1}}) .$$

If P is an object of $\mathcal{C}_k$, then

$$\mathrm{Hom}_{\mathcal{C}_k}(C_{p^{n+1}}, P) \times_{C_{p^{n+1}}} \xi_{n+1} \subseteq \mathrm{Hom}_{\mathcal{C}_k}(C_{p^{n+1}}, P) \times_{C_{p^{n+1}}} \mathrm{Hom}_{\mathcal{C}_k}(C_{p^n}, C_{p^{n+1}}) \times_{C_{p^n}} \xi_n ,$$

and

$$\mathrm{Hom}_{\mathcal{C}_k}(C_{p^{n+1}}, P) \times_{C_{p^{n+1}}} \mathrm{Hom}_{\mathcal{C}_k}(C_{p^n}, C_{p^{n+1}}) \subseteq \mathrm{Hom}_{\mathcal{C}_k}(C_{p^n}, P) .$$

Thus $I_{n+1} \subseteq I_n$. ∎

Lemma 2. *For $n \geq 2$, we have* $\mathrm{End}_{\mathcal{C}_k}(C_{p^n}) \times_{C_{p^n}} \xi_n$ *is a trivial* $\mathrm{End}_{\mathcal{C}_k}(C_{p^n})$*-module one-dimensional. Moreover, if K is an abelian p-group such that $|K| \leq p^n$ and $K \not\simeq C_{p^n}$, and if L is a subgroup of $K \times C_{p^n}$, then $(K \times C_{p^n} / L) \times_{C_{p^n}} \xi_n$ is zero.*

Proof. Let K be an abelian p-group such that $|K| \leq p^n$, and L be a subgroup of $K \times C_{p^n}$, we will prove that

$$(K \times C_{p^n}/L) \times_{C_{p^n}} \xi_n \subseteq k\xi_n .$$

The use of the Mackey formula for the $(K \times C_{p^n})$-biset $K \times C_{p^n}/L$ and the $(C_{p^n} \times 1)$-biset ξ_n, implies that the result is a $(K \times 1)$-biset, that is simply a K-set :

$$(K\times C_{p^n}/L)\times_{C_{p^n}}\xi_n = |C_{p^n}/p_2(L)| \cdot K/k_1(L) - |C_{p^n}/(p_2(L)\cdot C)| \cdot K/p_1(L\cap(K\times C)) .$$

If $p_2(L) \neq C_{p^n}$, then

$$(K \times C_{p^n}/L) \times_{C_{p^n}} \xi_n = 0 ,$$

since $|p_2(L)| < p^n$ and $|p_2(L) \cdot C| < p^n$.

Hence we can suppose that $p_2(L) = C_{p^n}$, so

$$(K \times C_{p^n}/L) \times_{C_{p^n}} \xi_n = K/k_1(L) - K/p_1(L \cap (K \times C)) .$$

There are two cases to consider.

Case 1. If $k_2(L) = 1$, then $p_1(L)/k_1(L) \simeq C_{p^n}$, thus $K \simeq C_{p^n}$ if $|K| \leq p^n$. We have $k_1(L) = 1$, so $k_1(L \cap (C_{p^n} \times C)) = 1$, and as

$$p_2(L \cap (C_{p^n} \times C))\Big/k_2(L \cap (C_{p^n} \times C)) \cong p_1(L \cap (C_{p^n} \times C))\Big/k_1(L \cap (C_{p^n} \times C)) ,$$

it follows that $|p_1(L \cap (C_{p^n} \times C))| \leq p$. In other words

$$(K \times C_{p^n}/L) \times_{C_{p^n}} \xi_n = C_{p^n}/1 - C_{p^n}/p_1(L \cap (C_{p^n} \times C)) ,$$

with $|p_1(L \cap (C_{p^n} \times C))| \leq p$.

Case 2. If $k_2(L) \neq 1$, then $C \subseteq k_2(L)$; let c be a generator of the subgroup C. If $(x, c) \in L$ then $(x, 1) \in L$, because $(x, c) = (x, 1) \cdot (1, c)$, so we obtain $k_1(L) = p_1(L \cap (K \times C))$. Hence

$$(K \times C_{p^n}/L) \times_{C_{p^n}} \xi_n = 0 .$$

Thus we have the following easy consequences :

$$\mathrm{End}_{\mathcal{C}_k}(C_{p^n}) \times_{C_{p^n}} \xi_n = k\xi_n ,$$

and if K is an abelian p-group such that $|K| < p^n$, then

$$(K \times C_{p^n}/L) \times_{C_{p^n}} \xi_n = 0 ,$$

since $k_2(L) \neq 1$ if $p_2(L) = C_{p^n}$, and we can be reduced to the second case. ∎

Proposition 1 : *Let J_n and J_1 be subfunctors of $k \otimes_{\mathbb{Z}} B$ defined, for an object P of $\mathcal{C}_k$ by :*

$$J_n(P) \ = \{u \in I_n(P) \mid \forall \varphi \in \mathrm{Hom}_{\mathcal{C}_k}(P, C_{p^n}) : \ \varphi \times_P u \ = \ 0\} ,$$

and

$$J_1(P) = \{u \in k \otimes_{\mathbb{Z}} B(P) \mid \forall \varphi \in \mathrm{Hom}_{\mathcal{C}_k}(P, 1) : \ \varphi \times_P u \ = \ 0\} ,$$

i.e.

$$J_1(P) = \{X \in k \otimes_{\mathbb{Z}} B(P) \mid \forall U \text{ subgroup of } P, \ |U \backslash X| = 0\} .$$

Then J_n is the unique maximal subfunctor of I_n, and J_1 is the unique maximal subfunctor of $k \otimes_{\mathbb{Z}} B$.

Proof. First we prove that J_n is a subfunctor of I_n :
Let P and P' be two objects of $\mathcal{C}_k$, let $\psi \in \mathrm{Hom}_{\mathcal{C}_k}(P', P)$, we prove

$$\psi \times_{P'} J_n(P') \subset J_n(P) .$$

Indeed we have

$$\forall u \in J_n(P'), \forall \psi' \in \mathrm{Hom}_{\mathcal{C}_k}(P, C_{p^n}), \ \psi' \times_P (\psi \times_{P'} u) = (\psi' \times_P \psi) \times_{P'} u .$$

However by the definition of $J_n(P')$:

$$\mathrm{Hom}_{\mathcal{C}_k}(P', C_{p^n}) \times_{P'} u = 0 ,$$

and $(\psi' \times_P \psi) \in \mathrm{Hom}_{\mathcal{C}_k}(P', C_{p^n})$, then $(\psi' \times_P \psi) \times_{P'} u = 0$, so $\psi' \times_P (\psi \times_{P'} u) = 0$, and J_n is a subfunctor of I_n. Moreover $J_n \neq I_n$, because for example $J_n(C_{p^n}) = \{0\}$ while $I_n(C_{p^n}) = \mathrm{End}_{\mathcal{C}_k}(C_{p^n}) \times_{C_{p^n}} \xi_n$ which is one-dimensional (see Lemma 2).

Now we prove that J_n is the unique maximal subfunctor of I_n. Let L be a subfunctor of I_n, in particular we have

$$L(C_{p^n}) \subset I_n(C_{p^n}) = \mathrm{End}_{\mathcal{C}_k}(C_{p^n}) \times_{C_{p^n}} \xi_n .$$

As $\mathrm{End}_{\mathcal{C}_k}(C_{p^n}) \times_{C_{p^n}} \xi_n$ is one-dimensional, there are two cases :
Case 1 : if $L(C_{p^n}) = 0$, then for an abelian finite p-group P we have

$$\forall u \in L(P), \forall \varphi \in \mathrm{Hom}_{\mathcal{C}_k}(P, C_{p^n}) : \ (\varphi \times_P u\ ) \in L(C_{p^n}) ,$$

thus $L \subset J_n$.
Case 2 : if $L(C_{p^n}) = \mathrm{End}_{\mathcal{C}_k}(C_{p^n}) \times_{C_{p^n}} \xi_n$, then for an abelian finite p-group P we have

$$\begin{aligned} I_n(P) \ &= \ \mathrm{Hom}_{\mathcal{C}_k}(C_{p^n}, P) \times_{C_{p^n}} \xi_n \\ &= \ \mathrm{Hom}_{\mathcal{C}_k}(C_{p^n}, P) \times_{C_{p^n}} L(C_{p^n}) . \end{aligned}$$

Since L is a functor, we have

$$\mathrm{Hom}_{\mathcal{C}_k}(C_{p^n}, P) \times_{C_{p^n}} L(C_{p^n}) \subset L(P) ,$$

then $L(P) = I_n(P)$. It follows that $L = I_n$.

Similarly, we prove that J_1 is the unique maximal subfunctor of $k \otimes_{\mathbb{Z}} B$. ∎

Proposition 2 : *Let P be an object of $\mathcal{C}_k$, for $n \geq 2$ we have*

$$I_n(P) \supseteq J_n(P) \oplus < P/R - P/Z, \text{ where } Z \supset R \text{ are subgroups of } P \text{ with}$$
$$P/R \simeq C_{p^n} \text{ and } |Z/R| = p >,$$

and

$$k \otimes_{\mathbb{Z}} B(P) \supseteq J_1(P) \oplus <P/P, P/M - P/P \text{ with } M \text{ a maximal subgroup of } P> .$$

Proof. First we have

$$\begin{aligned} I_n(P) \;\; \supseteq \;\; & < P/R - P/Z, \text{ where } Z \supset R \text{ are subgroups of } P \text{ with} \\ & P/R \simeq C_{p^n} \text{ and } |Z/R| = p >, \end{aligned}$$

because for any subgroup R such that $P/R \simeq C_{p^n}$ and $|Z/R| = p$, we have

$$P/R - P/Z = \mathrm{Inf}^P_{P/R} \, \mathrm{Iso}^{C_{p^n}}_{P/R} \, \xi_n \;\; \in \mathrm{Hom}_{\mathcal{C}_k}(C_{p^n}, P) \times_{C_{p^n}} \xi_n \, .$$

On the other hand

$$\begin{aligned} J_n(P) \;\; \cap \;\; & < P/R - P/Z, \text{ where } Z \supset R \text{ are subgroups of } P \text{ with} \\ & P/R \simeq C_{p^n} \text{ and } |Z/R| = p > = \{0\} \, . \end{aligned}$$

Indeed, let x be the following element

$$x = \sum_{\substack{R \\ P/R \simeq C_{p^n}}} \lambda_R (P/R - P/Z) \in J_n(P) \, .$$

Fixing a subgroup R of P such that $P/R \simeq C_{p^n}$, we now prove $\lambda_R = 0$:
Applying the functor $\mathrm{Def}^P_{P/R}$ to x, we obtain $\mathrm{Def}^P_{P/R}(x) \in J_n(P/R)$. By Proposition 1, we have $J_n(P/R) = \{0\}$. In other words

$$0 = \lambda_R \Big[(P/R)/(R/R) - (P/R)/(Z{\cdot}R/R) \Big] +$$

$$\sum_{\substack{P/R' \simeq C_{p^n} \\ |Z'/R'| = p \\ R' \neq R}} \lambda_{R'} \Big[(P/R)/(R'{\cdot}R/R) - (P/R)/(Z'{\cdot}R/R) \Big] .$$

In this equality $(P/R)/(R/R)$ is unique, so $\lambda_R = 0$. Since R is arbitrary, we obtain $x = 0$, it follows that

$$\begin{aligned} J_n(P) \;\; \cap \;\; & < P/R - P/Z, \text{ where } Z \supset R \text{ are subgroups of } P \text{ with} \\ & P/R \simeq C_{p^n} \text{ and } |Z/R| = p > = \{0\} \, . \end{aligned}$$

Similarly, we have

$$J_1(P) \cap <P/P, P/M - P/P \text{ with } M \text{ a maximal subgroup of } P> = \{0\} \, .$$

Let x be the following element

$$\Big(x = \lambda_P P/P + \sum_{\substack{M \\ |P/M|=p}} \lambda_M P/M \Big) \in J_1(P)\,,$$

so, by the Mackey formula, for any subgroup K of P we have

$$\lambda_P \,|K\backslash P/P| + \sum_{\substack{M \\ |P/M|=p}} \lambda_M \,|K\backslash P/M\,| = 0 \qquad (\clubsuit)\,.$$

In particular for $K = 1$, the equality ($\clubsuit$) becomes $\lambda_P \ |\,1\backslash P/P\,| = 0$, so $\lambda_P = 0$. Hence

$$x = \sum_{\substack{M \\ |P/M|=p}} \lambda_M \cdot P/M \ \in J_1(P)\,.$$

For the subgroup $K = P$, the equality ($\clubsuit$) becomes

$$\sum_{\substack{M \\ |P/M|=p}} \lambda_M = 0 \qquad (\spadesuit_1)\,.$$

We fix a subgroup M_0 of P such that $|\,P/M_0\,| = p$, and we apply ($\clubsuit$) to the subgroup $K = M_0$, we obtain

$$\sum_{\substack{M\neq M_0 \\ |P/M|=p}} \lambda_M = 0 \qquad (\spadesuit_2)\,.$$

From ($\spadesuit_1 - \spadesuit_2$), we deduce that $\lambda_{M_0} = 0$. Since M_0 is arbitrary, we obtain $x = 0$. Hence in $k \otimes_{\mathbb{Z}} B(P)$ we have

$$J_1(P) \cap \lt P/P, P/M - P/P \text{ with } M \text{ a maximal subgroup of } P\gt = \{0\}\,.$$

■

3 A unique maximal filtration of $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$

An important result of Ritter [3] and Segal [4] states that if P is a p-group, then the natural morphism s from the Burnside ring $B(P)$ of P to the Grothendieck ring $R_{\mathbb{Q}}(P)$ of rational representations of P, mapping a finite P-set X to the permutation module $\mathbb{Q}X$, is surjective. We shall denote $s(X)$ by $\overline{X}$.

Remark 1 : Let P be a finite p-group, $\dim_k(k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P))$ is equal to the number of conjugacy classes of cyclic subgroups of P (see [5], Chapitre 13, Théorème 29, Corollaire 1). Thus, if P is a cyclic group of order p^n, then $k\otimes_{\mathbb{Z}} B(P)$ and $k\otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P)$ are isomorphic. In particular, we have $\overline{\xi_n}$ is non-zero.

Theorem 1 : *The restriction of the functor of rational representations $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$ to abelian finite p-groups, has a unique maximal filtration*
$k \otimes_{\mathbb{Z}} R_{\mathbb{Q}} = \overline{I_1} \supseteq \overline{I_2} \supseteq \overline{I_3} \supseteq \cdots$

Proof. By Lemma 1, for any integer n we have $I_{n+1} \subseteq I_n$. Let P be an object of $\mathcal{C}_k$ and C_{p^n} be a cyclic group of order p^n, we will prove that $\overline{J_n}(P) = \overline{I_{n+1}}(P)$:

Indeed $\overline{J_n} \neq \overline{I_n}$, because for example $\overline{J_n}(C_{p^n}) = \{\overline{0}\}$ while $\overline{I_n}(C_{p^n}) = \overline{\xi_n}$ is non-zero (see Remark 1). Therefore

$$I_n(P)/J_n(P) \simeq \overline{I_n}(P)/\overline{J_n}(P)\,.$$

By Proposition 2, in $k \otimes_{\mathbb{Z}} B(P)$ we have

$$I_n(P) \supseteq J_n(P) \oplus < P/R - P/Z \text{ where } Z \supset R \text{ are subgroups of } P \text{ with}$$
$$P/R \simeq C_{p^n} \text{ and } |\,Z/R\,| = p >\,,$$

then in $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P)$ we have

$$\overline{I_n}(P) \supseteq \overline{J_n}(P) \oplus < \overline{P/R} - \overline{P/Z} \text{ where } Z \supset R \text{ are subgroups of } P \text{ with}$$
$$P/R \simeq C_{p^n} \text{ and } |\,Z/R\,| = p > \ .$$

Similarly, we find

$$k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P) \supseteq \overline{J_1}(P) \oplus <\overline{P/P}, \overline{P/M} - \overline{P/P} \text{ with } M \text{ a maximal subgroup of } P> \ .$$

Thus, the following set

$$\mathcal{L} = \{\overline{P/P}, \overline{P/M} - \overline{P/P} \text{ where } M \text{ is a maximal subgroup of } P, \overline{P/R} - \overline{P/Z}$$
$$\text{where } Z \supset R \text{ are subgroups of } P \text{ with } P/R \text{ is non-trivial cyclic and } |\,Z/R\,| = p\}$$

is linearly independent in the space $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P)$. As P is abelian, by duality $|\,\mathcal{L}\,|$ is the number of (conjugacy classes of) cyclic subgroups of P, which is exactly $\dim_k(k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P))$. Hence $\mathcal{L}$ is a basis of $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(P)$, and $\overline{J_n}(P) = \overline{I_{n+1}}(P)$ for $n \geq 1$.

We now show that, in the abelian case, the functors $(\overline{I_n})_{n\geq 2}$ are the unique non-zero proper subfunctors of $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$:

Let $\overline{F}$ be a proper subfunctor of $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$. The restriction of the functor of rational representations $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$ to abelian finite p-groups, has a unique maximal subfunctor $\overline{J_1}$ (see Proposition 1), then $\overline{F} \subseteq \overline{J_1}$. Moreover $\overline{J_1} = \overline{I_2}$, so $\overline{F} \subseteq \overline{I_2}$.

Each functor $\overline{I_n}$ admits a unique maximal (proper) subfunctor $\overline{J_n}$ (see Proposition 1), let n_0 be the maximal integer such that $\overline{F} \subseteq \overline{I_{n_0}}$. Since $\overline{I_{n_0}}$ admits a unique maximal subfunctor $\overline{J_{n_0}}$, and since $\overline{I_{n_0+1}} = \overline{J_{n_0}}$, it follows that

$$\overline{F} = \overline{I_{n_0}} \text{ or } \overline{F} \subseteq \overline{I_{n_0+1}}\,.$$

By the hypothesis about the integer n_0, we must have $\overline{F} = \overline{I_{n_0}}$. Hence, in the abelian case, the functors $(\overline{I_n})_{n\geq 2}$ are the unique non-zero proper subfunctors of $k \otimes_{\mathbb{Z}} R_{\mathbb{Q}}$.

To complete the proof of this theorem, let us now show that

$$\overline{I} = \bigcap_{n\geq 2} \overline{I_n} = \{0\}\,.$$

Assume that the proper subfunctor $\overline{I}$ is non-zero, then by the previous result $\overline{I}$ must be equal to $\overline{I_{n_0}}$, for a suitable integer n_0. In particular, we would have $\overline{I_{n_0}}(C_{p^{n_0}}) \subseteq \overline{I_{n_0+1}}(C_{p^{n_0}})$. By Remark 1 $\overline{I_{n_0}}(C_{p^{n_0}})$ is non-zero, while by Lemma 2

$$\begin{aligned} I_{n_0+1}(C_{p^{n_0}}) &= \mathrm{Hom}_{\mathcal{C}_k}(C_{p^{n_0+1}}, C_{p^{n_0}}) \times_{C_{p^{n_0+1}}} \xi_{n_0+1} \\ &= \{0\}\,, \end{aligned}$$

and consequently $\overline{I_{n_0+1}}(C_{p^{n_0}}) = \{0\}$. This contradiction shows that

$$\overline{I} = \bigcap_{n \geq 2} \overline{I_n} = \{0\}\,.$$

■

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