

# On abelian surfaces with potential quaternionic multiplication\*

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## Abstract

An abelian surface  $A$  over a field  $K$  has potential quaternionic multiplication if the ring  $\text{End}_{\bar{K}}(A)$  of geometric endomorphisms of  $A$  is an order in an indefinite rational division quaternion algebra. In this brief note, we study the possible structures of the ring of endomorphisms of these surfaces and we provide explicit examples of Jacobians of curves of genus two which show that our result is sharp.

## 1 The ring of endomorphisms of an abelian surface with potential quaternionic multiplication

**Definition 1.1.** Let  $K$  be a field and let  $\bar{K}$  be a separable closure of  $K$ . An abelian surface  $A$  over  $K$  has potential quaternionic multiplication if  $\text{End}_{\bar{K}}(A)$  is an order  $\mathcal{O}$  in an indefinite quaternion division algebra  $B$  over  $\mathbb{Q}$ .

In the literature, it is often required that abelian surfaces with quaternionic multiplication over a field  $K$  have all the endomorphisms defined over the base field. Under this hypothesis, these abelian surfaces are also sometimes called fake elliptic

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curves, since their arithmetic bear a strong analogy with the arithmetic of elliptic curves. We refer the reader to [Bu], [Jo1], [Jo2], [DiRo], [Oh] and [Ro]. However, in our definition we allow the endomorphisms to be defined over a field extension  $L/K$ , since this is precisely the phenomenon we wish to study and natural examples of them arise in the modular setting.

When working with Shimura curves as moduli spaces of abelian surfaces with quaternionic multiplication, one often considers abelian surfaces that *contain* a quaternion order. Hence, in our definition we are missing products of elliptic curves with complex multiplication. Indeed, according to Definition 1.1, our abelian surfaces are absolutely simple. This is not relevant for our purposes, since the arithmetic of elliptic curves with complex multiplication is better known.

Let us recall that an order  $\mathcal{O}$  in a quaternion algebra is *hereditary* if all its one-sided ideals are projective. An order  $\mathcal{O}$  is hereditary if and only if the reduced discriminant  $D = \text{disc}(\mathcal{O})$  is a square-free integer.

In [DiRo], the authors studied the possible fields of definition of the quaternionic multiplication on an abelian surface and the possible structures for the integral quaternion order  $\text{End}_{\bar{K}}(A)$ . In the following statement we recall and strengthen these results in the case that the base field is  $\mathbb{Q}$ .

**Theorem 1.2.** *Let  $A/\mathbb{Q}$  be an abelian surface with potential quaternionic multiplication by an hereditary order  $\mathcal{O}$  of discriminant  $D$  in a quaternion algebra  $B$ . Let  $L/\mathbb{Q}$  be the minimal field of definition of the quaternionic multiplication on  $A$  and  $F = \text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$ . Then, either*

- *$L$  is an imaginary quadratic field and  $F = \mathbb{Q}(\sqrt{m})$  is a real quadratic field such that  $B \simeq \left(\frac{-D\delta, m}{\mathbb{Q}}\right)$  for any possible degree  $\delta$  of a polarization on  $A/\mathbb{Q}$ , or*
- *$L$  is a purely imaginary dihedral extension of  $\mathbb{Q}$  of degree  $[L : \mathbb{Q}] \geq 4$  and  $F = \mathbb{Q}$ . If  $A$  is the Jacobian variety of a curve  $C/\mathbb{Q}$  of genus two, then  $[L : \mathbb{Q}] = 4$ .*

*Proof.* As it is shown in [DiRo], the field extension  $L/\mathbb{Q}$  is Galois and  $\text{Gal}(L/\mathbb{Q}) \simeq C_n$  or  $D_n$ ,  $n \leq 6$ , where  $C_n$  is the cyclic group of order  $n$  and  $D_n$  denotes the dihedral group of  $2n$  elements. Moreover, it is shown that if  $\text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$  is a totally real number field over  $\mathbb{Q}$ , then  $L/\mathbb{Q}$  is necessarily dihedral. In addition, if  $A$  admits a polarization over  $\mathbb{Q}$  of degree not equal to  $D$  nor  $3D$  up to squares, then  $|\text{Gal}(L/\mathbb{Q})| = 4$ . This is for instance the case when  $A = \text{Jac}(C)$  where  $C/\mathbb{Q}$  is a curve of genus two. The statement above now follows from the following proposition.

**Proposition 1.3.** *Let  $A/\mathbb{Q}$  be an abelian surface over  $\mathbb{Q}$  which is simple over  $\bar{\mathbb{Q}}$ . Then  $\text{End}_{\mathbb{Q}}(A)$  is either  $\mathbb{Z}$  or an order in a real quadratic field.*

*Proof.* By Albert's classification of involuting division algebras and Shimura's work on endomorphisms of abelian varieties,  $\text{End}_{\bar{\mathbb{Q}}}(A) \otimes \mathbb{Q}$  is either  $\mathbb{Q}$ , a real quadratic field or a totally indefinite quaternion algebra  $B$  over  $\mathbb{Q}$ . Assume that the latter holds; we will reach a contradiction.

It follows that  $\text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$  is either  $\mathbb{Q}$  or a real or imaginary quadratic field  $E$  that embeds in  $B$ . Indeed, the possibility  $\text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q} = B$  is not allowed since

otherwise there would exist an embedding  $B \hookrightarrow \text{End}_{\mathbb{Q}}(\Omega_A^1) \simeq M_2(\mathbb{Q})$ , which is not possible because  $B \not\cong M_2(\mathbb{Q})$ .

In addition,  $\text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$  can not be imaginary quadratic. Indeed, assume that  $\text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q} = E \subset B$  were an imaginary quadratic field. Then  $B = \text{End}_{\overline{\mathbb{Q}}}(A) = \text{End}_K(A)$  for a certain quadratic number field (cf. [DiRo]). For any prime  $\lambda$  on  $E$  over a rational prime  $\ell$ , let  $\delta_\lambda : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_\ell^*$  be the determinant of the  $\lambda$ -adic representation of  $GL_2$ -type attached to the  $\ell$ -power torsion of  $A$  (cf. [Ri92], Section 3) and let  $\chi_\ell$  be the  $\ell$ -adic cyclotomic character. By [Ri92], Lemma 3.1, there is a character of finite order  $\epsilon : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow E^*$  such that  $\delta_\lambda = \epsilon \chi_\lambda$ , which is trivial if and only if  $K$  is real. By [Ri92], Lemma 3.2,  $\epsilon$  is even, that is,  $\epsilon(\sigma) = +1$  for any complex conjugation  $\sigma$  on  $\overline{\mathbb{Q}}$ . Since  $E$  is imaginary, it turns out that  $\epsilon$  is trivial and hence  $K$  is real. Then we would obtain an abelian surface  $(A, \iota : B \hookrightarrow \text{End}_K(A) \otimes \mathbb{Q})$  that would represent a point on a Shimura curve rational over  $K$ . This again yields a contradiction with [Sh]. This shows that  $\text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$  is either  $\mathbb{Q}$  or a real quadratic field.  $\square$

Natural examples of abelian surfaces  $A$  with potential quaternionic multiplication over  $\mathbb{Q}$  and  $\text{End}_{\mathbb{Q}}(A)$  quadratic arise when considering modular abelian surfaces (with no character): simple two-dimensional factors of the Jacobian  $J_0(N)$  of the modular curve  $X_0(N)$  of level  $N \geq 1$ . Indeed, let  $A/\mathbb{Q}$  be a modular abelian surface. Although  $\text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q} = F$  is a quadratic number field, it may be the case that  $\text{End}_{\overline{\mathbb{Q}}}(A) \otimes \mathbb{Q}$  is a strictly larger algebra (cf. [Mo], [Ri92]). This is exactly the case when  $A \stackrel{\mathbb{Q}}{\sim} A_f$  is isogenous to the abelian variety attached by Shimura to a newform  $f \in S_2(\Gamma_0(N))$  without CM and with an *extra-twist*. Then,  $\text{End}_{\overline{\mathbb{Q}}}(A_f) \otimes \mathbb{Q}$  is necessarily a (possibly splitting) quaternion algebra  $B$  over  $\mathbb{Q}$ .

More precisely, if we let  $\chi$  the quadratic character attached to the extra-twist of  $f$ , then  $\text{End}_{\overline{\mathbb{Q}}}A$  is an order in the quaternion algebra  $(\frac{d, \chi(-1)r}{\mathbb{Q}})$ , where  $d = \text{disc}(F)$  and  $r$  is the conductor of  $\chi$ .

Computations due to Hasegawa ([Ha]) and Clark and Stein show that the only endomorphism algebras  $\text{End}_{\overline{\mathbb{Q}}}(A_f) \otimes \mathbb{Q}$  that occur for extra-twisting newforms  $f$  of level  $N < 4500$  and trivial Neben-typus are the quaternion algebras  $B$  over  $\mathbb{Q}$  of discriminant  $D = 1, 6, 10, 14$  and  $15$ .

We note that it very often holds that the ring  $\text{End}_{\mathbb{Q}}(A) = \mathbb{T} \simeq \mathbb{Z}\{a_n\}$  of Hecke operators acting on the modular abelian surface  $A$  is the maximal ring of integers of the number field  $\mathbb{Q}(\{a_n\})$  generated by the Fourier coefficients of  $f$ . Under this assumption and the assumptions of Theorem 1.2, it implies that  $\mathcal{O} = \text{End}_{\overline{\mathbb{Q}}}(A)$  contains a maximal quadratic order and is therefore a *primitive* quaternion order. In particular,  $\mathcal{O}$  is a *Bass* order.

## 2 Examples of non modular abelian surfaces with potential quaternionic multiplication

In this section, we exhibit examples of abelian surfaces with potential quaternionic multiplication which show that all the cases of Theorem 1.2 can occur.

In order to accomplish that, we consider several particular fibres of a family of curves of genus two whose Jacobian varieties have multiplication by a maximal order in the quaternion algebra  $B$  over  $\mathbb{Q}$  of reduced discriminant 6 obtained by

Hashimoto-Tsunogai in [HaTs]. We also refer the reader to [HaMu]. For these curves  $C/K$ , we compute the minimal field of definition  $L$  of all endomorphisms on  $J(C)$  and we apply a theorem proved by the authors in [DiRo] to conclude that the varieties  $J(C)/K$  are not of  $GL_2$ -type over the base field  $K$ .

**Theorem 2.1. I.** *Let  $C_1/\mathbb{Q}(\sqrt{2})$  be a smooth projective model of the curve*

$$Y^2 = (X^2 + 5)((-1/6 + \sqrt{2})X^4 + 20X^3 - 490/6X^2 + 100X + 25(-1/6 - \sqrt{2})).$$

*Then, the Jacobian variety  $A_1 = J(C_1)/\mathbb{Q}(\sqrt{2})$  of  $C_1$  has multiplication by a maximal order  $\mathcal{O}$  in the quaternion algebra  $B_6$  of discriminant 6 over the quartic extension  $L = \mathbb{Q}(\sqrt{2}, \sqrt{-1}, \sqrt{-5})$  of  $\mathbb{Q}(\sqrt{2})$ . Moreover,*

- $\text{End}_{\mathbb{Q}(\sqrt{2}, \sqrt{-5})}(A_1) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{2})$
- $\text{End}_{\mathbb{Q}(\sqrt{2}, \sqrt{5})}(A_1) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{3})$
- $\text{End}_{\mathbb{Q}(\sqrt{2}, \sqrt{-1})}(A_1) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-6})$ .

and  $\text{End}_{\mathbb{Q}(\sqrt{2})}(A_1) = \mathbb{Z}$ .

**II.** *Let  $C_2/\mathbb{Q}$  be a smooth projective model of the curve*

$$Y^2 = (X^2 + 7/2)(83/30X^4 + 14X^3 - 1519/30X^2 + 49X - 1813/120)$$

and let  $A_2 = J(C_2)/\mathbb{Q}$  be its Jacobian variety. Then,

- $\text{End}_L(A_2) = \mathcal{O}$  is a maximal order in  $B_6$  for  $L = \mathbb{Q}(\sqrt{-6}, \sqrt{-14})$ .
- $\text{End}_{\mathbb{Q}(\sqrt{-14})}(A_2) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{2})$
- $\text{End}_{\mathbb{Q}(\sqrt{21})}(A_2) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{3})$
- $\text{End}_{\mathbb{Q}(\sqrt{-6})}(A_2) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-6})$
- $\text{End}_{\mathbb{Q}}(A_2) = \mathbb{Z}$ .

*Proof.* By [HaTs], Lemma 4.5, putting  $(\tau_1, \sigma_1) = (\sqrt{-2}, \sqrt{-1})$  and  $(\tau_2, \sigma_2) = (\sqrt{-3/2}, \sqrt{-3/2})$ , we know that  $\text{End}_{\overline{\mathbb{Q}}}(A_i) \otimes \mathbb{Q}$ ,  $i = 1, 2$  contains the quaternion algebra  $B_6$ . Moreover, as it was shown in [HaMu] for an isomorphic family of curves,  $\text{End}_{\overline{\mathbb{Q}}}(A_i)$  contains a maximal order in  $B_6$ .

The curve  $C_1$  is defined over  $K = \mathbb{Q}(\sqrt{2})$  and the primes of bad reduction of the curve all divide  $N = 30$ .

For any prime number  $\ell$ , let  $\rho_\ell$  be the Galois representation given by the action of the Galois group  $\text{Gal}(\bar{K}/K)$  on the  $\ell$ -adic Tate module of  $A_1$ . For the primes  $\wp$  in  $K$  above the rational primes  $p = 7, 17, 23, 31, 41, 47, 71, 73, 79, 89$  and  $97$ , we computed the characteristic polynomial of  $\rho_\ell(\text{Frob } \wp)$  ( $\ell \neq p$ ) and we obtained that the values

of the *half-traces*  $(a_\wp, b_\wp)$  are  $\pm 2\sqrt{2}, \pm 2\sqrt{-6}, \pm 4\sqrt{2}, \pm 2\sqrt{3}, (8, 8), (0, 0), \pm 8\sqrt{3}, \pm 4\sqrt{-6}, \pm 2\sqrt{3}, (-16, -16)$  and  $(0, 0)$ , respectively.

By *half-traces* we mean that the values  $a_\wp$  and  $b_\wp$  correspond to the factorization

$$(x^2 - a_\wp x + p)(x^2 - b_\wp x + p)$$

of the characteristic polynomial of  $\rho_\ell(\text{Frob } \wp)$ , except for the rational primes  $p = 17, 73, 97$ , namely those that are a square mod 4 and a non-square mod 5. The latter correspond to the factorization

$$(x^2 - a_\wp x - p)(x^2 - b_\wp x - p).$$

Above, the term  $-p$  is due to the fact that  $\sqrt{-6}$  is not real. Indeed, as we explained in [DiRo], Section 5.1, when an imaginary quadratic field shows up in the endomorphism algebra, the Galois representations are reducible but the determinant of the two-dimensional irreducible components is not  $\chi$ .

The endomorphism algebra of an abelian surface containing a quaternion algebra  $B$  is either  $B$  itself or a matrix algebra  $M_2(K)$  for  $K$  an imaginary quadratic field. Let us explain why the latter does not occur for  $A_1$  and thus  $\text{End}_{\overline{\mathbb{Q}}}(A_1)$  is a maximal order in  $B_6$ . To see this, we will show that for a suitable prime  $\ell$ , the representation  $\rho_\ell$  is not potentially abelian, thus eliminating the case of the product of two elliptic curves with complex multiplication.

The values of the traces computed show that  $\text{End}_K(A_1) \otimes \mathbb{Q}$  can not contain a real or imaginary quadratic field. Thus, we have  $\text{End}_K(A_1) = \mathbb{Z}$ . Hence, the abelian surface  $A_1$  acquires multiplication by a quadratic field over a quadratic extension  $K'$  of  $K$  unramified outside 30. Looking at the values of the half-traces we see that they fall in a fixed quadratic field only if we restrict to the Galois group of one of the following quadratic extensions  $K'$  of  $K$  (among those with the above specified ramifying places):  $K(\sqrt{-1}), K(\sqrt{-5})$  and  $K(\sqrt{5})$ . Thus we know that over one of these fields,  $\text{End}_{K'}(A_1) \otimes \mathbb{Q}$  contains a quadratic field and in particular the representations  $\rho_\ell$  restricted to the corresponding Galois group decompose and have two two-dimensional irreducible components:

$$\rho_\ell|_H \cong \sigma_\ell \oplus \sigma'_\ell$$

where  $H = \text{Gal}(\mathbb{Q}/K')$  and  $K'$  is one of above the three fields.

The next and last step is to check that for a suitable  $\ell$  the images of these two-dimensional components are maximal. We have checked this for the three possibilities for  $K'$ . Here we give the details only for the case  $K' = K(\sqrt{5})$ , the other two being similar. In this case, when we restrict to  $H$  the traces of the two-dimensional components fall in  $\mathbb{Q}(\sqrt{3})$ , so we are assuming that  $A_1$  has real multiplication by this field, defined over  $K(\sqrt{5})$ . We take the prime  $\ell = 11$  and observe that it decomposes in  $\mathbb{Q}(\sqrt{3})$ . Let us show that the image of  $\sigma_{11}$  is the full  $GL(2, \mathbb{Z}_{11})$ . Thanks to a well-known lemma of Serre, it is enough to show that the residual representation  $\bar{\sigma}_{11}$  has maximal image  $GL(2, \mathbb{F}_{11})$ . There is a well-known procedure to prove this maximality computationally, we recall it for the reader's convenience: first, compute the reduction modulo 11 of the characteristic polynomials of several Frobenius elements (corresponding to elements in  $H$ ), and observe that some have two different roots in  $\mathbb{F}_{11}$ , one of them of maximal order  $\ell - 1 = 10$ , and on the other hand some

have irreducible characteristic polynomials whose roots are elements of high order in  $\mathbb{F}_{121}^*$ . We have checked this using the Frobenius elements for the following primes (all splitting totally in  $K(\sqrt{5})$ ):  $p = 31, 41, 71, 79$  and  $89$ . Using the classification of maximal subgroups of  $GL_2(\mathbb{F}_\ell)$  due to L. E. Dickson, one concludes that the image of  $\bar{\sigma}_{11}$  is the full  $GL(2, \mathbb{F}_{11})$ . Therefore, the image of  $\sigma_{11}$  is also maximal. The same holds if we take the other two quadratic extensions  $K'$  of  $K$  mentioned above. Thus, we conclude that  $A_1$  is not in the potentially CM case, and so it has only potential quaternionic multiplication.

We know that  $\text{End}_{\bar{\mathbb{Q}}}(A_1)$  is a maximal order in  $B_6$ , and that  $\text{End}_K(A_1) = \mathbb{Z}$ . In consequence, Theorem 1.3 in [DiRo] yields that  $[L : K] = 4$ . By considering all possible quadratic extensions unramified outside  $\{2, 3, 5\}$  and by applying Lemma 5.2 in [DiRo], we conclude that the traces above only match with the fields of definition and intermediate endomorphism algebras claimed in part **I** of our theorem. A similar argument and similar computations yield our statement for curve  $C_2$ .  $\square$

**Corollary 2.2.** *The Jacobian variety  $J(C_2)/\mathbb{Q}$  is a non modular abelian surface with potential quaternionic multiplication over  $\mathbb{Q}$ .*

Note that, since  $B_6 = \left(\frac{-6,2}{\mathbb{Q}}\right) = \left(\frac{-6,3}{\mathbb{Q}}\right)$ , the three possible intermediate endomorphism algebras  $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{-6}) \subset B_6$  allowed by the Theorem 1.3 in [DiRo] arise both on  $A_1$  and  $A_2$ .

### 3 Fake elliptic curves over quadratic imaginary fields

Finally, let us conclude by exhibiting a pair of examples of abelian surfaces  $A/K$  over a quadratic imaginary field  $K$  such that all quaternionic endomorphisms of  $A$  are defined over  $K$  itself. These are therefore examples of what we can call fake elliptic curves over  $K$ . Our examples are non trivial, since it can be shown that they are not the base extension to  $K$  of an abelian surface over  $\mathbb{Q}$ .

As a by-product, we also show that all computations performed in [HaTs] supporting an analogue of the Sato-Tate conjecture for these surfaces are unconditionally correct (cf. [HaTs] for details).

In order to accomplish that, assume that for a particular curve  $C/K$ , computations suggest that the minimal field of definition of the ring of correspondences of  $C$  is  $L = K$ . This can be the case if the characteristic polynomial of Frob  $\wp$ , for  $\wp$  a prime in  $K$  of good reduction of  $A$  and residual degree 1, factorizes as

$$(x^2 - a_\wp x + p)^2 \tag{2.1}$$

with  $a_\wp \in \mathbb{Z}$ .

In order to eliminate the quadratic case  $\text{Gal}(L/K) = C_2$  allowed by [DiRo], Theorem 1.3, we may compute all quadratic extensions  $L$  of  $K$  ramifying only at the primes of bad reduction of  $A$  and, for each of them, exhibit a prime  $\wp$  of  $K$  inert in this quadratic extension verifying (2.1) with  $a_\wp \neq 0$ . This contradicts formula (4.5) in [DiRo] and we thus conclude that the field  $L$  of definition of the quaternionic multiplication can not be a quadratic extension of  $K$ .

Moreover, applying [DiRo], Lemma 5.2, we see that the above fact is also incompatible with the case  $\text{Gal}(L/K) = D_2$  because it violates the trace 0 condition for those primes that do not totally decompose in  $L/K$ . As examples of this phenomenon we can exhibit the following: in [HaTs], (3.1), a family  $S_6(t, s)$  of QM-curves  $C_{(t,s)}$  of genus 2 is given. It is such that, for every rational value of the parameter  $t$ , the curve  $C_{(t,s)}$  is defined over the imaginary quadratic field  $\mathbb{Q}(s) = \mathbb{Q}(\sqrt{-3 + 14t^2 - 27t^4})$ . For several rational values of  $t$ , the authors considered the action of  $\text{Gal}(\bar{\mathbb{Q}}/K)$  on the Tate modules of  $J(C_{(t,s)})$  and they observed that, for the first 30 primes of residual degree 1 in  $K$ , formula (2.1) is verified. In consequence, they suggested that all endomorphisms are defined over  $K$ , that is,  $L = K$ . In fact, the experimental verification of the Sato-Tate conjecture that they obtain in their article depends on this assumption (cf. the examples 1 and 2 in [HaTs], page 1655, corresponding to  $(t, s) = (2, \sqrt{-379})$  and  $(2/3, \sqrt{-19}/3)$ , respectively, and the tables and figures of p. 1658, 1659).

For these two particular examples, we computed all quadratic extensions of  $K$  ramifying only at primes of bad reduction and for each of them we found a prime  $\wp$  of  $K$  inert in this extension verifying (2.1) with  $a_\wp \neq 0$ . This eliminates all the cases listed in [DiRo], Theorem 1.3, except  $L = K$ . We thus conclude

**Theorem 3.1.** *Let  $C_3/K_3, C_4/K_4$  be the fibres of the Hashimoto-Tsunogai's family  $S_6(t, s)$  at the values  $t_3 = 2$  and  $t_4 = 2/3$  over  $K_3 = \mathbb{Q}(\sqrt{-379})$  and  $K_4 = \mathbb{Q}(\sqrt{-19})$ , respectively. Then,  $\text{End}_{K_i}(J(C_i)) \simeq \mathcal{O}$  is a maximal order in  $B_6$ .*

In particular, all computations performed in [HaTs] for these two examples supporting the Sato-Tate conjecture for these surfaces are unconditionally correct.

**Remark 3.2.** The computation of the absolute Igusa invariants of these curves show that there exists no curve  $C/\mathbb{Q}$  such that  $C \simeq C_3$  nor  $C \simeq C_4$  over  $\bar{\mathbb{Q}}$ . See [CaQu] for details.

**Remark 3.3.** The importance of the property  $L = K$  is that it implies that (2.1) holds for every prime and it is then enough to compute the number of rational points on the curve over  $\mathbb{F}_p$  in order to determine the characteristic polynomial of  $\text{Frob } \wp$  for any prime  $\wp \mid p$  in  $K$ , while in general it is necessary to compute also the number of rational points over  $\mathbb{F}_{p^2}$ . This enormously speeds the computations.

**Remark 3.4.** The above two curves  $C_3$  and  $C_4$  and also the respective rings of correspondences on them are defined over completely imaginary fields  $L = K$ . This is not a coincidence since there do not exist curves  $C/K$  of genus 2 over a number field  $K$  admitting a real archimedean place such that  $\text{End}_K(J(C))$  is a quaternion order. Indeed, this follows from Shimura's result that the set of real points on Shimura curves is the empty set ([Sh]).

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