q-Generating functions for one and two variables

Thomas Ernst

Abstract

We use a multidimensional extension of Bailey's transform to derive two very general q-generating functions, which are q-analogues of a paper by Exton [10]. These expressions are then specialized to give more practical formulae, which are q-analogues of generating relations for Karlssons generalized Kampé de Fériet function. A number of examples are given including q-Laguerre polynomials of two variables.

1 Preliminaries

The purpose of this paper is to continue the study of q-special functions by the method outlined in [2]– [7]. The paper is a q-analogue of Exton [10]. All of Exton's results are obtained as the special case $q = 1$ by this method for $n \leq 2$.

We begin with a few definitions.

Definition 1. The power function is defined by $q^a = e^{alog(q)}$. We always use the principal branch of the logarithm.

The q-analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q = \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\}, \tag{1}$$

$$\{n\}_q! = \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! = 1,\ q \in \mathbb{C}, \tag{2}$$

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Definition 2. The q-hypergeometric series was developed by Heine 1846 as a generalization of the hypergeometric series:

$$ {}_2\phi_1(a,b;c|q,z) = \sum_{n=0}^{\infty} \frac{\langle a;q\rangle_n \langle b;q\rangle_n}{\langle 1;q\rangle_n \langle c;q\rangle_n} z^n, \tag{3}$$

with the notation for the q-shifted factorial (compare [12, p.38])

$$\langle a;q\rangle_n = \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\ldots, \end{cases} \tag{4}$$

which is introduced in this paper.

Remark 1. The Watson notation [11] will also be used.

$$(a;q)_n = \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-aq^m), & n=1,2,\ldots, \end{cases} \tag{5}$$

Definition 3. Furthermore,

$$(a;q)_\infty = \prod_{m=0}^{\infty}(1-aq^m),\ 0<|q|<1. \tag{6}$$

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},\ a \neq q^{-m-\alpha}, m=0,1,\ldots. \tag{7}$$

Definition 4. In the following, $\frac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers mod $\frac{2\pi i}{\log q}$. This is isomorphic to the cylinder $\mathbb{R}\times e^{2\pi i\theta},\ \theta\in\mathbb{R}$. The operator

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{8}$$

Furthermore we define

$$\widetilde{\langle a;q\rangle_n} = \langle \widetilde{a};q\rangle_n. \tag{9}$$

By (8) it follows that

$$\widetilde{\langle a;q\rangle_n} = \prod_{m=0}^{n-1}(1+q^{a+m}), \tag{10}$$

where this time the tilde denotes an involutioninvolution which changes a minus sign to a plus sign in all the n factors of $\langle a;q\rangle_n$.

The following simple rules follow from (8). Clearly the first two equations are applicable to q-exponents. Compare [24, p. 110].

$$\widetilde{a} \pm b \equiv \widetilde{a \pm b} \bmod \frac{2\pi i}{\log q}, \tag{11}$$

$$\widetilde{a} \pm \widetilde{b} \equiv a \pm b \bmod \frac{2\pi i}{\log q}, \tag{12}$$

$$q^{\widetilde{a}} = -q^{a}, \tag{13}$$

where the second equation is a consequence of the fact that we work mod$\frac{2\pi i}{\log q}$.

We will use the following abbreviation

$$\langle (a); q\rangle_n \equiv \langle a_1, \ldots, a_A; q\rangle_n = \prod_{j=1}^{A} \langle a_j; q\rangle_n. \tag{14}$$

Definition 5. Generalizing Heine's series, we shall define a q-hypergeometric series by (compare [11, p.4]):

$$\begin{aligned} &{}_p\phi_r(\hat{a_1}, \ldots, \hat{a_p}; \hat{b_1}, \ldots, \hat{b_r}|q, z) \equiv {}_p\phi_r \left[\begin{matrix} \hat{a_1}, \ldots, \hat{a_p} \\ \hat{b_1}, \ldots, \hat{b_r} \end{matrix} |q, z \right] = \\ &= \sum_{n=0}^{\infty} \frac{\langle \hat{a}_1, \ldots, \hat{a}_p; q\rangle_n}{\langle 1, \hat{b_1}, \ldots, \hat{b_r}; q\rangle_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+r-p} z^n, \end{aligned} \tag{15}$$

where $q \neq 0$ when $p > r + 1$, and

$$\widehat{a} = \begin{cases} a \\ \widetilde{a} \end{cases} \tag{16}$$

We will skip the $\hat{a}$ for the rest of the paper.

Definition 6. The following generalization of (15) will sometimes be used:

$$\begin{aligned} &{}_{p+p'}\phi_{r+r'}(a_1, \ldots, a_p; b_1, \ldots, b_r|q, z||s_1, \ldots, s_{p'}; t_1, \ldots, t_{r'}) \equiv \\ &{}_{p+p'}\phi_{r+r'} \left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_r \end{matrix} |q, z|| \begin{matrix} s_1, \ldots, s_{p'} \\ t_1, \ldots, t_{r'} \end{matrix} \right] = \\ &= \sum_{n=0}^{\infty} \frac{\langle a_1; q\rangle_n \ldots \langle a_p; q\rangle_n}{\langle 1; q\rangle_n \langle b_1; q\rangle_n \ldots \langle b_r; q\rangle_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+r+r'-p-p'} \times \\ &z^n \prod_{k=1}^{p'} (s_k; q)_n \prod_{k=1}^{r'} (t_k; q)_n^{-1}, \end{aligned} \tag{17}$$

where $q \neq 0$ when $p + p' > r + r' + 1$.

Remark 2. Equation (17) is used in certain special cases when we need factors $(t; q)_n$ in the q-series.

Definition 7. Let the q-Pochhammer symbol $\{a\}_{n,q}$ be defined by

$$\{a\}_{n,q} = \prod_{m=0}^{n-1} \{a + m\}_q. \tag{18}$$

An equivalent symbol is defined in [9, p.18] and is used throughout that book.

This quantity can be very useful in some cases where we are looking for q-analogues and it is included in the new notation.

If $|q| > 1$, or

$0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function $E_q(z)$ was defined by Jackson [13] 1904, and by Exton [9]

$$E_q(z) = \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{19}$$

Two q-analogues of the trigonometric functions are defined by

$$Sin_q(x) = \frac{1}{2i}(E_q(ix) - E_q(-ix)), \tag{20}$$

and

$$Cos_q(x) = \frac{1}{2}(E_q(ix) + E_q(-ix)). \tag{21}$$

The following multidimensional generalization of Bailey's transform was given by Exton [8, p.139].

Theorem 1.1. *If*

$$\gamma_{m_1,\ldots,m_n} = \sum_{p_1=m_1,\ldots,p_n=m_n}^{\infty} \delta_{p_1,\ldots,p_n} u_{p_1-m_1,\ldots,p_n-m_n} v_{p_1+m_1,\ldots,p_n+m_n}, \tag{22}$$

$$\beta_{m_1,\ldots,m_n} = \sum_{p_1,\ldots,p_n=0}^{m_1,\ldots,m_n} \alpha_{p_1,\ldots,p_n} u_{m_1-p_1,\ldots,m_n-p_n} v_{p_1+m_1,\ldots,p_n+m_n}, \tag{23}$$

then formally

$$\sum_{\vec{m}} \alpha_{m_1,\ldots,m_n} \gamma_{m_1,\ldots,m_n} = \sum_{\vec{m}} \beta_{m_1,\ldots,m_n} \delta_{m_1,\ldots,m_n}. \tag{24}$$

We assume that α, δ, u, v are functions of $m_1, \ldots, m_n$ only. The notation $\sum_{\vec{m}}$ denotes a multiple summation with the indices $m_1, \ldots, m_n$ running over all non-negative integer values.

Definition 8. The following notation will be convenient.

$$\mathrm{QE}(x) = q^x. \tag{25}$$

When there are several q:s, we generalize this to

$$\mathrm{QE}(x, q_i) = q_i^x. \tag{26}$$

We will only need one q-Lauricella function, which is defined by

$$\begin{aligned} &\Phi_D^{(n)}(a, b_1, \ldots, b_n; c|q; x_1, \ldots, x_n) = \\ &= \sum_{\vec{m}} \frac{\langle a; q\rangle_{m_1+\ldots+m_n} \langle b_1; q\rangle_{m_1} \ldots \langle b_n; q\rangle_{m_n} \prod_{j=1}^n x_j^{m_j}}{\langle c; q\rangle_{m_1+\ldots+m_n} \prod_{j=1}^n \langle 1; q\rangle_{m_j}}. \end{aligned} \tag{27}$$

The following reduction theorem is a q-analogue of Appell and Kampé de Fériet [1, p. 116]. The proof uses one of the q-Vandermonde summation formulas. Because there are two such formulas, there is a quite similar equation, which was published for $n = 2$ in [22, p. 224, 4.18].

Theorem 1.2.

$$\begin{aligned}&\Phi_D^{(n)}(a, b_1, \ldots, b_n; c|q; x, xq^{-b_2}, xq^{-b_2-b_3}, \ldots, xq^{-b_2-\ldots-b_n}) = \\ &= {}_2\phi_1(a, b_1 + \ldots + b_n; c|q, xq^{-b_2-\ldots-b_n}).\end{aligned} \tag{28}$$

Proof. In the LHS of (28) we change summation indices to $\{k_l\}_{l=1}^n$, where

$$k_l = \sum_{s=l}^{n} m_s. \tag{29}$$

By matrix inversion, this is equivalent to

$$m_l = k_l - k_{l+1},\ 1 \le l \le n-1,\ m_n = k_n. \tag{30}$$

$$\begin{aligned}
LHS &= \sum_{m_i} \frac{\langle a;q\rangle_{m_1+\ldots+m_n}\langle b_1;q\rangle_{m_1}\ldots\langle b_n;q\rangle_{m_n} x^{m_1+\ldots+m_n}\prod_{j=2}^n q^{-m_j(b_2+\ldots+b_j)}}{\langle c;q\rangle_{m_1+\ldots+m_n}\prod_{j=1}^n\langle 1;q\rangle_{m_j}} = \\
&\sum_{k_i} \frac{\langle a;q\rangle_{k_1}\prod_{j=1}^{n-1}\langle b_j;q\rangle_{k_j-k_{j+1}}\langle b_n;q\rangle_{k_n} x^{k_1}\prod_{j=2}^{n-1} q^{(k_{j+1}-k_j)(b_2+\ldots+b_j)} q^{(-k_n)(b_2+\ldots+b_n)}}{\langle c;q\rangle_{k_1}\prod_{j=1}^{n-1}\langle 1;q\rangle_{k_j-k_{j+1}}\langle 1;q\rangle_{k_n}} \\
&= \sum_{k_i} \frac{\langle a;q\rangle_{k_1}\prod_{j=1}^{n-1}\langle b_j;q\rangle_{k_j}\langle -k_j;q\rangle_{k_{j+1}} q^{(k_{j+1})(-b_j+1)}\langle b_n;q\rangle_{k_n} x^{k_1}}{\langle c;q\rangle_{k_1}\prod_{j=1}^{n-1}\langle 1-b_j-k_j;q\rangle_{k_{j+1}}\langle 1;q\rangle_{k_j}\langle 1;q\rangle_{k_n}} \times \\
&\prod_{j=2}^{n-1} q^{(k_{j+1}-k_j)(b_2+\ldots+b_j)} q^{(-k_n)(b_2+\ldots+b_n)} \\
&= \sum_{k_1,\ldots,k_{n-1}} \frac{\langle a, b_1;q\rangle_{k_1} x^{k_1}}{\langle c, 1;q\rangle_{k_1}} \prod_{j=2}^{n-2} \frac{\langle b_j, -k_{j-1};q\rangle_{k_j} q^{(k_j)(1-b_j-b_{j-1})}}{\langle 1, 1-b_{j-1}-k_{j-1};q\rangle_{k_j}} \times \\
&\frac{\langle b_{n-1}, -k_{n-2}, b_n + b_{n-1};q\rangle_{k_{n-1}} q^{(k_{n-1})(1-b_{n-2}-b_{n-1}-b_n)}}{\langle 1, 1-b_{n-2}-k_{n-2}, b_{n-1};q\rangle_{k_{n-1}}} = \\
&= \sum_{k_1,\ldots,k_{n-2}} \frac{\langle a, b_1;q\rangle_{k_1} x^{k_1}}{\langle c, 1;q\rangle_{k_1}} \prod_{j=2}^{n-2} \frac{\langle b_j, -k_{j-1};q\rangle_{k_j} q^{(k_j)(1-b_j-b_{j-1})}}{\langle 1, 1-b_{j-1}-k_{j-1};q\rangle_{k_j}} \times \\
&\frac{\langle 1-b_n-b_{n-1}-b_{n-2}-k_{n-2};q\rangle_{k_{n-2}}}{\langle 1-b_{n-2}-k_{n-2};q\rangle_{k_{n-2}}} = \\
&= \sum_{k_1,\ldots,k_{n-2}} \frac{\langle a, b_1;q\rangle_{k_1} x^{k_1}}{\langle c, 1;q\rangle_{k_1}} \prod_{j=2}^{n-2} \frac{\langle b_j, -k_{j-1};q\rangle_{k_j} q^{(k_j)(1-b_j-b_{j-1})}}{\langle 1, 1-b_{j-1}-k_{j-1};q\rangle_{k_j}} \times \\
&\frac{\langle b_n + b_{n-1} + b_{n-2};q\rangle_{k_{n-2}} q^{-k_{n-2}(b_n+b_{n-1})}}{\langle b_{n-2};q\rangle_{k_{n-2}}}.
\end{aligned} \tag{31}$$

We can continue this process to obtain the final result. ■

The following generalization of (28) is a q-analogue of [17] and [18]. Compare [20, (4.3), p. 107] for an important example of the general case.

Theorem 1.3. *If* $\{C_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ *are sequences of arbitrary complex numbers then*

$$\sum_{\mathbf{m}} \frac{C_{m_1+\ldots+m_n} \prod_{j=1}^{n} x^{m_j} \langle \alpha_j; q\rangle_{m_j}}{\prod_{j=1}^{n} \langle 1; q\rangle_{m_j}} \mathrm{QE}\left(-\sum_{k=1}^{n} m_k \sum_{l=2}^{k} \alpha_l\right) = \sum_{N=0}^{\infty} \frac{C_N x^N \langle \sum_{k=1}^{n} \alpha_k; q\rangle_N}{\langle 1; q\rangle_N} \mathrm{QE}\left(-N \sum_{l=2}^{n} \alpha_l\right). \tag{32}$$

Proof. We use induction. Suppose that (32) is true for $n > 1$, and denote the LHS by Δ_n, then

$$\begin{aligned}\Delta_{n+1} = &\sum_{m_{n+1}=0}^{\infty} \langle \alpha_{n+1}; q\rangle_{m_{n+1}} \frac{x^{m_{n+1}}}{\langle 1; q\rangle_{m_{n+1}}} \mathrm{QE}\left(-m_{n+1}(\alpha_2 + \ldots + \alpha_{n+1})\right) \times \\ &\sum_{m_1,\ldots,m_n} C_{m_1+\ldots+m_{n+1}} \frac{x^{m_1+\ldots+m_n}}{\prod_{i=1}^{n} \langle 1; q\rangle_{m_i}} \prod_{k=1}^{n} \langle \alpha_k; q\rangle_{m_k} \mathrm{QE}\left(-m_k \sum_{l=2}^{k} \alpha_l\right) \\ = &\sum_{m_{n+1}=0}^{\infty} \langle \alpha_{n+1}; q\rangle_{m_{n+1}} \frac{x^{m_{n+1}}}{\langle 1; q\rangle_{m_{n+1}}} \mathrm{QE}\left(-m_{n+1} \sum_{l=2}^{n+1} \alpha_l\right) \times \\ &\sum_{N=0}^{\infty} C_{N+m_{n+1}} \langle \sum_{k=1}^{n} \alpha_k; q\rangle_N \frac{x^N}{\langle 1; q\rangle_N} \mathrm{QE}\left(-N \sum_{l=2}^{n} \alpha_l\right) \\ = &\sum_{N=0}^{\infty} C_N x^N \mathrm{QE}\left(-N \sum_{l=2}^{n+1} \alpha_l\right) \frac{\langle \sum_{k=1}^{n+1} \alpha_k; q\rangle_N}{\langle 1; q\rangle_N},\end{aligned} \tag{33}$$

where in the last step we used the induction hypothesis for $n = 2$, with the following values of the parameters in (32).

$$m_1 \to N,\ m_2 \to m_n + 1,\ \alpha_2 \to \alpha_n + 1,\ x \to x\mathrm{QE}\left(-\sum_{l=2}^{n} \alpha_l\right). \tag{34}$$

∎

Remark 3. There is a dual too.

2 One variable

We begin with the case $n = 1$. In the rest of this paper, we assume that $|t| < 1, |q| < 1$.

Theorem 2.1. *If* $C(m)$ *is any arbitrary function, then, formally*

$$\sum_{m} \frac{C(m)\langle d; q\rangle_m t^m (tq^{d+m}; q)_\infty}{(t; q)_\infty} = \sum_{m} \frac{\langle d; q\rangle_m t^m}{\langle 1; q\rangle_m} \sum_{p=0}^{\infty} C(p) \langle -m; q\rangle_p (-1)^p q^{mp-\binom{p}{2}}. \tag{35}$$

Proof. In theorem 1.1 put

$$\alpha_m = C(m), \tag{36}$$

$$u_m = \frac{1}{\langle 1;q\rangle_m}, \tag{37}$$

$$v_m = 1 \tag{38}$$

and

$$\delta_m = \langle d;q\rangle_m t^m. \tag{39}$$

Now (22) and (23) imply that

$$\begin{aligned}\beta_m &= \sum_{p=0}^{m} \frac{C(p)}{\langle 1;q\rangle_{m-p}} = \\ &= \sum_{p=0}^{m} \frac{C(p)\langle -m;q\rangle_p}{\langle 1;q\rangle_m}(-1)^p q^{mp-\binom{p}{2}}.\end{aligned} \tag{40}$$

and

$$\begin{aligned}\gamma_m &= \sum_{p=m}^{\infty} \frac{\langle d;q\rangle_p t^p}{\langle 1;q\rangle_{p-m}} = \sum_{p=0}^{\infty} \frac{\langle d;q\rangle_{p+m} t^{p+m}}{\langle 1;q\rangle_p} = \\ &= \langle d;q\rangle_m t^m \sum_{p=0}^{\infty} \frac{\langle d+m;q\rangle_p t^p}{\langle 1;q\rangle_p} = \\ &= \langle d;q\rangle_m t^m \; {}_1\phi_0(d+m;-|q,t) \\ &= \langle d;q\rangle_m t^m \frac{(tq^{d+m};q)_\infty}{(t;q)_\infty}.\end{aligned} \tag{41}$$

The proof is completed by substituting (40) and (41) into (24). ∎

Theorem 2.2. *If $C(m)$ is any arbitrary function of m, then, formally*

$$\begin{aligned}&E_q(t)\sum_m C(m)t^m(1-q)^m = \\ &\sum_m \frac{t^m(1-q)^m}{\langle 1;q\rangle_m}\sum_{p=0}^{\infty} C(p)\langle -m;q\rangle_p(-1)^p q^{mp-\binom{p}{2}}.\end{aligned} \tag{42}$$

Proof. Let $d \to \infty$ in (35). ∎

The theorems 2.1 and 2.2 are much too general for many practical purposes when deriving generating functions for various classes of q-hypergeometric polynomials. A more convenient form is obtained by considering the following special case.

$$C(m) = \frac{\langle (a),(f);q\rangle_m (-x)^m q^{\theta(m)}}{\langle (h),(g),1;q\rangle_m}, \tag{43}$$

where $\theta(m)$ is an arbitrary function.

Theorem 2.1 can be written as

$$\begin{aligned}&\sum_m \frac{\langle (a),(f),d;q\rangle_m(-x)^m}{\langle (h),(g),1;q\rangle_m}\frac{t^m q^{\theta(m)}}{(t;q)_{d+m}} =\\ &=\sum_m \frac{\langle d;q\rangle_m t^m}{\langle 1;q\rangle_m}\sum_{p=0}^{\infty}\frac{\langle (a),(f);q\rangle_p (x)^p q^{\theta(p)}}{\langle (h),(g),1;q\rangle_p}\times\\ &\langle -m;q\rangle_p q^{mp-\binom{p}{2}}.\end{aligned} \tag{44}$$

Remark 4. Equation (44) is a q-analogue of [21, p. 328, (1.2)] and closely resembles [23, p. 108, (3.33)].

The confluent form

$$\begin{aligned}&E_q(t)\sum_m \frac{\langle (a),(f);q\rangle_m(-x)^m}{\langle (h),(g),1;q\rangle_m}t^m(1-q)^m q^{\theta(m)} =\\ &=\sum_m \frac{t^m(1-q)^m}{\langle 1;q\rangle_m}\sum_{p=0}^{\infty}\frac{\langle (a),(f);q\rangle_p (x)^p q^{\theta(p)}}{\langle (h),(g),1;q\rangle_p}\times\\ &\langle -m;q\rangle_p q^{mp-\binom{p}{2}}\end{aligned} \tag{45}$$

follows similarly from theorem 2.2.

Another special case is obtained by putting

$$C(m) = \frac{\langle (a),(f),x;q\rangle_m(-1)^m q^{\theta(m)}}{\langle (h),(g),1;q\rangle_m}, \tag{46}$$

where $\theta(m)$ is an arbitrary function.

Theorem 2.1 can then be written as

$$\begin{aligned}&\sum_m \frac{\langle (a),(f),d,x;q\rangle_m(-1)^m}{\langle (h),(g),1;q\rangle_m}\frac{t^m q^{\theta(m)}}{(t;q)_{d+m}} =\\ &=\sum_m \frac{\langle d;q\rangle_m t^m}{\langle 1;q\rangle_m}\sum_{p=0}^{\infty}\frac{\langle (a),(f),x;q\rangle_p q^{\theta(p)}}{\langle (h),(g),1;q\rangle_p}\times\\ &\langle -m;q\rangle_p q^{mp-\binom{p}{2}}.\end{aligned} \tag{47}$$

The confluent form

$$\begin{aligned}&E_q(t)\sum_m \frac{\langle (a),(f),x;q\rangle_m(-1)^m}{\langle (h),(g),1;q\rangle_m}t^m(1-q)^m q^{\theta(m)} =\\ &=\sum_m \frac{t^m(1-q)^m}{\langle 1;q\rangle_m}\sum_{p=0}^{\infty}\frac{\langle (a),(f),x;q\rangle_p q^{\theta(p)}}{\langle (h),(g),1;q\rangle_p}\times\\ &\langle -m;q\rangle_p q^{mp-\binom{p}{2}}\end{aligned} \tag{48}$$

follows similarly from theorem 2.2.

2.1 Special cases

Put $A = F = G = 0,\ H = 1,\ \theta(m) = m^2$. in (44) to obtain

$$\begin{aligned}&\sum_{m=0}^{\infty} \frac{(-x)^m}{\langle h;q\rangle_m \langle 1;q\rangle_m} \frac{\langle d;q\rangle_m t^m q^{m^2}}{(t;q)_{d+m}} = \\ &= \sum_{m=0}^{\infty} \frac{\langle d;q\rangle_m t^m}{\langle 1;q\rangle_m} \sum_{p=0}^{\infty} \frac{x^p}{\langle h;q\rangle_p \langle 1;q\rangle_p} \times \\ &\langle -m;q\rangle_p q^{mp+p^2-\binom{p}{2}}.\end{aligned} \tag{49}$$

By a change of variables $x \to\ xq^{h-1}(1-q)$ this is equivalent to

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{\{c\}_{n,q} L_{n,q}^{(\alpha)}(x) t^n}{\{1+\alpha\}_{n,q}} = \sum_{n=0}^{\infty} \frac{\{c\}_{n,q}\ q^{n^2+\alpha n}(-xt)^n}{\{n\}_q!\{1+\alpha\}_{n,q}(t;q)_{c+n}} \\ &\equiv \frac{1}{(t;q)_c} {}_1\phi_2(c;1+\alpha|q;-xtq^{1+\alpha}(1-q)||-;tq^c).\end{aligned} \tag{50}$$

This is a well known generating function for the q-Laguerre polynomials.

Put $A = H = 1,\ F = G = 0,\ \theta(m) = \binom{m}{2}$ in (45). The little q-Jacobi polynomials are defined by

$$P_n(x;a,b|q) = \sum_{p=0}^{n} \frac{\langle a+b+n+1,-n;q\rangle_p x^p q^p}{\langle a+1,1;q\rangle_p}. \tag{51}$$

Then we obtain the following generating function for the little q-Jacobi polynomials:

$$\begin{aligned}&\sum_n \frac{t^n(1-q)^n}{\langle 1;q\rangle_n} P_n(xq^{n-1};a,b|q) \\ &= E_q(t)\ {}_1\phi_1(a+b+n+1;a+1|q,xt(1-q)).\end{aligned} \tag{52}$$

Denote

$${}_4\phi_7(\alpha) \equiv {}_4\phi_7\left[\begin{array}{c} \frac{a+b+n+1}{2}, \widetilde{\frac{a+b+n+1}{2}}, \frac{a+b+n+2}{2}, \widetilde{\frac{a+b+n+2}{2}} \\ \frac{1+a}{2}, \widetilde{\frac{1+a}{2}}, \frac{2+a}{2}, \widetilde{\frac{2+a}{2}}, \frac{1}{2}, \widetilde{\frac{1}{2}}, \widetilde{1} \end{array} |q, q(1-q)^2x^2t^2\right], \tag{53}$$

$${}_4\phi_7(\beta) \equiv {}_4\phi_7\left[\begin{array}{c} \frac{a+b+n+2}{2}, \widetilde{\frac{a+b+n+2}{2}}, \frac{a+b+n+3}{2}, \widetilde{\frac{a+b+n+3}{2}} \\ \frac{2+a}{2}, \widetilde{\frac{2+a}{2}}, \frac{3+a}{2}, \widetilde{\frac{3+a}{2}}, \frac{3}{2}, \widetilde{\frac{3}{2}}, \widetilde{1} \end{array} |q, q^3(1-q)^2x^2t^2\right]. \tag{54}$$

Making use of the decomposition of a series into even and odd parts from [19, p.200,208], we can rewrite (52) in the form

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{P_{2n}(xq^{2n-1};a,b|q)t^{2n}}{\{2n\}_q!} + \sum_{n=0}^{\infty} \frac{P_{2n+1}(xq^{2n};a,b|q)t^{2n+1}}{\{2n+1\}_q!} \\ &= E_q(t)\left[{}_4\phi_7(\alpha) - xt\frac{\{1+a+b+n\}_q}{\{1+a\}_q} {}_4\phi_7(\beta)\right],\end{aligned} \tag{55}$$

and replacing t in (55) by it, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}P_{2n}(xq^{2n-1};a,b|q)}{\{2n\}_q!}+i\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}P_{2n+1}(xq^{2n};a,b|q)}{\{2n+1\}_q!}\\&=(Cos_q(t)+iSin_q(t))\left[{}_4\phi_7(\alpha)-ixt\frac{\{1+a+b+n\}_q}{\{1+a\}_q}{}_4\phi_7(\beta)\right].\end{aligned}\tag{56}$$

Next equate real and imaginary parts from both sides to arrive at the generating functions

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}P_{2n}(xq^{2n-1};a,b|q)}{\{2n\}_q!}\\&=Cos_q(t){}_4\phi_7(\alpha)+xtSin_q(t)\frac{\{1+a+b+n\}_q}{\{1+a\}_q}{}_4\phi_7(\beta)\end{aligned}\tag{57}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}P_{2n+1}(xq^{2n};a,b|q)}{\{2n+1\}_q!}\\&=Sin_q(t){}_4\phi_7(\alpha)-xtCos_q(t)\frac{\{1+a+b+n\}_q}{\{1+a\}_q}{}_4\phi_7(\beta).\end{aligned}\tag{58}$$

Finally put $A=F=G=0,\ H=1,\ \theta(m)=\binom{m}{2}$ in (47) to obtain the following generating function for q-Meixner polynomials from [23, p. 103, (3.10)]

$$\sum_m\frac{\langle x,d;q\rangle_m(-t)^m q^{\binom{m}{2}}}{\langle 1,h;q\rangle_m(t;q)_{d+m}}=\sum_m\frac{\langle d;q\rangle_m t^m}{\langle 1;q\rangle_m}{}_2\phi_1(x,-m_1;h|q,q^{m_1}).\tag{59}$$

3 Two variables

We can generalize (35) to two variables.

Theorem 3.1. *If $C(m_1,m_2)$ is any arbitrary function of m_1,m_2, then, formally*

$$\begin{aligned}&\sum_{\vec{m}}\frac{C(m_1,m_2)\langle d;q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j;q\rangle_{m_j}t^{m_1+m_2}}{\langle k_1+k_2;q\rangle_{m_1+m_2}}\times\\&\frac{(tq^{d+m_1-k_2};q)_\infty}{(tq^{-k_2-m_2};q)_\infty}=\\&=\sum_{\vec{m}}\frac{\langle d;q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j;q\rangle_{m_j}t^{m_1+m_2}q^{\frac{m_2^2}{2}}}{\langle k_1+k_2;q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle 1;q\rangle_{m_j}}\times\\&\sum_{p_1,p_2=0}^{\infty}C(p_1,p_2)\prod_{j=1}^{2}\langle -m_j;q\rangle_{p_j}(-1)^{p_1+p_2}\mathrm{QE}\left(-\sum_{j=1}^{2}\binom{p_j}{2}\right.\\&\left.-k_2(m_2-p_2)+m_1p_1+p_2^2\right).\end{aligned}\tag{60}$$

Proof. In theorem 1.1 put

$$\alpha_{m_1,m_2} = C(m_1, m_2), \tag{61}$$

$$u_{m_1,m_2} = \frac{\mathrm{QE}(\frac{3}{4}m_2^2 - k_2 m_2)}{\prod_{j=1}^{2}\langle 1; q\rangle_{m_j}}, \tag{62}$$

$$v_{m_1,m_2} = q^{\frac{1}{4}m_2^2} \tag{63}$$

and

$$\delta_{m_1,m_2} = \frac{q^{-m_2^2}\langle d; q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j; q\rangle_{m_j} t^{m_1+m_2}}{\langle k_1+k_2; q\rangle_{m_1+m_2}}. \tag{64}$$

Now (22) and (23) imply that

$$\begin{aligned}
&\beta_{m_1,m_2} = \sum_{p_1,p_2=0}^{m_1,m_2} \frac{C(p_1,p_2)}{\prod_{j=1}^{2}\langle 1; q\rangle_{m_j-p_j}} \\
&\times \mathrm{QE}(\tfrac{3}{4}(m_2-p_2)^2 - k_2(m_2-p_2) + \tfrac{1}{4}(m_2+p_2)^2) = \\
&= \sum_{p_1,p_2=0}^{m_1,m_2} \frac{C(p_1,p_2)\prod_{j=1}^{2}\langle -m_j; q\rangle_{p_j}}{\prod_{j=1}^{2}\langle 1; q\rangle_{m_j}} (-1)^{p_1+p_2}\mathrm{QE}\left(-\sum_{j=1}^{2}\binom{p_j}{2}\right) \\
&\mathrm{QE}(-k_2(m_2-p_2) + m_2^2 + m_1 p_1 + p_2^2),
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
&\gamma_{m_1,m_2} = \sum_{p_1=m_1,p_2=m_2}^{\infty} \frac{\langle d; q\rangle_{p_1+p_2}\prod_{j=1}^{2}\langle k_j; q\rangle_{p_j} t^{p_1+p_2}}{\prod_{j=1}^{2}\langle 1; q\rangle_{p_j-m_j}\langle k_1+k_2; q\rangle_{p_1+p_2}} \\
&\times \mathrm{QE}(-p_2^2 + \tfrac{3}{4}(p_2-m_2)^2 - k_2(p_2-m_2) + \tfrac{1}{4}(m_2+p_2)^2) = \\
&= \sum_{p_1,p_2=0}^{\infty} \frac{\langle d; q\rangle_{p_1+m_1+p_2+m_2}\prod_{j=1}^{2}\langle k_j; q\rangle_{p_j+m_j} t^{p_1+p_2+m_1+m_2}}{\prod_{j=1}^{2}\langle 1; q\rangle_{p_j}\langle k_1+k_2; q\rangle_{p_1+p_2+m_1+m_2}} \\
&\times \mathrm{QE}(-p_2(k_2+m_2)) = \\
&= \frac{\langle d; q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j; q\rangle_{m_j} t^{m_1+m_2}}{\langle k_1+k_2; q\rangle_{m_1+m_2}} \sum_{p_1,p_2=0}^{\infty} \mathrm{QE}(-p_2(k_2+m_2))\times \\
&\frac{\langle d+m_1+m_2; q\rangle_{p_1+p_2}\prod_{j=1}^{2}\langle k_j+m_j; q\rangle_{p_j} t^{p_1+p_2}}{\prod_{j=1}^{2}\langle 1; q\rangle_{p_j}\langle k_1+k_2+m_1+m_2; q\rangle_{p_1+p_2}} = \\
&= \frac{\langle d; q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j; q\rangle_{m_j} t^{m_1+m_2}}{\langle k_1+k_2; q\rangle_{m_1+m_2}} \Phi_D^{(2)}(d+m_1+m_2, \\
&, k_1+m_1, k_2+m_2; k_1+m_1+k_2+m_2|q; t, tq^{-k_2-m_2}) \\
&= \frac{\langle d; q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j; q\rangle_{m_j} t^{m_1+m_2}}{\langle k_1+k_2; q\rangle_{m_1+m_2}} \times \\
&{}_1\phi_0(d+m_1+m_2; -|q, tq^{-k_2-m_2}) \\
&= \frac{\langle d; q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j; q\rangle_{m_j} t^{m_1+m_2}}{\langle k_1+k_2; q\rangle_{m_1+m_2}} \frac{1}{(tq^{-k_2-m_2}; q)_{d+m_1+m_2}}.
\end{aligned} \tag{66}$$

The proof is completed by substituting (65) and (66) into (24). ∎

Theorem 3.2. *If $C(m_1, m_2)$ is any arbitrary function of m_1, m_2, then, formally*

$$
\begin{aligned}
&\sum_{\vec{m}} \frac{E_q(tq^{-k_2-m_2})C(m_1,m_2)\prod_{j=1}^{2}\langle k_j;q\rangle_{m_j} t^{m_1+m_2}(1-q)^{m_1+m_2}}{\langle k_1+k_2;q\rangle_{m_1+m_2}} = \\
&= \sum_{\vec{m}} \frac{\prod_{j=1}^{2}\langle k_j;q\rangle_{m_j} t^{m_1+m_2}(1-q)^{m_1+m_2} q^{\frac{m_2^2}{2}}}{\langle k_1+k_2;q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle 1;q\rangle_{m_j}} \times \\
&\sum_{p_1,p_2=0}^{\infty} C(p_1,p_2)\prod_{j=1}^{2}\langle -m_j;q\rangle_{p_j}(-1)^{p_1+p_2}\mathrm{QE}\left(-\sum_{j=1}^{2}\binom{p_j}{2}\right) \\
&\mathrm{QE}(-k_2(m_2-p_2)+m_1p_1+p_2^2).
\end{aligned} \tag{67}
$$

Proof. Let $d \to \infty$ in (60). ∎

The theorems 3.1 and 3.2 are much too general for many practical purposes when deriving generating functions for various classes of hypergeometric polynomials. A more convenient form is obtained by considering the following special case.

$$
C(m_1,m_2) = \frac{\langle (a);q\rangle_{m_1+m_2} q^{\theta(m_1,m_2)}\prod_{j=1}^{2}\langle (f_j);q\rangle_{m_j}(-x_j)^{m_j}}{\langle (h);q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle (g_j),1;q\rangle_{m_j}}, \tag{68}
$$

where $\theta(m_1, m_2)$ is an arbitrary function.

Theorem 3.1 can be written as

$$
\begin{aligned}
&\sum_{\vec{m}} \frac{\langle (a);q\rangle_{m_1+m_2} q^{\theta(m_1,m_2)}\prod_{j=1}^{2}\langle (f_j);q\rangle_{m_j}(-x_j)^{m_j}}{\langle (h),k_1+k_2;q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle (g_j),1;q\rangle_{m_j}} \times \\
&\frac{\langle d;q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j;q\rangle_{m_j} t^{m_1+m_2}(tq^{d+m_1-k_2};q)_\infty}{(tq^{-k_2-m_2};q)_\infty} = \\
&= \sum_{\vec{m}} \frac{\langle d;q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle k_j;q\rangle_{m_j} t^{m_1+m_2} q^{\frac{m_2^2}{2}}}{\langle k_1+k_2;q\rangle_{m_1+m_2}\prod_{j=1}^{2}\langle 1;q\rangle_{m_j}} \sum_{p_1,p_2=0}^{\infty} \\
&\frac{\langle (a);q\rangle_{p_1+p_2} q^{\theta(p_1,p_2)}\prod_{j=1}^{2}\langle (f_j),-m_j;q\rangle_{p_j} x_j^{p_j}}{\langle (h);q\rangle_{p_1+p_2}\prod_{j=1}^{2}\langle (g_j),1;q\rangle_{p_j}} \times \\
&\mathrm{QE}\left(-k_2(m_2-p_2)+m_1p_1+p_2^2-\sum_{j=1}^{2}\binom{p_j}{2}\right).
\end{aligned} \tag{69}
$$

The following confluent form follows similarly from theorem (3.2).

$$\sum_{\vec{m}} \frac{E_q(tq^{-k_2-m_2})\langle(a);q\rangle_{m_1+m_2} q^{\theta(m_1,m_2)}}{\langle(h),k_1+k_2;q\rangle_{m_1+m_2}\prod_{j=1}^2\langle(g_j),1;q\rangle_{m_j}}\times$$
$$t^{m_1+m_2}(1-q)^{m_1+m_2}\prod_{j=1}^2\langle(f_j),k_j;q\rangle_{m_j}(-x_j)^{m_j} =$$
$$= \sum_{\vec{m}} \frac{\prod_{j=1}^2\langle k_j;q\rangle_{m_j} t^{m_1+m_2}(1-q)^{m_1+m_2} q^{\frac{m_2^2}{2}}}{\langle k_1+k_2;q\rangle_{m_1+m_2}\prod_{j=1}^2\langle 1;q\rangle_{m_j}} \sum_{p_1,p_2=0}^{\infty} \tag{70}$$
$$\frac{\langle(a);q\rangle_{p_1+p_2} q^{\theta(p_1,p_2)}\prod_{j=1}^2\langle(f_j),-m_j;q\rangle_{p_j} x_j^{p_j}}{\langle(h);q\rangle_{p_1+p_2}\prod_{j=1}^2\langle(g_j),1;q\rangle_{p_j}}\times$$
$$\mathrm{QE}\left(-k_2(m_2-p_2)+m_1p_1+p_2^2-\sum_{j=1}^2\binom{p_j}{2}\right).$$

3.1 Special cases

Put $A=F=G=0,\ H=1,\ \theta(m_1,m_2)=m_1^2$ in (69). Then

$$\sum_{\vec{m}} \frac{q^{m_1^2}\langle d;q\rangle_{m_1+m_2} t^{m_1+m_2}\prod_{j=1}^2(-x_j)^{m_j}\langle k_j;q\rangle_{m_j}}{\langle h,k_1+k_2;q\rangle_{m_1+m_2}(tq^{-k_2-m_2};q)_{d+m_1+m_2}\prod_{j=1}^2\langle 1;q\rangle_{m_j}} =$$
$$\sum_{\vec{m}} \frac{\langle d;q\rangle_{m_1+m_2}\prod_{j=1}^2\langle k_j;q\rangle_{m_j} t^{m_1+m_2} q^{\frac{m_2^2}{2}}}{\langle k_1+k_2;q\rangle_{m_1+m_2}\prod_{j=1}^2\langle 1;q\rangle_{m_j}} \sum_{p_1,p_2=0}^{m_1,m_2} \frac{\mathrm{QE}(-k_2(m_2-p_2))}{\langle h;q\rangle_{p_1+p_2}}\times \tag{71}$$
$$\frac{\prod_{j=1}^2\langle -m_j;q\rangle_{p_j}(x_j)^{p_j}}{\prod_{j=1}^2\langle 1;q\rangle_{p_j}}\mathrm{QE}\left(m_1p_1+\sum_{j=1}^2-\binom{p_j}{2}+p_j^2\right).$$

By a change of variables $x_1\to\ x_1q^{h-1}(1-q),\ x_2\to\ x_2(1-q)$ this is equivalent to (compare [10, A.17])

$$\sum_{\vec{m}} \frac{q^{m_1^2+m_1(h-1)}\langle d;q\rangle_{m_1+m_2} t^{m_1+m_2}\prod_{j=1}^2(-x_j)^{m_j}\langle k_j;q\rangle_{m_j}(1-q)^{m_1+m_2}}{\langle h,k_1+k_2;q\rangle_{m_1+m_2}(tq^{-k_2-m_2};q)_{d+m_1+m_2}\prod_{j=1}^2\langle 1;q\rangle_{m_j}}$$
$$= \sum_{\vec{m}} \frac{\langle d;q\rangle_{m_1+m_2}\prod_{j=1}^2\langle k_j;q\rangle_{m_j} t^{m_1+m_2}}{\langle k_1+k_2,h;q\rangle_{m_1+m_2}} L_{m_1,m_2,k_2,q}^{h-1}(x_1,x_2), \tag{72}$$

where $L_{m_1,m_2,k_2,q}^{\alpha}(x_1,x_2)$ is the q-Laguerre polynomial in two variables given by

$$L_{m_1,m_2,k_2,q}^{\alpha}(x_1,x_2) = \frac{\langle\alpha+1;q\rangle_{m_1+m_2}}{\prod_{j=1}^2\langle 1;q\rangle_{m_j}} \sum_{p_1,p_2=0}^{m_1,m_2} \frac{q^{p_1^2+\alpha p_1}\prod_{j=1}^2\langle -m_j;q\rangle_{p_j}(x_j)^{p_j}}{\langle 1+\alpha;q\rangle_{p_1+p_2}\prod_{j=1}^2\langle 1;q\rangle_{p_j}}$$
$$\mathrm{QE}(m_1p_1+\tfrac{1}{2}m_2^2+p_2^2-k_2(m_2-p_2))(1-q)^{p_1+p_2}\prod_{j=1}^2\mathrm{QE}\left(-\binom{p_j}{2}\right). \tag{73}$$

By letting $d=h,\ d=k_1+k_2$ and $d\to\infty$ in (72), we obtain q-analogues of [10, A19-A21] for two variables.

Put $F = G = H = 0,\ A = 1,\ \theta(m_1, m_2) = m_1^2$ in (69). Then

$$\sum_{\vec{m}} \frac{q^{m_1^2}\langle a, d; q\rangle_{m_1+m_2} t^{m_1+m_2} \prod_{j=1}^2 (-x_j)^{m_j} \langle k_j; q\rangle_{m_j}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} (tq^{-k_2-m_2}; q)_{d+m_1+m_2} \prod_{j=1}^2 \langle 1; q\rangle_{m_j}} \cong$$
$$\sum_{\vec{m}} \frac{\langle d; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle k_j; q\rangle_{m_j} t^{m_1+m_2} q^{\frac{m_2^2}{2}}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle 1; q\rangle_{m_j}} \sum_{p_1,p_2=0}^{m_1,m_2} \mathrm{QE}(-k_2(m_2 - p_2))\times$$
$$\frac{\langle a; q\rangle_{p_1+p_2} \prod_{j=1}^2 \langle -m_j; q\rangle_{p_j} (x_j)^{p_j}}{\prod_{j=1}^2 \langle 1; q\rangle_{p_j}} \mathrm{QE}\left(m_1 p_1 + \sum_{j=1}^2 -\binom{p_j}{2} + p_j^2\right) \cong \tag{74}$$
$$\sum_{\vec{m}} \frac{\langle d; q\rangle_{m_1+m_2} t^{m_1+m_2} q^{\frac{m_2^2}{2}} \langle a; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle k_j; q\rangle_{m_j} (-x_j)^{m_j}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle 1; q\rangle_{m_j}} \sum_{p_1,p_2=0}^{m_1,m_2}$$
$$\frac{\prod_{j=1}^2 \langle -m_j; q\rangle_{p_j} (-x_j)^{-p_j}}{\langle -a + 1 - m_1 - m_2; q\rangle_{p_1+p_2} \prod_{j=1}^2 \langle 1; q\rangle_{p_j}} \mathrm{QE}\Big(-k_2(p_2) - m_1 p_1 + m_1^2$$
$$+ p_1 p_2 - (p_1 + p_2)(m_1 + m_2) + \sum_{j=1}^2 p_j^2 - (a - 1)p_j\Big).$$

This is a q-analogue of the corrected form of [10, A22] for two variables. The symbol $\cong$ denotes that the equality is purely formal.

Put $A = H = 0,\ F = G = 1,\ \theta(m_1, m_2) = -m_2^2 + \sum_{j=1}^2 \binom{m_j}{2}$ in (69). Then (compare [10, A29])

$$\sum_{\vec{m}} \frac{q^{-m_2^2} \prod_{j=1}^2 \langle f_j; q\rangle_{m_j} (-x_j)^{m_j} q^{\binom{m_j}{2}}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle g_j, 1; q\rangle_{m_j}} \times$$
$$\frac{\langle d; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle k_j; q\rangle_{m_j} t^{m_1+m_2} (tq^{d+m_1-k_2}; q)_\infty}{(tq^{-k_2-m_2}; q)_\infty} = \tag{75}$$
$$= \sum_{\vec{m}} \frac{\langle d; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle k_j; q\rangle_{m_j} t^{m_1+m_2} q^{\frac{m_2^2}{2} - k_2 m_2}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle 1; q\rangle_{m_j}}\ {}_2\phi_1(f_1, -m_1; g_1|q, x_1 q^{m_1})\times$$
$${}_2\phi_1(f_2, -m_2; g_2|q, x_2 q^{k_2}).$$

By letting $k_i = g_i,\ d = g_1 + g_2,\ d \to \infty$ and $(k_i = g_i,\ d \to \infty)$ in (75), we obtain q-analogues of [10, A30-A33].

Put $A = H = G = 0,\ F = 1,\ \theta(m_1, m_2) = -m_2^2 + \sum_{j=1}^2 \binom{m_j}{2}$ in (69). Then (compare [10, A34])

$$\sum_{\vec{m}} \frac{q^{-m_2^2} \prod_{j=1}^2 \langle f_j; q\rangle_{m_j} (-x_j)^{m_j} q^{\binom{m_j}{2}}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle 1; q\rangle_{m_j}} \times$$
$$\frac{\langle d; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle k_j; q\rangle_{m_j} t^{m_1+m_2} (tq^{d+m_1-k_2}; q)_\infty}{(tq^{-k_2-m_2}; q)_\infty} = \tag{76}$$
$$= \sum_{\vec{m}} \frac{\langle d; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle k_j; q\rangle_{m_j} t^{m_1+m_2} q^{\frac{m_2^2}{2} - k_2 m_2}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^2 \langle 1; q\rangle_{m_j}}\ {}_2\phi_1(f_1, -m_1; \infty|q, x_1 q^{m_1})\times$$
$${}_2\phi_1(f_2, -m_2; \infty|q, x_2 q^{k_2}).$$

By letting $d \to \infty$ in (76), we obtain a q-analogue of [10, A36].

Put $A = F = H = 0$, $G = 1$, $\theta(m_1, m_2) = m_1^2$ in (69). Then (compare [10, A38])

$$\sum_{\vec{m}} \frac{q^{m_1^2} \langle d; q\rangle_{m_1+m_2} t^{m_1+m_2} \prod_{j=1}^{2} (-x_j)^{m_j} \langle k_j; q\rangle_{m_j}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} (tq^{-k_2-m_2}; q)_{d+m_1+m_2} \prod_{j=1}^{2} \langle g_j, 1; q\rangle_{m_j}} =$$

$$\sum_{\vec{m}} \frac{\langle d; q\rangle_{m_1+m_2} t^{m_1+m_2} q^{\frac{m_2^2}{2}} \prod_{j=1}^{2} \langle k_j; q\rangle_{m_j}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^{2} \langle 1; q\rangle_{m_j}} \sum_{p_1,p_2=0}^{m_1,m_2} \mathrm{QE}(-k_2(m_2 - p_2)) \times \tag{77}$$

$$\frac{\prod_{j=1}^{2} \langle -m_j; q\rangle_{p_j} (x_j)^{p_j}}{\prod_{j=1}^{2} \langle g_j, 1; q\rangle_{p_j}} \mathrm{QE}(m_1 p_1 + \sum_{j=1}^{2} -\binom{p_j}{2} + p_j^2).$$

By a change of variables $x_j \to x_j q^{g_j-1}(1-q)$, $j = 1, 2$ this is equivalent to

$$\sum_{\vec{m}} \frac{q^{m_1^2} \langle d; q\rangle_{m_1+m_2} t^{m_1+m_2} (1-q)^{m_1+m_2} \prod_{j=1}^{2} (-x_j)^{m_j} q^{m_j(g_j-1)} \langle k_j; q\rangle_{m_j}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} (tq^{-k_2-m_2}; q)_{d+m_1+m_2} \prod_{j=1}^{2} \langle g_j, 1; q\rangle_{m_j}}$$

$$= \sum_{\vec{m}} \frac{\langle d; q\rangle_{m_1+m_2} t^{m_1+m_2} q^{\frac{m_2^2}{2}} \prod_{j=1}^{2} \langle k_j; q\rangle_{m_j}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^{2} \langle 1; q\rangle_{m_j}} \sum_{p_1,p_2=0}^{m_1,m_2} \mathrm{QE}(-k_2(m_2 - p_2)) \times$$

$$\frac{(1-q)^{p_1+p_2} \prod_{j=1}^{2} \langle -m_j; q\rangle_{p_j} (x_j)^{p_j} q^{p_j(g_j-1)}}{\prod_{j=1}^{2} \langle g_j, 1; q\rangle_{p_j}} \mathrm{QE}(m_1 p_1 + \sum_{j=1}^{2} -\binom{p_j}{2} + p_j^2) \tag{78}$$

$$= \sum_{\vec{m}} \frac{\langle d; q\rangle_{m_1+m_2} t^{m_1+m_2} q^{\frac{m_2^2}{2} - k_2 m_2} \prod_{j=1}^{2} \langle k_j; q\rangle_{m_j}}{\langle k_1 + k_2; q\rangle_{m_1+m_2} \prod_{j=1}^{2} \langle g_j; q\rangle_{m_j}} L_{m_1,q}^{g_1-1}(x_1) L_{m_2,q}^{g_2-1}(x_2 q^{k_2-m_2}).$$

By letting $k_i = g_i$, $d = g_1 + g_2$, $d \to \infty$ and ($k_i = g_i$, $d \to \infty$) in (78), we obtain q-analogues of [10, A39-A42] for two variables

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Department of Mathematics
Uppsala University
P.O. Box 480, SE-751 06 Uppsala, Sweden
email: Thomas.Ernst@math.uu.se