On a lower bound for the L-S category of a rationally elliptic space

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Abstract

We discuss a formula that the dimension of the rational cohomology gives a lower bound for the L-S category of a rationally elliptic space.

1 Introduction

In this paper, X is a simply connected finite cell complex. Recall that the L-S (Lusternik-Schnirelmann) category of X, $cat(X)$, is the least integer n such that X can be covered by $n+1$ open subsets contractible in X. The rational L-S category, $cat_0(X)$, is the least integer n such that $X \simeq_0 Y$ and $cat(Y) = n$. A simply connected space Y is called *(rationally) elliptic* if the rank of the homotopy group $\pi_*(Y)$ is finite and the rational cohomology $H^*(Y;Q)$ is finite dimensional.

Recently G.Lupton showed that $2cat_0(X) \leq \dim H^*(X;Q)$ for certain elliptic spaces X [8, Corollary 2.6]. In the case of elliptic spaces X, it seems, roughly speaking, that $cat(X)$ becomes larger when $\dim H^*(X;Q)$ does. So we want to give a lower bound for the L-S category of an elliptic space X in terms of its cohomology. We propose a

Problem. *If X is elliptic, then* $\dim H^*(X;Q) \leq 2^{cat(X)}$ *?*

Note that $cat_0(X) \leq N = \max\{i|H^i(X;Q) \neq 0\}$ in general [5, p.386] and $\dim H^*(X;Q) \leq 2^N$ if X is elliptic [2, p.61].

Recall that the rational cup length, $cup_0(X)$, is the largest integer n such that the n-product of $H^+(X;Q)$ is not zero. Also the rational Toomer invariant of X,

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$e_0(X)$, is given by using the Sullivan minimal model [9] $M(X) = (\Lambda V, d)$ as $sup\{n|$ there is an element $\alpha \in \Lambda^{\geq n}V$ such that $0 \neq [\alpha] \in H^*(X;Q)\}$. If $H^*(X;Q)$ is a Poincaré duality algebra, it is proved that $e_0(X) = cat_0(X)$ [4, Theorem 3]. If X is elliptic, $H^*(X;Q)$ has a Poincaré duality. Therefore if X is elliptic

$$cup_0(X) \leq e_0(X) = cat_0(X) \leq cat(X),$$

in which $cup_0(X) = e_0(X)$ if X is formal [5]. For example, there is an 11-dimensional manifold X whose rational homotopy type is given by the Sullivan minimal model $M(X) = (\Lambda(x, y, z), d)$ with the degrees $deg(x) = deg(y) = 3$, $deg(z) = 5$ and the differentials $d(x) = d(y) = 0$, $d(z) = xy$. We see that $H^*(X;Q) \cong \Lambda(x, y) \otimes Q[u, w]/(xy, xu, yw, u^2, uw, w^2, xw+yu)$ with $deg(u) = deg(w) = 8$. Notice that it is isomorphic to the cohomology of the non-elliptic manifold $Y = (S^3 \times S^8)\sharp(S^3 \times S^8)$, where S^n is the n-dimensional sphere and $\sharp$ is the connected sum. Then

$$2^{cat(Y)} = 2^2 < \dim H^*(X;Q) = 6 < 2^3 = 2^{e_0(X)} \leq 2^{cat(X)}.$$

Hence the formula of the problem does not hold in non-elliptic cases in general.

Recall that the toral rank of X, $rk(X)$, is the largest integer n such that an n-torus can act continuously on X with all its isotropy subgroups finite. Also $rk_0(X)$ is defined as $max\{rk(Y)|Y \simeq_0 X\}$. In [7], S.Halperin conjectured that $2^{rk(X)} \leq \dim H^*(X;Q)$ in general. If X is elliptic, by [1] and [3],

$$rk(X) \leq rk_0(X) \leq -\chi_\pi(X) \leq rank\ \pi_{odd}(X) \leq e_0(X),$$

where $\chi_\pi(X) = \sum_i(-1)^i rank\ \pi_i(X)$. Thus our problem does not contradict his conjecture. Especially note that $2^{rk(X)} = \dim H^*(X;Q) = 2^{cat(X)}$ if X is the product space of some odd-dimensional spheres. In this paper, we give a partial answer to our problem;

Theorem 1.1. *If X is elliptic with rank $\pi_{even}(X) \leq 1$, $\dim H^*(X;Q) \leq 2^{cat(X)}$.*

Note that even if the formula of the problem holds for elliptic spaces X and Y, it does not for the non-elliptic space $X \vee Y$, one point union of X and Y, in general. Indeed $\dim H^*(X \vee Y;Q) = \dim H^*(X;Q) + \dim H^*(Y;Q) - 1$ but $cat(X \vee Y) = max\{cat(X), cat(Y)\}$.

In the following sections, we use Sullivan minimal model of a simply connected space Y. It is a free Q-commutative differential graded algebra $(\Lambda V, d)$ with a graded vector space $V = \oplus_{i>1}V^i$ and a minimal differential, i.e., $d(V^i) \subset (\Lambda^+V \cdot \Lambda^+V)^{i+1}$. Especially note that $V^i \cong Hom(\pi_i(Y), Q)$ and $H^*(\Lambda V, d) \cong H^*(Y;Q)$. Refer [5] for a general introduction and notations, for example, rational fibration and K-S extension. Since cup_0, e_0 and cat_0 are rational homotopy invariants, we denote often $cup_0(\Lambda V, d)$, $e_0(\Lambda V, d)$ and $cat_0(\Lambda V, d)$ for a minimal model $(\Lambda V, d)$ of a space. For an element v, we denote $deg(v)$ as $|v|$.

We give the proof of Theorem 1.1 in Section 2 and some examples of the case of rank $\pi_{even}(X) = 2$ in Section 3.

2 Proof

Lemma 2.1. *Let X be the total space of the rational fibration $F \to X \to S^{2n+1}$ with an elliptic space F satisfying* $\dim H^*(F;Q) \leq 2^{e_0(F)}$. *Then* $\dim H^*(X;Q) \leq 2^{e_0(X)}$.

Proof. By [2, Lemme 5.6.3], $e_0(F) < e_0(X)$. Therefore we have $\dim H^*(X;Q) \leq \dim H^*(F;Q) \cdot \dim H^*(S^{2n+1};Q) = 2\dim H^*(F;Q) \leq 2 \cdot 2^{e_0(F)} = 2^{e_0(F)+1} \leq 2^{e_0(X)}$. ∎

Proof of Theorem 1.1. If rank $\pi_{even}(X) = 0$, Then $M(X) = (\Lambda(v_1, \cdots, v_m), d)$ with $|v_i|$ are odd. Note the element $v_1 \cdots v_m$ is non-exact d-cocycle. Thus $e_0(X) = m$ and $\dim H^*(X;Q) \leq \sharp\{v_1^{\epsilon_1} \cdots v_m^{\epsilon_m} | \epsilon_i = 0, 1\} = 2^m$

Let rank $\pi_{even}(X) = 1$. Up to isomorphisms, we can put the Sullivan minimal model of X as

$$M(X) = (\Lambda(x, z, v_1, \cdots, v_m), d) \quad ;$$

(i) $|x|$ is even, the other elements are odd and $i < j$ if $|v_i| < |v_j|$,
(ii) $d(z) = x^n + f$ for some $n > 1$ and $f \in I(v_1, \cdots, v_m)$,
(iii) $d(v_i) \in I(v_1, \cdots v_{i-1})$ for any $1 \leq i \leq m$,
where $I(*)$ is the ideal of $\Lambda(x, z, v_1, \cdots, v_m)$ generated by $*$. Factor out the differential graded ideal generated by v_i inductively for $i = 1, \cdots, m$ as the K-S extensions

$$(\Lambda(v_i), 0) \to (\Lambda(x, z, v_i, \cdots, v_m), \overline{d}) \to (\Lambda(x, z, v_{i+1}, \cdots, v_m), \overline{d}),$$

in which the differential $\overline{d}$ of the right side is induced from one of the left. Put $W_i = Q\{x, z, v_{i+1}, \cdots, v_m\}$ for $i = 0, \cdots, m$. Note that $\dim H^*(\Lambda W_i, \overline{d}) < \infty$ for $i = 1, .., m$ [6]. Finally we have $(\Lambda W_m, \overline{d}) = (\Lambda(x, z), \overline{d})$ with $\overline{d}(z) = x^n$ and $\overline{d}(x) = 0$. Then $e_0(\Lambda(x, z), \overline{d}) = cup_0(\Lambda(x, z), \overline{d}) = n-1$. and $\dim H^*(\Lambda(x, z), \overline{d}) = n \leq 2^{n-1} = 2^{e_0(\Lambda(x,z),\overline{d})}$. By using Lemma 2.1 inductively, we have

$$\dim H^*(\Lambda W_i, \overline{d}) \leq 2^{e_0(\Lambda W_i, \overline{d})}$$

for $i = m-1, \cdots, 1$ and 0. Finally we have $\dim H^*(X;Q) \leq 2^{e_0(X)} = 2^{cat_0(X)} \leq 2^{cat(X)}$. ∎

3 case of $rank\ \pi_{even}(X) = 2$

In this section, we give some examples of elliptic spaces X of rank $\pi_{even}(X) = 2$ such that the formula of our problem holds.

(1) $H^*(X;Q) = Q[x, y]/(f, g)$. Let $cup_0(X) = n$. Then $\dim Q[x, y]/(f, g) \leq \sharp\{x^i y^j | 0 \leq i + j < n\} + 1 = \frac{n(n+1)}{2} + 1 \leq 2^n = 2^{cup_0(X)} = 2^{e_0(X)}$.

(2) $M(X) = (\Lambda(x, y, v_1, v_2, v_3), d)$ with $|x| = |y| = 2$, $|v_1| = 2n+3$, $|v_2| = |v_3| = 3$ and $dv_1 = x^{n+2}$, $dv_2 = xy$, $dv_3 = y^2$. Then $H^*(X;Q) \cong Q\{1, x, x^2, \cdots, x^{n+1}, y, [yv_1 - x^{n+1}v_2], x^i[yv_2 - xv_3]$ for $i = 0, \cdots, n+1\} \cong H^*((CP^{n+1} \times S^5)\sharp(S^2 \times S^{2n+5}); Q)$, Thus $e_0(X) = n+3$ and $\dim H^*(X;Q) = 2n+6 < 2^{n+3} = 2^{e_0(X)}$. Here CP^{n+1} is

the complex $n+1$-dimensional projective space.

(3) $M(X) = (\Lambda(x, y, z, a, b, c),\ d)$ with $|x| = 2, |y| = 3, |z| = 3, |a| = 4, |b| = 5, |c| = 7$ and $d(x) = d(y) = 0, d(z) = x^2, d(a) = xy, d(b) = xa + yz, d(c) = a^2 + 2yb$. Then $H^*(X;Q) \cong Q\{1, x, y, [ya], [xb-za], [x^2c-xab+yzb], [3xyc+a^3], [3x^2yc+xa^3]\}$. Thus $e_0(X) = 4$ and $\dim H^*(X;Q) = 8 < 2^4 = 2^{e_0(X)}$.

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