On a method of approximation by Jacobi polynomials

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Abstract

Convolution structure for Jacobi series allows end point summability of Fourier-Jacobi expansions to lead an approximation of function by a linear combination of Jacobi polynomials. Thus, using Cesàro summability of some orders > 1 at $x = 1$, we prove a result of approximation of functions on $[-1, 1]$ by operators involving Jacobi polynomials. Precisely, we pick up functions from a Lebesgue integrable space and then study its representation by Jacobi polynomials under different conditions.

1 Introduction

We write X to denote either of the spaces $L^p_{\alpha,\beta}$ or C. The space is p-power Lebesgue integrable functions with weight $w(x)$ on $-1 \leq x \leq 1$, and C is space of all continuous functions on $[-1, 1]$. The expansion of the function $f(x) \in X$ at $x = \cos\theta$, in the form of Jacobi series is given by

$$\begin{aligned} f(\cos\theta) &\sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(\cos\theta) \qquad (1.1) \\ &\equiv \sum U_n(\cos\theta) \qquad \text{(say)} \end{aligned}$$

where

$$a_n = \frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}} \cdot \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \qquad (1.2)$$

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$$\int_0^\pi (1-\cos\omega)^\alpha (1+\cos\omega)^\beta P_n^{(\alpha,\beta)}(\cos\omega)\sin\omega f(\cos\omega)d\omega$$

$P_n^{(\alpha,\beta)}(\cos\theta), \alpha > -1, \beta > -1$, is n-th Jacobi polynomial of order (α,β) and the existence of the integral in (1.2) is presumed.

We write

$$f(\omega) = \{f(\cos\omega) - A\}$$

$$T_n^\delta = \frac{1}{A_n^\delta}\sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} g_\nu P_\nu^{(\alpha,\beta)}(1) P_\nu^{(\alpha,\beta)}(\cos\omega)\sin\omega \tag{1.3}$$

$$L_\nu^\delta = \frac{1}{A_n^\delta}\sum_{k}^{\nu} A_{n-k}^{\delta-1} g_k P_k^{(\alpha,\beta)}(1) P_k^{(\alpha,\beta)}(\cos\omega)\sin\omega \tag{1.4}$$

where

$$g_\nu = \frac{2\nu+\alpha+\beta+1}{2^{\alpha+\beta+1}}.\frac{\Gamma(\nu+1)\Gamma(\nu+\alpha+\beta+1)}{\Gamma(\nu+\alpha+1)\Gamma(\nu+\beta+1)} = O(\nu) \tag{1.5}$$

$$g_{n,\alpha} = \frac{2^{-\alpha-\beta-1}.\,\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \cong n^{\alpha+1} \tag{1.6}$$

2 Preliminaries

Cesàro summability of the series (1.1) has been discussed in detail by Kogbetliantz, E.[2], Szegö, G.[5] and Obrechkoff, N.[3]. In a recent paper Pandey, G.S.[4] has proved the following theorems on Cesàro summability of the series (1.1) at the frontier as well as at the internal points of the interval $[-1,+1]$. These theorems are:

Theorem A. *If, for* $-\frac{1}{2} < k < \frac{1}{2}$, $\alpha \geq -\frac{1}{2}$, $\beta \geq \alpha$

$$f(x) \in \mathrm{lip}(1/2 - k) \tag{2.1}$$

then the series (1.1) *is summable* $(C, k-\frac{1}{2})$ *to the sum* $f(x)$ *at an interior point* x *of the interval* $[-1+\epsilon,\ 1-\epsilon]$, $\epsilon > 0$, *but fixed.*

Theorem B. *The series (1.1) is summable* (C,k) *for* $\alpha - \frac{1}{2} < k < \alpha + \frac{1}{2}$, $-\frac{1}{2} < \alpha < \frac{1}{2}$, $\beta \geq \alpha$ *at* $\theta = 0$ *to the sum* A *provided that*

$$f(\omega) \in \mathrm{lip}(\alpha + 1/2 - k) \tag{2.2}$$

The following result is due to Yadav [6] .

Theorem C. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent. Then the series* $\sum\{a_n P_n^{(\alpha,\beta)}(x)\lambda_n\}$, $-\frac{1}{2} < \alpha < \frac{1}{2}$, $\beta \geq \alpha$, *is absolutely summable* (C,δ), $1 < \delta < 2$ *at the point* $x = 1$ *of the interval* $[-1,1]$ *provided*

$$f(\omega) \in \mathrm{lip}(2-\delta) \tag{2.3}$$

3 Main Result

The object of the present paper is to discuss the approximation of a function of X by an operator formed by (C,δ) mean of the series (1.1) which is a linear combination of Jacobi polynomials. At first we prove the convergence of (C,δ) mean and later an approximation theorem. Our first theorem on convergence of Cesàro operators is as follows:

Theorem 1. *If $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent then $(C,\delta)_{1<\delta<2}$ transform $T_n^\delta(\cos\theta, f, X)$ of the series (1.1) with factor $\{\lambda_n\}$ converges at $\theta = 0$, for -$\frac{1}{2} < \alpha < \frac{1}{2}$, $\beta \geq \alpha$,, or at the point $x = 1$ of the interval $[-1,1]$ provided*

$$f(\omega) \in \mathrm{lip}(2-\delta) \tag{3.1}$$

Our approximation theorem is as follows:

Theorem 2. *There exists a linear combination $T_n^\delta(\cos\theta, f, X)$ of Jacobi polynomials such that for $\alpha = \beta \geq -\frac{1}{2}$ and $-\frac{1}{2} < \alpha < \frac{1}{2}$*

$$||T_n^\delta(\cos\theta, f, X) - f(\cos\theta)|| \to 0 \ \ as, \ \ n \to \infty \tag{3.2}$$

Uniformly in $[0,\pi]$ under the conditions of theorem 1, where $T_n^\delta(\cos\theta, f, X)$ is Cesàro mean of order δ, $(1 < \delta < 2)$) of the Jacobi series (1.1) i.e.

$$T_n^\delta(\cos\theta, f, X) = \sum_{\nu=0}^{n} \left(\frac{A_{n-\nu}^{\delta-1}}{A_n^\delta}\right) S_\nu(\cos\theta) \tag{3.3}$$

and

$$S_\nu(\cos\theta) = \sum_{k=0}^{\nu} \lambda_k a_k P_k^{(\alpha,\beta)}(\cos\theta) \tag{3.4}$$

is partial sum of the Jacobi series(1.1).

4 Lemmas

To prove the theorems, we use the following results given as lemmas.

Lemma 1. *For $0 \leq \omega \leq \gamma_n$, $\gamma_n = n^{-(2\alpha+2)(4+2\alpha-\delta)^{-1}}$ we have*

$$T_n^\delta = O(n^{2\alpha+1}\omega) \tag{4.1}$$

Proof: We have (see (1.3)),

$$\begin{aligned} T_n^\delta &= \frac{1}{A_n^\delta} \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} g_\nu P_\nu^{(\alpha,\beta)}(1) P_\nu^{(\alpha,\beta)}(\cos\omega)\sin\omega \\ &= n^{-\delta}.A \sum_{\nu=0}^{n} n^{\delta-1}\nu^{2\alpha+1}\omega \end{aligned}$$

where A is an independent constant.

$$= O(n^{2\alpha+1}\omega), \qquad \text{since } \delta > 1$$

Hence the lemma.

Lemma 2. *For $\pi - n^{-1} \leq \omega \leq \pi$, we have*

$$T_n^\delta = O(n^{\alpha+\beta+1}) \sin\omega \tag{4.2}$$

Proof:

$$\begin{aligned} T_n^\delta &= \frac{1}{A_n^\delta} \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} g_\nu P_\nu^{(\alpha,\beta)}(1) P_\nu^{(\alpha,\beta)}(\cos\omega) \sin\omega \\ &= \left\{ n^{-\delta}.A \sum_{\nu=0}^{n} A_{n-\nu}^{\delta-1} \nu^{\alpha+\beta+1} \right\} \sin\omega \\ &= O(n^{\alpha+\beta+1}) \sin\omega \qquad \text{since } \delta > 1 \end{aligned}$$

which proves the lemma.

Lemma 3. *For $0 \leq \omega \leq \gamma_n$, $\gamma_n = n^{-(2\alpha+2)(4+2\alpha-\delta)^{-1}}$ we have*

$$L_\nu^\delta = O(\nu^{2\alpha+1}) \sin\omega \tag{4.3}$$

Proof:

$$\begin{aligned} L_\nu^\delta &= \frac{1}{A_n^\delta} \sum_{k=0}^{\nu} A_{n-k}^{\delta-1} g_k P_k^{(\alpha,\beta)}(1) P_k^{(\alpha,\beta)}(\cos\omega) \sin\omega \\ &= O(n^{-\delta}) \left[\sum_{k=0}^{\nu} A_{n-k}^{\delta-1} O(k^{2\alpha+1}) \right] \sin\omega \end{aligned}$$

(since $|P_n^{(\alpha,\beta)}(\cos\omega)| = O(n^\alpha)$, for $\alpha \geq \frac{1}{2}$ and $P_k^{(\alpha,\beta)}(1) = k^\alpha$).

$$\begin{aligned} &= O(n^{-\delta}) O(n^{\delta-1}) \nu^{2\alpha+2} \sin\omega \\ &= O(\nu^{2\alpha+1}) \sin\omega, \qquad \text{since } \nu \leq n. \end{aligned}$$

Lemma 4. *For $\pi - 1/n \leq \omega \leq \pi$, we have*

$$L_\nu^\delta = O(\nu^{\alpha+\beta+1}) \sin\omega \tag{4.4}$$

Proof of the lemma is as that of lemma 3.

Lemma 5. *If $\gamma_n \leq \omega \leq \pi - 1/n$, $(\gamma_n \geq 1/n)$, and E_n and G_n are respectively real and imaginary parts of*

$$E \equiv \{M(\omega)\} e^{-\frac{\pi}{2} i(\alpha+\frac{1}{2})} \int_{-\infty}^{\omega} (\omega - t)^{-\alpha-\frac{3}{2}} \times$$

$$\sum_{\nu=1}^{n} A_{n-\nu}^{\delta-2}\left\{e^{i(\nu+\frac{\alpha+\beta}{2}+1)\omega} - e^{i(\nu+\frac{\alpha+\beta}{2}+1)t}\right\}dt \tag{4.5}$$

such that

$$M(\omega) = (\sin\omega/2)^{-\alpha-\frac{1}{2}}(\cos\omega/2)^{-\beta+\frac{1}{2}}$$

then

$$T_n^\delta = (A_n^\delta)^{-1}E_n + (A_n^\delta)^{-1}G_n + O\left(n^{\alpha-3/2}\sin^{-\alpha-3/2}\frac{\omega}{2}\cos^{-\beta-1/2}\frac{\omega}{2}\right) \tag{4.6}$$

Proof of the lemma follows by direct calculation substituting the orders of Jacobi polynomials and asymptotic values given by Szegö ([5], p. 71 and p. 196).

Lemma 6. *If $\gamma_n \le \omega \le \pi - 1/n$, $(\gamma_n \ge 1/n)$, we have*

$$E \equiv M(\omega)e^{i\frac{\pi}{2}(\alpha+\frac{1}{2})}e^{i(n+\frac{\alpha+\beta}{2}+1)\omega}\psi(\omega) \tag{4.7}$$

where

$$\psi(\omega) \equiv \int_0^\infty u^{\alpha-3/2}\left[k_n(\omega) - k_n(\omega-u)e^{i(n+\frac{\alpha+\beta}{2}+1)u}\right]du \tag{4.8}$$

such that

$$k_n(\omega) = \sum_{m=0}^{n} A_m^{\delta-2}e^{im\omega} \tag{4.9}$$

Proof: Proof the lemma is parallel to that of lemma 5.

Lemma 7. *We have*

$$\psi(\omega) = O(n^{\alpha+1/2}\omega^{1-\delta}) \tag{4.10}$$

and

$$\psi(\omega+\mu_n) - \psi(\omega) = O(n^{\delta+\alpha-3/2}\log n\omega^{-1}) \tag{4.11}$$

where

$$\mu_n = \frac{\pi}{n+\frac{\alpha+\beta}{2}+1}$$

Proof: Proof of the lemma is a consequence of direct calculations.

Lemma 8. *Combining lemmas 5 and 6, we have for If $\gamma_n \le \omega < \pi - 1/n$, $(\gamma_n \ge 1/n)$,*

$$\begin{aligned} T_n^\delta &= R_n\left\{M(\omega)e^{-\frac{\pi}{2}i(\alpha+\frac{1}{2})}e^{i(n+\frac{\alpha+\beta}{2}+1)\omega}n^{-\delta}\psi(\omega)\right\} \\ &+ I_n\left\{M(\omega)e^{-i\frac{\pi}{2}(\alpha+\frac{1}{2})}e^{i(n+\frac{\alpha+\beta}{2}+1)\omega}n^{-\delta}\psi(\omega)\right\} \\ &+ O\left(n^{-\alpha-3/2}\sin^{-\alpha-3/2}\frac{\omega}{2}\cos^{-\beta-1/2}\frac{\omega}{2}\right) \end{aligned} \tag{4.12}$$

where

$$\psi(\omega) = O(n^{\alpha+1/2}\omega^{1-\delta})$$

and

$$\psi(\omega+\mu_n) - \psi(\omega) = O(n^{\delta+\alpha-3/2}\log n\omega^{-1})$$

Lemma 9. *If, $\gamma_n \leq \omega \leq \pi - 1/n,\ (\gamma_n \geq 1/n)$,$E_n^1$ and G_n^1 and are respectively real and imaginary parts of*

$$E^1 = \{M(\omega)\}e^{i(\alpha+1/2)\frac{\pi}{2}} \int_{-\infty}^{\omega} (\omega - t)^{-\alpha-3/2} \sum_{k=0}^{\nu-1} A_{n-k}^{\delta-2} \times \left\{ e^{i(k+\frac{\alpha+\beta}{2}+1)\omega} - e^{i(k+\frac{\alpha+\beta}{2}+1)t} \right\} dt$$

where

$$M(\omega) = (\sin\omega/2)^{-\alpha-\frac{1}{2}}(\cos\omega/2)^{-\beta+\frac{1}{2}}$$

then

$$T_n^\delta = (A_n^\delta)^{-1}E_n^1 + (A_n^\delta)^{-1}G_n^1 + O\left[n^{-\delta}\nu^{\alpha+\delta-3/2}\left(\sin\frac{\omega}{2}\right)^{-\alpha-\frac{3}{2}}\left(\cos\frac{\omega}{2}\right)^{-\beta-\frac{1}{2}} \right] \quad (4.13)$$

Proof: Proof of the lemma is similar to that of lemma 5.

Lemma 10. *If, $\gamma_n \leq \omega \leq \pi - 1/n,\ (\gamma_n \geq 1/n)$,*

$$\mu_n = \frac{\pi}{n + \frac{\alpha+\beta}{2} + 1}$$

We have

$$E^1 = O\{M(\omega)\}e^{i(\alpha+1/2)\frac{\pi}{2}} e^{i\left(n+\frac{\alpha+\beta}{2}+1\right)\omega}\ \phi(\omega) \quad (4.14)$$

such that

$$\phi(\omega) = O(n^{\alpha+1/2}\omega^{1-\delta})$$

and

$$\phi(\omega+\mu_n) - \phi(\omega) = O(n^{\delta+\alpha-3/2}\log n\omega^{-1})$$

Proof: Again the proof is similar to that of lemma 5.

Lemma 11. *If, $\gamma_n \leq \omega \leq \pi - 1/n,\ (\gamma_n \geq 1/n)$, we have*

$$\begin{aligned} L_\nu^\delta &= R_n\left\{ M(\omega)e^{-i(\alpha+\frac{1}{2})\pi/2}e^{i(n+\frac{\alpha+\beta}{2}+1)\omega}n^{-\delta}\psi(\omega) \right\} \\ &+ I_n\left\{ M(\omega)e^{-i(\alpha+\frac{1}{2})\pi/2}e^{i(n+\frac{\alpha+\beta}{2}+1)\omega}n^{-\delta}\psi(\omega) \right\} \\ &+ O\left(n^{-\delta}\nu^{\alpha+\delta-3/2}\left(\sin\frac{\omega}{2}\right)^{-\alpha-3/2}\left(\cos\frac{\omega}{2}\right)^{-\beta-1/2} \right\} \end{aligned} \quad (4.15)$$

such that

$$\psi(\omega) = O(n^{\alpha+1/2})\omega^{1-\delta}$$

and

$$\psi(\omega + \mu_n) - \psi(\omega) = O(n^{\delta+\alpha-3/2} \log n\omega^{-1})$$

Proof: Combining lemmas 9 and 10, we have the result of lemma 11.

Proof of Theorem 1.

Calculating on the lines of Yadav [6] and using lemma 1 to 11. We find the bounded variation of $T_n^\delta(\cos\theta, f, X)$ at $\theta = 0$ when $1 < \delta < 2$.

Thus the sequence $\{T_n^\delta(\cos\theta, f, X)\}$ converges at $\theta = 0$ i.e. $x = 1$ of the interval $[-1, 1]$. This completes the proof of Theorem 1.

Proof of Theorem 2.

We have

$$T_n^\delta(\cos\theta, f, X) = \frac{1}{A_n^\delta} \sum_{k=0}^{n} A_{n-k}^{\delta-1} S_k(\cos\theta)$$

where

$$S_k(\cos\theta) = \sum_{i=0}^{\nu} a_i P_i^{(\alpha,\beta)}(\cos\theta)\lambda_i$$

Since the end point convergence of the sequence of operators $T_n^\delta(\cos\theta, f, X)$ implies its convergence in the whole interval $[-1, 1]$ (or $[0, \pi]$) by the theorem proved by Yadav[7] for $\alpha \geq \beta \geq -1/2$. Thus

$$\left|\left|T_n^\delta(\cos\theta, f, X)\right|\right|_X \leq \left|\left|f(\cos\theta)\right|\right|_X$$

and

$$||T_n^\delta(\cos\theta, f, X) - f(\cos\theta)|| \ \rightarrow 0.$$

This completes the proof of theorem 2.

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