Compact endomorphisms of certain analytic Lipschitz algebras

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Abstract

Let X be a compact plane set. $A(X)$ denotes the uniform algebra of all continuous complex-valued functions on X which are analytic on $\mathrm{int}X$. For $0 < \alpha \leq 1$, Lipschitz algebra of order α, $\mathrm{Lip}(X, \alpha)$ is the algebra of all complex-valued functions f on X for which $p_\alpha(f) = \sup\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in X, x \neq y\} < \infty$. Let $\mathrm{Lip}_A(X, \alpha) = A(X) \bigcap \mathrm{Lip}(X, \alpha)$, and $\mathrm{Lip}^n(X, \alpha)$ be the algebra of complex-valued functions on X whose derivatives up to order n are in $\mathrm{Lip}(X, \alpha)$. $\mathrm{Lip}_A(X, \alpha)$ under the norm $\|f\| = \|f\|_X + p_\alpha(f)$, and $\mathrm{Lip}^n(X, \alpha)$ for a certain plane set X under the norm $\|f\| = \sum_{k=0}^n \frac{\|f^{(k)}\|_X + p_\alpha(f^{(k)})}{k!}$ are natural Banach function algebras, where $\|f\|_X = \sup_{x\in X} |f(x)|$.

In this note we study endomorphisms of algebras $\mathrm{Lip}_A(X, \alpha)$ and $\mathrm{Lip}^n(X, \alpha)$ and investigate necessary and sufficient conditions for which these endomorphisms to be compact. Finally, we determine the spectra of compact endomorphisms of these algebras.

1 Introduction and Preliminaries

Let B be a unital commutative semi-simple Banach algebra with a maximal ideal space $\mathcal{M}(B)$. If T is a nonzero bounded endomorphism of B, then there exists a map $\varphi : \mathcal{M}(B) \to \mathcal{M}(B)$ such that $\widehat{Tf} = \hat{f} \circ \varphi$ for all $f \in B$. In fact φ is equal to the adjoint T^* restricted to $\mathcal{M}(B)$. If a Banach function algebra B on a compact

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Hausdorff space X is natural, meaning that its maximal ideal space is X, then every nonzero endomorphism T of B has the form $Tf = f \circ \varphi$ for some self-map φ of X. If X is a compact plane set and B contains the coordinate map Z, then obviously $\varphi \in B$. It is interesting to know that under what conditions such φ induce compact endomorphisms. H. Kamowitz in [6] showed that if T is an endomorphism of disc algebra $A(\mathbb{D})$, T is compact if and only if φ is constant or $\|\varphi\|_{\mathbb{D}} < 1$. H. Kamowitz and S. Scheinberg [8] showed that an endomorphism T of Lipschitz algebras induced by a map φ, is compact if and only if φ is supercontraction. In [1] it has been shown that an endomorphism of D^n, the algebra of functions on unit disc $\mathbb{D}$ with continuous nth derivatives, is compact if and only if φ is constant or $\|\varphi\|_{\mathbb{D}} < 1$.

In this note we investigate compact endomorphisms of Banach function algebras $\mathrm{Lip}_A(X,\alpha) = A(X) \bigcap \mathrm{Lip}(X,\alpha)$, $(0 < \alpha \leq 1)$ and $\mathrm{Lip}^n(X,\alpha)$ when X is a certain compact plane set. First we show that when X is a compact (perfect) plane set, a self-map $\varphi : X \to X$ in these algebras induces a compact endomorphism of them, if φ is constant or $\varphi(X) \subseteq \mathrm{int}X$, and in the case $\alpha = 1$ these conditions are also necessary for a certain compact plane set X. We then determine the spectra of compact endomorphisms of these algebras. In the former, we need some definitions and Julia-Caratheodory Theorem.

Definition 1.1. (a) A sector in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ at a point $\omega \in \partial\mathbb{D}$ is the region between two straight lines in $\mathbb{D}$ that meet at ω and are symmetric about the radius to ω.

(b) If f is a function defined on $\mathbb{D}$ and $\omega \in \partial\mathbb{D}$, then $\angle \lim_{z\to\omega} f(z) = L$ means that $f(z) \to L$ as $z \to \omega$ through any sector at ω. When this happens, we say L is the angular (or non-tangential) limit of f at ω.

(c) An analytic map $\varphi : \mathbb{D} \to \mathbb{D}$ has an angular derivative at a point $\omega \in \partial\mathbb{D}$ if for some $\eta \in \mathbb{D}$,

$$\angle \lim_{z\to\omega} \frac{\eta - \varphi(z)}{\omega - z}$$

exists(finitely). We call the limit the angular derivative of φ at ω, and denote it by $\varphi'(\omega)$.

Theorem 1.2. *(Julia-Caratheodory Theorem) Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is a non-constant analytic map and $\omega \in \partial\mathbb{D}$. Then the followings are equivalent:*

(i) $\liminf_{z\to\omega}(1 - |\varphi(z)|)/(1 - |z|) = \delta < \infty$,

(ii) $\angle \lim_{z\to\omega}(\eta - \varphi(z))/(\omega - z)$ *exists for some* $\eta \in \partial\mathbb{D}$,

(iii) $\angle \lim_{z\to\omega} \varphi'(z)$ *exists, and* $\angle \lim_{z\to\omega} \varphi(z) = \eta \in \partial\mathbb{D}$.

The boundary point η in (ii) and (iii) are the same, and in (i) $\delta > 0$. Also, the limit of the difference quotient in (ii) coincides with the limit of the derivative in (iii), and both are equal to $\omega\bar{\eta}\delta$. For further details and proof see [10] and [2, Sections 295-300].

We also need some results about the algebras of continuously differentiable functions, which were introduced by Dales and Davie [3].

Definition 1.3. Let X be a perfect compact plane set. A complex-valued function f on X is called complex-differentiable at a point $a \in X$ if

$$f'(a) = \lim_{\substack{z \to a \\ z \in X}} \frac{f(z) - f(a)}{z - a}$$

exists. We call $f'(a)$ the complex derivative of f at a. Let $D^1(X)$ be the algebra of continuously differentiable functions on X. Then $D^1(X)$ under the norm $\|f\| = \|f\|_X + \|f'\|_X$ is a normed algebra which is not necessarily complete. However, under some conditions $D^1(X)$ is complete.

Definition 1.4. Let X be a compact plane set which is connected by rectifiable arcs, and let $\delta(z, w)$ be the geodesic metric on X, the infimum of the lengths of the arcs joining z and w.

(i) X is called regular if for each $z_0 \in X$ there exists a constant C such that for all $z \in X$, $\delta(z, z_0) \le C|z - z_0|$.

(ii) X is called uniformly regular if there exists a constant C such that for all $z, w \in X$, $\delta(z, w) \le C|z - w|$.

As proved in [4], $D^1(X)$ is complete if and only if for each $z_0 \in X$ there exists a constant C such that for every $z \in X$ and $f \in D^1(X)$,

$$|f(z) - f(z_0)| \le C|z - z_0|(\|f\|_X + \|f'\|_X).$$

Moreover, Completeness of $D^1(X)$ is is concluded from the following condition which we use frequently, so it is called $(*)$-condition:

$(*)$ *There exists a constant C such that for every $z, w \in X$ and $f \in D^1(X)$,*

$$|f(z) - f(w)| \le C|z - w|(\|f\|_X + \|f'\|_X).$$

It is also shown in [9], that if X is a finite union of regular sets then $D^1(X)$ is complete.

The following lemma is easy to see but it is important and we will be using in the sequel.

Lemma 1.5. *Let K, X be two compact plane sets and $K \subseteq \text{int}X$. Then there exists a finite union of uniformly regular sets in* int$X$ *containing K, namely Y and then a constant C such that for every analytic function f in intX and any $z, w \in K$, $|f(z) - f(w)| \le C|z - w|(\|f\|_Y + \|f'\|_Y)$.*

Let X be a compact plane set with a nonempty interior and $A(X)$ denote the uniform algebra of all continuous complex-valued functions on X which are analytic on int$X$. As a first application of Lemma 1.5, we give sufficient conditions for which endomorphisms of every natural uniform subalgebra B of $A(X)$ to be compact. For this, let $\varphi : X \to X$ be a function in B. Obviously, constant φ induces a compact endomorphism of B. Let φ be non-constant and $\varphi(X) \subseteq \text{int}X$. We show that such φ induces a compact endomorphism of B. By Functional Calculus Theorem $f \circ \varphi \in B$, for all $f \in B$. Let T be an endomorphism with $Tf = f \circ \varphi$. For compactness of T, suppose $\{f_n\}$ is a bounded sequence in B with $\|f_n\|_X \leq 1$. Let Y be a compact set obtained from Lemma 1.5 such that $\varphi(X) \subseteq Y \subseteq \text{int}X$. By Montel Theorem, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which is uniformly convergent on every compact subset of int$X$, in particular on Y. So $\|Tf_{n_k} - Tf_{n_j}\|_X = \|f_{n_k} \circ \varphi - f_{n_j} \circ \varphi\|_X \leq \|f_{n_k} - f_{n_j}\|_Y \to 0$ as $k, j \to \infty$. By completeness of B, $Tf_{n_k} = f_{n_k} \circ \varphi$ is convergent in B. That is, T is compact.

2 Compact Endomorphisms of $\text{Lip}_A(X, \alpha)$

Let X be a compact plane set and $0 < \alpha \leq 1$. Lipschitz algebra of order α, $\text{Lip}(X, \alpha)$ is the algebra of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup\{\frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in X, z \neq w\} < \infty.$$

Let $\text{Lip}_A(X, \alpha) = A(X) \bigcap \text{Lip}(X, \alpha)$. Then $\text{Lip}_A(X, \alpha)$ is a Banach function algebra on X, if equipped with the norm $\|f\| = \|f\|_X + p_\alpha(f)$. Jarosz [5] proved that for $0 < \alpha \leq 1$ the maximal ideal space of $\text{Lip}_A(X, \alpha)$ is X. It is well known every analytic function in a neighborhood of X is in $\text{Lip}(X, \alpha)$, so it is in $\text{Lip}_A(X, \alpha)$.

In this section we discuss compact endomorphisms of $\text{Lip}_A(X, \alpha)$. As we know if the interior of X is empty, then $\text{Lip}_A(X, \alpha) = \text{Lip}(X, \alpha)$, and the compact endomorphisms of $\text{Lip}(X, \alpha)$ have been studied in [8]. Thus in this section we assume that X has a nonempty interior. We recall that for every endomorphism T of $\text{Lip}_A(X, \alpha)$, there exists a self-map φ on X such that $Tf = f \circ \varphi$ and $\varphi \in \text{Lip}_A(X, \alpha)$ since $\text{Lip}_A(X, \alpha)$ contains the coordinate map Z. In the next theorem we consider sufficient conditions that φ induces a compact endomorphism.

Theorem 2.1. *Let X be a compact plane set, let φ be a self-map on X which belongs to $\text{Lip}_A(X, \alpha)$. If φ is constant or $\varphi(X) \subseteq \text{int}X$, then φ induces a compact endomorphism of $\text{Lip}_A(X, \alpha)$.*

Proof. If φ is constant, it is clear. Let $\varphi(X) \subseteq \text{int}X$, by Functional Calculus Theorem $f \circ \varphi \in \text{Lip}_A(X, \alpha)$, for every $f \in \text{Lip}_A(X, \alpha)$, so $Tf = f \circ \varphi$ is an endomorphism of $\text{Lip}_A(X, \alpha)$. For compactness of T, we assume that $\{f_n\}$ is a bounded sequence in $\text{Lip}_A(X, \alpha)$ with $\|f_n\| = \|f_n\|_X + p_\alpha(f_n) \leq 1$. Then $\{f_n\}$ is bounded and equicontinuous in $C(X)$. By Arzela-Ascoli Theorem, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that converges in $C(X)$. Since $A(X)$ is closed in $C(X)$, $\{f_{n_k}\}$ is convergent in $A(X)$. One can also by Montel Theorem say $\{f_{n_k}\}$ and their derivatives $\{f'_{n_k}\}$ are uniformly convergent in every compact subset of int$X$. We claim that $\{f_{n_k} \circ \varphi\}$ is convergent in $\text{Lip}_A(X, \alpha)$. Using Lemma 1.5, for all $z, w \in X$, with $\varphi(z) \neq \varphi(w)$ and $k, j \in \mathbb{Z}_+$ we have

$$\begin{aligned}
&\frac{|(f_{n_k} \circ \varphi - f_{n_j} \circ \varphi)(w) - (f_{n_k} \circ \varphi - f_{n_j} \circ \varphi)(z)|}{|w - z|^\alpha} \\
&= \frac{|\varphi(w) - \varphi(z)|}{|w - z|^\alpha} \frac{|(f_{n_k} - f_{n_j})(\varphi(w)) - (f_{n_k} - f_{n_j})(\varphi(z))|}{|\varphi(w) - \varphi(z)|} \\
&\leq C p_\alpha(\varphi)(\|f_{n_k} - f_{n_j}\|_Y + \|f'_{n_k} - f'_{n_j}\|_Y),
\end{aligned}$$

since each $f_{n_k} - f_{n_j}$ is analytic in int$X$. Therefore $\{f_{n_k} \circ \varphi\}$ is a Cauchy sequence in $\mathrm{Lip}_A(X, \alpha)$, hence T is compact. ∎

We will show that the above conditions are also necessary, for $\alpha = 1$ and for certain plane sets X. For notational convenience, in the case $\alpha = 1$, we set $p(f) = p_1(f)$ and $\mathrm{Lip}_A(X) = \mathrm{Lip}_A(X, 1)$. We now introduce the type of plane sets which we shall consider.

Definition 2.2. A plane set X at $c \in \partial X$ has an internal circular tangent if there exists a disc D such that $c \in \partial D$ and $\bar{D} \backslash \{c\} \subseteq \mathrm{int}X$. A plane set X has smooth boundary if it has an internal circular tangent at each point of its boundary.

Such sets include the closed unit disc $\bar{\mathbb{D}}$ and $\bar{D}(z_0, r) \backslash \cup_{k=1}^{n} D(z_k, r_k)$ where closed discs $\bar{D}(z_k, r_k)$ are mutually disjoint in $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

We say a compact plane set X has peak boundary with respect to $B \subseteq C(X)$ if for each $c \in \partial X$ there exists a non-constant function $h \in B$ such that $\|h\|_X = h(c) = 1$.

For example, the sets mentioned the above have peak boundaries with respect to the algebra of all analytic functions on a neighborhood of X, $H(X)$. Since $H(X) \subseteq \mathrm{Lip}_A(X)$, those sets have peak boundaries with respect to $\mathrm{Lip}_A(X)$. In fact, if X is a compact plane set such that $\mathbb{C} \setminus X$ has smooth boundary, then X has peak boundary with respect to $H(X)$ and hence with respect to every subset of $C(X)$ which contains $H(X)$. For this, suppose $c \in \partial X$. Then there exists a disc $D = D(z_0, r)$ such that $c \in \partial D$ and $\bar{D} \setminus \{c\} \subseteq \mathbb{C} \setminus X$. The function $h(z) = \dfrac{r}{z - z_0}$ satisfies the definition of peak boundary.

We now give the necessary conditions.

Theorem 2.3. *Let Ω be a bounded domain in plane and $X = \overline{\Omega}$. Let B be a natural uniform algebra on X such that $B \subseteq A(X)$ and let X have peak boundary with respect to B. If T is a nonzero compact endomorphism of B induced by φ, then φ is constant or $\varphi(X) \subseteq$ int$X$*

Proof. Suppose that $\varphi(c) \in \partial X$ for some $c \in \partial X$. By hypothesis there exists a non-constant function $h \in B$ such that $1 = h(\varphi(c)) = \|h\|_X$. Define $f_n(z) = h^n(z)$. By the compactness of T, the bounded sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $Tf_{n_k} = f_{n_k} \circ \varphi = (h \circ \varphi)^{n_k}$ converges to a function f in B. Since $h \circ \varphi(c) = 1$, $f(c) = 1$. If φ is not constant, the maximum modulus principle implies that $|h \circ \varphi(z)| < 1$ whenever $z \in \mathrm{int}X$. So $h^{n_k}(\varphi(z)) \to 0$ when $z \in \mathrm{int}X$. Therefore $f(z) = 0$ on int$X$ and so on X. This is a contradiction to $f(c) = 1$. ∎

Similar to Definition 1.1, one can define the angular derivative for a non-constant bounded analytic function φ on an open disc $D = D(z_0, r)$. By applying a suitable linear transformation to D, it can be shown that the existence of the angular derivative φ at $w \in \partial D$, according to Julia-Caratheodory Theorem, is equivalent to $\liminf_{z \to w} \frac{\|\varphi\|_D - |\varphi(z)|}{r - |z - z_0|} < \infty$. In this case, the angular derivative at ω also is nonzero.

Proposition 2.4. *Let $D = D(z_0, r)$, $c \in \partial D$ and $\varphi \in \mathrm{Lip}_A(D)$ be a non-constant function such that $|\varphi(c)| = \|\varphi\|_D$. Then the angular derivative of φ at c exists and is nonzero.*

Proof. Let $\Gamma = \{z \in D : \dfrac{|z-c|}{r-|z-z_0|} < 2\}$. For every $z \in \Gamma$ we have

$$\frac{\|\varphi\|_D - |\varphi(z)|}{r-|z-z_0|} = \frac{|\varphi(c)| - |\varphi(z)|}{r-|z-z_0|} \leq \frac{|z-c|}{r-|z-z_0|}\frac{|\varphi(z)-\varphi(c)|}{|z-c|} < 2p(\varphi).$$

Then $\liminf_{z\to c} \dfrac{\|\varphi\|_D - |\varphi(z)|}{r-|z-z_0|} < \infty$, and by Julia-Caratheodory Theorem, the proof is complete. ■

Theorem 2.5. *Let Ω be a bounded domain in plane and $X = \overline{\Omega}$ have smooth and peak boundary with respect to $\mathrm{Lip}_A(X)$. Let $0 \neq T$ be a compact endomorphism of $\mathrm{Lip}_A(X)$ induced by φ. Then φ is constant or $\varphi(X) \subseteq intX$.*

Proof. Let $0 \neq T$ be compact and there exists $c \in \partial X$ such that $\varphi(c) \in \partial X$. By hypothesis, there exists a non-constant function $h \in \mathrm{Lip}_A(X)$ such that $h(\varphi(c)) = \|h\|_X = 1$. Let $F_n(z) = \dfrac{h^n(z)}{n}$, then $\|F_n\| \leq \dfrac{1}{n} + p(h)$. Therefore $\{F_n\}$ is a bounded sequence in $\mathrm{Lip}_A(X)$. By the compactness of T, there exists a subsequence $\{F_{n_k}\}$ such that $TF_{n_k} = F_{n_k} \circ \varphi$ is convergent in $\mathrm{Lip}_A(X)$. Since $F_{n_k} \longrightarrow 0$ uniformly on X, $F_{n_k} \circ \varphi \longrightarrow 0$ in $\mathrm{Lip}_A(X)$. Thus

$$p(F_{n_k} \circ \varphi) = \sup_{\substack{z,w\in X\\ z\neq w}} \left| \frac{(h\circ\varphi)^{n_k}(w) - (h\circ\varphi)^{n_k}(z)}{n_k(w-z)} \right| \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$$

In particular,

$$\frac{1}{n_k} \sup_{\substack{z\in X\\ z\neq c}} \left| \frac{(h\circ\varphi)^{n_k}(z) - (h\circ\varphi)^{n_k}(c)}{z-c} \right| < \epsilon, \tag{1}$$

for arbitrary $\epsilon > 0$ and some n_k.

By hypothesis, there exists an open disc D such that $c \in \partial D$ and $\bar{D} \backslash \{c\} \subseteq \mathrm{int}X$. We have $h\circ\varphi \in \mathrm{Lip}_A(D)$ and $(h\circ\varphi)(c) = \|h\circ\varphi\|_D = 1$. Then, by Proposition 2.4, the angular derivative of $h\circ\varphi$ restricted to D at c exists and is nonzero, if $h\circ\varphi$ is not constant on D. But by (1) this derivative is zero, so $h\circ\varphi$ is constant on D. Since h and φ are analytic on $\mathrm{int}X$ and h is non-constant, φ must be constant. ■

We conjecture that the same conditions are necessary for compactness of an endomorphism T of $\mathrm{Lip}_A(X,\alpha), 0 < \alpha < 1$.

3 Compact endomorphisms of $\mathrm{Lip}^n(X,\alpha)$

Let $\mathrm{Lip}^n(X,\alpha)$, $(0 < \alpha \leq 1,\ n > 1)$ be the algebra of all functions f on a perfect compact plane set X whose derivatives up to order n exist and for each k $(0 \leq k \leq n)$, $f^{(k)} \in \mathrm{Lip}(X,\alpha)$. Note that, $f \in \mathrm{Lip}^n(X,\alpha)$ is analytic on $\mathrm{int}X$. These algebras are normed algebras if equipped with the norm

$$\|f\| = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X + p(f^{(k)})}{k!}, \qquad (f \in \mathrm{Lip}^n(X)).$$

These algebras were introduced by T. Honary and H. Mahyar [4]. It was shown that if $D^1(X)$ is complete, $\mathrm{Lip}^n(X,\alpha)$ is a natural Banach function algebra [4].

Let X be a compact perfect plane set satisfying $(*)$-condition. Then every self-map $\varphi : X \to X$ in $\mathrm{Lip}^1(X,\alpha)$ induces an endomorphism of $\mathrm{Lip}^1(X,\alpha)$, that is $f \circ \varphi \in \mathrm{Lip}^1(X,\alpha)$ for every $f \in \mathrm{Lip}^1(X,\alpha)$. Because for every $z, w \in X$ and every $f \in \mathrm{Lip}^1(X,\alpha)$ we have

$$\begin{aligned}\frac{|f\circ\varphi(z) - f\circ\varphi(w)|}{|z-w|^\alpha} &= \frac{|f(\varphi(z)) - f(\varphi(w))|}{|\varphi(z)-\varphi(w)|^\alpha}\left|\frac{\varphi(z)-\varphi(w)}{z-w}\right|^\alpha \\ &\le p_\alpha(f)C^\alpha(\|\varphi\|_X + \|\varphi'\|_X)^\alpha \\ &\text{or} \\ &\le C(\|f\|_X + \|f'\|_X)p_\alpha(\varphi)\end{aligned}$$

and

$$\begin{aligned}\frac{|(f\circ\varphi)'(z) - (f\circ\varphi)'(w)|}{|z-w|^\alpha} &= \frac{|f'(\varphi(z))\varphi'(z) - f'(\varphi(w))\varphi'(w)|}{|z-w|^\alpha} \\ &\le \frac{|f'(\varphi(z)) - f'(\varphi(w))|}{|\varphi(z)-\varphi(w)|^\alpha}|\varphi'(z)|\left|\frac{\varphi(z)-\varphi(w)}{z-w}\right|^\alpha \\ &\quad + \frac{|\varphi'(z)-\varphi'(w)|}{|z-w|^\alpha}|f'(\varphi(w))| \\ &\le p_\alpha(f')\|\varphi'\|_X C^\alpha(\|\varphi\|_X + \|\varphi'\|_X)^\alpha + p_\alpha(\varphi')\|f'\|_X.\end{aligned}$$

We now consider sufficient conditions that such φ induce compact endomorphisms of $\mathrm{Lip}^1(X,\alpha)$.

Theorem 3.1. *Let X be a compact plane set with a nonempty interior and satisfying the $(*)$-condition, and a self-map φ on X be in* $\mathrm{Lip}^1(X,\alpha)$*. If φ is constant or $\varphi(X) \subseteq \mathrm{int}X$, then φ induces a compact endomorphism of* $\mathrm{Lip}^1(X,\alpha)$.

Proof. When φ is constant, it is clear. Let $\varphi(X) \subseteq \mathrm{int}X$. For compactness of T we assume that $\{f_n\}$ is a bounded sequence in $\mathrm{Lip}^1(X,\alpha)$ with $\|f_n\| = \|f_n\|_X + \|f_n'\|_X + p_\alpha(f_n) + p_\alpha(f_n') \le 1$. Then $\{f_n\}$ and $\{f_n'\}$ are bounded and equicontinuous in $C(X)$. By Arzela-Ascoli Theorem and the $(*)$-condition, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges in $D^1(X)$, and also by Montel Theorem their derivatives of each order, $\{f_{n_k}^{(r)}\}$ are uniformly convergent in every compact subset of $\mathrm{int}X$, since each f_{n_k} is analytic in $\mathrm{int}X$. We denote this subsequence by $\{f_n\}$ again. We want to show that $\{f_n\circ\varphi\}$ is convergent in $\mathrm{Lip}^1(X,\alpha)$. Clearly, $\|f_n\circ\varphi\|_X \le \|f_n\|_X$, $\|(f_n\circ\varphi)'\|_X \le \|f_n'\|_X\|\varphi'\|_X$ and $p_\alpha(f_n\circ\varphi) \le C(\|f_n\|_X + \|f_n'\|_X)p_\alpha(\varphi)$. Thus the sequences $\{\|f_n\circ\varphi\|_X\}, \{\|(f_n\circ\varphi)'\|_X\}$ and $\{p_\alpha(f_n\circ\varphi)\}$ are Cauchy. Moreover, using Lemma 1.5, we obtain a finite union of uniformly regular sets Y in $\mathrm{int}X$ containing $\varphi(X)$, and a constant C such that for all $z, w \in X$ with $\varphi(z) \neq \varphi(w)$

and all positive integers m, n we have

$$\begin{aligned}
&\frac{|((f_n \circ \varphi)' - (f_m \circ \varphi)')(z) - ((f_n \circ \varphi)' - (f_m \circ \varphi)')(w)|}{|z-w|^\alpha} \\
&\leq \frac{|(f'_n - f'_m)(\varphi(z)) - (f'_n - f'_m)(\varphi(w))|}{|\varphi(z) - \varphi(w)|} \frac{|\varphi(z) - \varphi(w)|}{|z-w|^\alpha} |\varphi'(z)| \\
&+ \frac{|\varphi'(z) - \varphi'(w)|}{|z-w|^\alpha} |(f'_n - f'_m)(\varphi(w))| \\
&\leq C(\|f'_n - f'_m\|_Y + \|f''_n - f''_m\|_Y) p_\alpha(\varphi) \|\varphi'\|_X + p_\alpha(\varphi') \|f'_n - f'_m\|_X.
\end{aligned}$$

Hence $\{p_\alpha((f_n \circ \varphi)')\}$ is also a Cauchy sequence. By completeness of $\mathrm{Lip}^1(X, \alpha)$, the Cauchy sequence $\{f_n \circ \varphi\}$ in $\mathrm{Lip}^1(X, \alpha)$ is convergent. ∎

Remark 3.2. Using the similar way, one can conclude Theorem 3.1 for $\mathrm{Lip}^n(X, \alpha)$, $n \geq 1, 0 < \alpha \leq 1$.

We show that for certain plane sets X in the case $\alpha = 1$ the conditions in Theorem 3.1 are also necessary. For this we need the following lemma due to Julia [2, part six].

Lemma 3.3. *Let f be a continuously differentiable complex-valued function on the closed unit disc $\bar{\mathbb{D}}$. If f is non-constant and $f(1) = 1 = \|f\|_{\mathbb{D}}$, then $|f'(1)|$ is a strictly positive number.*

In the next theorem, we write $\mathrm{Lip}^1(X)$ instead of $\mathrm{Lip}^1(X, 1)$ and we use the above lemma for any disc $D = D(z_0, r)$.

Theorem 3.4. *Let Ω be a bounded domain in plane and $X = \overline{\Omega}$ satisfying the $(*)$-condition have smooth and peak boundary with respect to* $\mathrm{Lip}^1(X)$. *Let $0 \neq T$ be a compact endomorphism of* $\mathrm{Lip}^1(X)$ *induced by φ. Then φ is constant or $\varphi(X) \subseteq$ int$X$.*

Proof. Let $0 \neq T$ be compact and $\varphi(c) \in \partial X$ for some $c \in \partial X$. By hypothesis, there exists a non-constant function $h \in \mathrm{Lip}^1(X)$ such that $h(\varphi(c)) = \|h\|_X = 1$. Let $F_n(z) = \dfrac{h^n(z)}{n(n-1)}$. Then

$$\begin{aligned}
\|F_n\| &= \|F_n\|_X + \|F'_n\|_X + p(F_n) + p(F'_n) \\
&\leq \frac{1}{n(n-1)} + \frac{\|h'\|_X}{n-1} + \frac{p(h)}{n-1} + \frac{p(h')}{n-1} + p(h)\|h'\|_X.
\end{aligned}$$

Therefore $\{F_n\}$ is a bounded sequence in $\mathrm{Lip}^1(X)$. By compactness of T, there exists a subsequence $\{F_{n_k}\}$ such that $TF_{n_k} = F_{n_k} \circ \varphi$ is convergent in $\mathrm{Lip}^1(X)$. Since $F_{n_k} \to 0$ uniformly on X, $F_{n_k} \circ \varphi \to 0$ in $\mathrm{Lip}^1(X)$. Thus

$$\begin{aligned}
p((F_{n_k} \circ \varphi)') &= \sup_{\substack{z,w \in X \\ z \neq w}} \frac{|(h \circ \varphi)'(z) h^{n_k-1}(\varphi(z)) - (h \circ \varphi)'(w) h^{n_k-1}(\varphi(w))|}{(n_k - 1)|z-w|} \\
&= \frac{1}{n_k - 1} \sup_{\substack{z,w \in X \\ z \neq w}} \Big| \frac{h^{n_k-1}(\varphi(z)) - h^{n_k-1}(\varphi(w))}{z-w} (h \circ \varphi)'(z) \\
&+ \frac{(h \circ \varphi)'(z) - (h \circ \varphi)'(w)}{z-w} h^{n_k-1}(\varphi(w)) \Big| \to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Considering

$$\frac{1}{n_k-1}\sup_{\substack{z,w\in X\\ z\neq w}}\left|\frac{(h\circ\varphi)'(z)-(h\circ\varphi)'(w)}{z-w}h^{n_k-1}(\varphi(w))\right|\leq\frac{1}{n_k-1}p((h\circ\varphi)')$$

$$\to 0 \quad \text{as} \quad k\to\infty,$$

we have

$$\frac{1}{n_k-1}\sup_{\substack{z,w\in X\\ z\neq w}}\left|\frac{h^{n_k-1}(\varphi(z))-h^{n_k-1}(\varphi(w))}{z-w}(h\circ\varphi)'(z)\right|\to 0 \quad \text{as} \quad k\to\infty.$$

In particular

$$\frac{1}{n_k-1}\sup_{\substack{z\in X\\ z\neq c}}\left|\frac{h^{n_k-1}(\varphi(z))-h^{n_k-1}(\varphi(c))}{z-c}(h\circ\varphi)'(z)\right|\to 0 \quad \text{as} \quad k\to\infty,$$

and hence

$$\frac{1}{n_k-1}\sup_{\substack{z\in X\\ z\neq c}}\left|\frac{(h\circ\varphi)^{n_k-1}(z)-(h\circ\varphi)^{n_k-1}(c)}{z-c}(h\circ\varphi)'(z)\right|<\epsilon,$$

for arbitrary $\epsilon>0$ and some n_k. Then

$$\frac{1}{n_k-1}\lim_{\substack{z\to c\\ z\in X}}\left|\frac{(h\circ\varphi)^{n_k-1}(z)-(h\circ\varphi)^{n_k-1}(c)}{z-c}(h\circ\varphi)'(z)\right|\leq\epsilon,$$

so $|(h\circ\varphi)'(c)|^2=|((h\circ\varphi)'(c))^2(h\circ\varphi)^{n_k-2}(c)|\leq\epsilon$, for any $\epsilon>0$. Hence $(h\circ\varphi)'(c)=0$.

On the other hand, by hypothesis X has smooth boundary, then there exists an open disc D such that $c\in\partial D$ and $\bar{D}\setminus\{c\}\subseteq\text{int}X$. Since $h\circ\varphi\in \text{Lip}^1(X)$, $h\circ\varphi$ has continuous complex derivative on X, in particular on $\bar{D}$, and $(h\circ\varphi)(c)=\|h\circ\varphi\|_D=\|h\|_X=1$. Then by Lemma 3.3, $h\circ\varphi$ is constant on $\bar{D}$. Since $h\circ\varphi$ is analytic on int$X$ and int$X$ is connected, $h\circ\varphi$ is constant on X. As we know h is not constant, so φ must be constant. ■

Remark 3.5. (a) Considering the function $\frac{h^n(z)}{n(n-1)\ldots(n-m)}$ rather than $\frac{h^n(z)}{n(n-1)}$ in the proof of theorem 3.4 , this theorem is valid for $\text{Lip}^m(X), m\geq 1$.

(b) Suppose a set X satisfies the conditions of theorems 2.5 and 3.4 except for the requirement that X to be connected. Let $X=\bigcup_{i=1}^n\bar{\Omega}_i$, where each Ω_i is a bounded domain in plane. Suppose that T is a compact endomorphism of B ($B=\text{Lip}_A(X)$ or $\text{Lip}^n(X)$) induced by φ. One can prove that if $\varphi(\bar{\Omega}_i)\bigcap\partial X\neq\phi$, then φ is constant on $\bar{\Omega}_i$.

As a consequence of Theorems 2.1, 2.5, 3.1 and 3.4 we have the following corollary.

Corollary 3.6. *Let B be one of the algebras $\text{Lip}_A(\mathbb{D})$ or $\text{Lip}^1(\mathbb{D})$ and T be an endomorphism of B induced by φ. Then T is compact if and only if φ is constant or $\|\varphi\|_{\mathbb{D}}<1$.*

When $X = [0,1]$, we have:

Theorem 3.7. *Every nonzero compact endomorphism* T *of* $\mathrm{Lip}^1([0,1])$ *has the form* $Tf = f(x_0)1$ *for some* $x_0 \in [0,1]$

Proof. Let T be a compact endomorphism of $\mathrm{Lip}^1([0,1])$ induced by φ. Then $\varphi \in \mathrm{Lip}^1([0,1])$. We show that φ is a constant function. Let $f_n(x) = \dfrac{e^{inx}}{n^2}$. Then $f_n \in \mathrm{Lip}^1([0,1])$ and $\|f_n\| \le 4$ for all n. By compactness of T, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{Tf_{n_k}\}$ is convergent in $\mathrm{Lip}^1([0,1])$. Since $Tf_{n_k} = f_{n_k}\circ\varphi$ is uniformly convergent to zero on $[0,1]$, $Tf_{n_k} = f_{n_k}\circ\varphi \to 0$ in $\mathrm{Lip}^1([0,1])$. Hence $p((f_{n_k}\circ\varphi)') \to 0$ as $k \to \infty$. Similarly to the proof of Theorem 3.4 we conclude that

$$\frac{1}{n_k} \sup_{\substack{0\le x,y\le 1\\ x\ne y}} \left| \frac{e^{in_k\varphi(x)} - e^{in_k\varphi(y)}}{x-y}\varphi'(y)\right| \to 0 \quad \text{as} \quad k\to\infty.$$

Hence

$$\frac{1}{n_k} \sup_{\substack{0\le x,y\le 1\\ x\ne y}} \left| \frac{e^{in_k\varphi(x)} - e^{in_k\varphi(y)}}{x-y}\varphi'(y)\right| < \epsilon,$$

for arbitrary $\epsilon > 0$ and some n_k. Then for any $y \in [0,1]$ we have

$$\frac{1}{n_k} \lim_{x\to y} \left| \frac{e^{in_k\varphi(x)} - e^{in_k\varphi(y)}}{x-y}\varphi'(y)\right| < \epsilon,$$

that is, $|\varphi'(y)|^2 = 0$. Hence $\varphi' = 0$ and φ is a constant function. Therefore, $Tf = f(x_0)1$ for some $x_0 \in [0,1]$. ∎

We remark, one can show that every nonzero compact endomorphism of $\mathrm{Lip}^n([0,1])$ has the form $Tf = f(x_0)1$ for some $x_0 \in [0,1]$.

4 Spectra of compact endomorphisms

In this section we determine the spectrum of a compact endomorphism of algebras $\mathrm{Lip}_A(X,\alpha)$ and $\mathrm{Lip}^n(X,\alpha)$ $(0 < \alpha \le 1,\ n > 1)$. H. Kamowitz in [7] proved that if T is a nonzero compact endomorphism of commutative semi-simple Banach algebra B induced by $\varphi : \mathcal{M}(B) \longrightarrow \mathcal{M}(B)$, then $\cap\varphi_n(\mathcal{M}(B))$ is finite and if $\mathcal{M}(B)$ is connected and B is unital, $\cap\varphi_n(\mathcal{M}(B))$ is singleton where φ_n is nth iterate of φ, i.e, $\varphi_0(x) = x$ and $\varphi_n(x) = \varphi(\varphi_{n-1}(x))$. If $\cap\varphi_n(\mathcal{M}(B)) = \{x_0\}$, then x_0 is a fixed point for φ. In fact, if $F = \cap\varphi_n(\mathcal{M}(B))$, then $\varphi(F) = F$.

Theorem 4.1. *Let* B *be a natural Banach function algebra on a compact plane set* X *containing the coordinate function* Z, *and* $B \subseteq A(X)$. *Let* T *be a compact endomorphism of* B *induced by* φ. *If* $\varphi(X) \subseteq intX$ *and* z_0 *is a fixed point of* φ, *then* $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbb{Z}_+\} \cup \{0,1\}$.

Proof. Clearly 0 and also $1 \in \sigma(T)$ since $T(1) = 1$. If φ is constant, then the proof is complete. Let $\lambda \in \sigma(T)\backslash\{0,1\}$. By compactness of T there exists $f \neq 0$ in B such that $Tf = f \circ \varphi = \lambda f$. Since $\varphi(z_0) = z_0 \in \text{int}X$, $f(z_0) = 0$. Let n be a positive integer number such that for every $0 \leq k < n$, $f^{(k)}(z_0) = 0$ but $f^{(n)}(z_0) \neq 0$. We show that $\lambda = \varphi'(z_0)^n$. By n times differentiation of $f \circ \varphi = \lambda f$, we have $\varphi'(z_0)^n f^{(n)}(\varphi(z_0)) = \lambda f^{(n)}(z_0)$, therefore $\lambda = \varphi'(z_0)^n$. Then $\sigma(T)\backslash\{0,1\} \subseteq \{\varphi'(z_0)^n : n \in \mathbb{Z}_+\}$.

Conversely, first we show that if $\lambda \in \sigma(T)$ with $|\lambda| = 1$, then $\lambda = 1$. Let $\lambda \in \sigma(T)$ and $|\lambda| = 1$. By compactness of T there exists $0 \neq f \in B$ such that $f \circ \varphi = \lambda f$. Since $\varphi(X) \subseteq \text{int}X$ by maximum modulus principle $\lambda = 1$. We claim that $\varphi'(z_0) \in \sigma(T)$. If $\varphi'(z_0) \notin \sigma(T)$, then there exists $g \in B$ such that $g \circ \varphi - \varphi'(z_0)g = z - z_0$. By differentiation at z_0, we have $0 = \varphi'(z_0)g'(\varphi(z_0)) - \varphi'(z_0)g'(z_0) = 1$, this is a contradiction. We now show that for every $n \in \mathbb{Z}_+$, $\varphi'(z_0)^n \in \sigma(T)$. If $\varphi'(z_0) = 0$ or $|\varphi'(z_0)| = 1$, the proof is complete. Suppose $\varphi'(z_0) \neq 0$ and $|\varphi'(z_0)| \neq 1$. If $\varphi'(z_0)^m \notin \sigma(T)$ for some $m > 1$, then there exists $f \in B$ such that $f \circ \varphi - \varphi'(z_0)^m f = (z - z_0)^m$. By $m-1$ times differentiation at z_0, we have $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$, and m times differentiation at z_0, we have $0 = \varphi'(z_0)^m f^{(m)}(\varphi(z_0)) - \varphi'(z_0)^m f^{(m)}(z_0) = m!$, this is a contradiction. ∎

Corollary 4.2. *Let B and T satisfy the conditions of Theorem 4.1. Let F be a finite set such that $\varphi(F) = F$. Then there exist $z_0 \in F$ and $k \in \mathbb{Z}_+$ such that $\sigma(T)^k = \{\varphi_k'(z_0)^n : n \in \mathbb{Z}_+\} \cup \{0,1\}$.*

Proof. Since F is a finite set and $\varphi(F) = F$, there exists $z_0 \in F$ and $k \in \mathbb{Z}_+$ such that $\varphi_k(z_0) = z_0$. Since $\varphi(X) \subseteq \text{int}X$, so $z_0 \in \text{int}X$. If φ is constant, the proof is complete. When φ is not constant, let $\tilde{T}f = f \circ \varphi_k$ for $f \in B$. Therefore $\tilde{T}$ is a compact endomorphism induced by φ_k and $\varphi_k(z_0) = z_0$. By Theorem 4.1, $\sigma(\tilde{T}) = \{\varphi_k'(z_0)^n : n \in \mathbb{Z}_+\} \cup \{0,1\}$. But $\tilde{T} = T^k$, by Spectral Mapping Theorem $\sigma(T)^k = \sigma(T^k) = \sigma(\tilde{T})$. Therefore $\sigma(T)^k = \{\varphi_k'(z_0)^n : n \in \mathbb{Z}_+\} \cup \{0,1\}$. ∎

Remark 4.3. Let X be a compact plane set and B be one of the natural Banach function algebras $\text{Lip}_A(X,\alpha)$ and $\text{Lip}^n(X,\alpha)$. In the case $\text{Lip}^n(X,\alpha)$, we assume that X satisfies the $(*)$-condition. Let T be a nonzero compact endomorphism of B induced by φ and $\varphi(X) \subseteq \text{int}X$. Then the spectrum $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbb{Z}_+\} \cup \{0,1\}$, where z_0 is a fixed point of φ.

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