The p-adic Finite Fourier Transform and Theta Functions

G. Van Steen

A polarization on an abelian variety A induces an isogeny between A and its dual variety $\hat{A}$. The kernel of this isogeny is a direct sum of two isomorphic subgroups. If A is an analytic torus over a non-archimedean valued field then it is possible to associate with each of these subgroups a basis for a corresponding space of theta functions, cf. [5], [6].
The relation between these bases is given by a finite Fourier transform. Similar results hold for complex abelian varieties, cf. [3].

The field k is algebraically closed and complete with respect to a non-archimedean absolute value. The residue field with respect to this absolute value is $\overline{k}$.

1 The finite Fourier transform

In this section we consider only finite abelian groups whose order is not divisible by $char(\overline{k})$.
For such a group A we denote by $\widehat{A}$ the group of k-characters of A, i.e. $\widehat{A} = Hom(A, k^*)$. The vector space of k valued functions on A is denoted as $V(A)$.

Lemma 1.1 *Let A_1 and A_2 be finite abelian groups. Then $(\widehat{A_1 \times A_2})$ is isomorphic with $\widehat{A_1} \times \widehat{A_2}$.*

Proof The map $\theta : \widehat{A_1} \times \widehat{A_2} \to \widehat{A_1 \times A_2}$, defined by $\theta(\chi, \tau)(a_1, a_2) = \chi(a_1).\tau(a_2)$ is an isomorphism. ∎

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The vectorspace $V(A)$ is a banach space with respect to the norm

$$||f|| = max\left\{|f(a)| \;\Big|\; a \in A\right\}$$

For each $a \in A$ the function $\delta_a \in V(A)$ is defined by $\delta_a(b) = 0$ if $a \neq b$ and $\delta_a(a) = 1$. The functions $(\delta_a)_{a\in A}$ form an orthonormal basis for $V(A)$, i.e.

$$\begin{cases} ||\sum_{a\in A}\lambda_a\delta_a|| = max\left\{|\lambda_a| \;\Big|\; a \in A\right\} \\ ||\delta_a|| = 1 \text{ for all } a \in A \end{cases}$$

Definition 1.1 *Let m be the order of the finite group A. The finite Fourier transform F_A on $V(A)$ is the linear map $F_A : V(A) \to V(\widehat{A})$ defined by $F_A(\delta_a) = (1/\sqrt{m})\sum_{\chi\in\widehat{A}}\chi(a)\delta_\chi$.*

Proposition 1.2 *Let A_1 and A_2 be finite abelian groups and let $F_1 = F_{A_1}$ and $F_2 = F_{A_2}$ be the finite Fourier transforms.*

1. *The map $\phi : V(A_1)\otimes V(A_2) \to V(A_1 \times A_2)$, defined by $\phi(\delta_{a_1}\otimes\delta_{a_2}) = \delta_{(a_1,a_2)}$ is an isomorphism. Furthermore $\phi(f_1\otimes f_2)(a_1,a_2) = f_1(a_1)f_2(a_2)$.*

2. *Let $\widehat{\theta} : V(\widehat{A_1\times A_2}) \to V(\widehat{A}_1\times\widehat{A}_2)$ be the linear map induced by the homomorphism $\theta : \widehat{A}_1\times\widehat{A}_2 \to \widehat{A_1\times A_2}$, (cf 1.1).*
   The following diagram is then commutative :

$$\begin{array}{ccccc} V(A_1)\otimes V(A_2) & & \xrightarrow{\phi} & & V(A_1\times A_2) \\ {\scriptstyle F_1\otimes F_2}\big\downarrow & & & & \big\downarrow{\scriptstyle F_{A_1\times A_2}} \\ V(\widehat{A}_1)\otimes V(\widehat{A}_2) & \xrightarrow{\hat{\phi}} & V(\widehat{A}_1\times\widehat{A}_2) & \xrightarrow{\hat{\theta}^{-1}} & V(\widehat{A_1\times A_2}) \end{array}$$

Proof Straightforward calculation. ∎

Proposition 1.3 *The finite Fourier transform F_A is a unitary operator on $V(A)$, i.e. $||F_A|| = 1$. Furthermore $F_A(f)(\tau) = (1/\sqrt{m})\sum_{a\in A} f(a)\tau(a)$.*

Proof For $f \in V(A)$ we have :

$$\begin{aligned} F_A(f)(\tau) &= F_A(\textstyle\sum_{a\in A} f(a)\delta_a)(\tau) \\ &= \textstyle\sum_{a\in A} f(a)\Big((1/\sqrt{m})\sum_{\chi\in\widehat{A}}\chi(a)\delta_\chi(\tau)\Big) \\ &= (1/\sqrt{m})\textstyle\sum_{a\in A} f(a)\tau(a) \end{aligned}$$

The norm on F_A is defined by $||F_A|| = max\left\{||F_A(f)|| \;\Big|\; f \in V(A) \text{ and } ||f|| \le 1\right\}$. Hence

$$\begin{aligned} ||F_A|| &= max\left\{||F_A\Big(\textstyle\sum_{a\in A}\lambda_a\delta_a\Big)|| \;\Big|\; \lambda_a \in k \text{ and } |\lambda_a| \le 1\right\} \\ &= max\left\{|(1/\sqrt{m})|.||\textstyle\sum_{a\in A}\sum_{\chi\in\widehat{A}}(\lambda_a\chi(a))|| \;\Big|\; 1 \ge |\lambda_a|\right\} \\ &\le max\left\{|\lambda_a\chi(a)| \;\Big|\; \chi\in\widehat{A}, a \in A \text{ and } |\lambda_a| \le 1\right\} \\ &\le 1 \quad \text{since } |\chi(a)| = 1 \text{ for all } \chi, a \end{aligned}$$

Since $||F_A(\delta_a)|| = 1$ we have $||F_A|| \ge 1$. ∎

Consider now the special case that $A = \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, (with $char(\overline{k}) \not| m$).
The group k_m^* of points of order m in k^* is a cyclic group of order m. Let ξ be a fixed generator for k_m^*.

Lemma 1.4 *The map* $\chi : \mathbb{Z}_m \to \widehat{\mathbb{Z}_m}$ *defined by* $\chi(\overline{a})(\overline{b}) = \xi^{ab}$ *is an isomorphism.*

Proof Easy calculation. ∎

We denote $\chi_{\overline{a}} = \chi(\overline{a})$.

Proposition 1.5 $\Big(\chi_{\overline{a}}\Big)_{\overline{a}\in\mathbb{Z}_m}$ *is an orthonormal basis for* $V(\mathbb{Z}_m)$.

Proof The characters $\Big(\chi_{\overline{a}}\Big)_{\overline{a}\in\mathbb{Z}_m}$ are linearly independant (standard algebra). Since $dim\Big(V(\mathbb{Z}_m)\Big) = m$ the characters form a basis.

Let $\tau = \sum_{\overline{a}\in\mathbb{Z}_m} \lambda_{\overline{a}}\chi_{\overline{a}}$ with $\lambda_{\overline{a}} \in k$.
We have $||\tau|| = max\Big\{|\tau(\overline{b})| \;\Big|\; \overline{b} \in \mathbb{Z}_m\Big\}$. It follows that $||\chi_{\overline{a}}|| = 1$ and since

$$\begin{aligned}\textstyle\sum_{b=0}^{m-1} \tau(\overline{b}) &= m\lambda_{\overline{0}} + \textstyle\sum_{a=1}^{m-1} \lambda_{\overline{a}}\Big(\sum_{b=0}^{m-1} \chi_{\overline{a}}(\overline{b})\Big)\\ &= m\lambda_{\overline{0}} + \textstyle\sum_{a=1}^{m-1} \lambda_{\overline{a}}\Big(\sum_{b=0}^{m-1} \xi^{ab}\Big)\\ &= m\lambda_{\overline{0}}\end{aligned}$$

we find that

$$|\lambda_{\overline{0}}| = |m\lambda_{\overline{0}}| \leq max\Big\{|\tau(\overline{b})| \;\Big|\; \overline{b} \in \mathbb{Z}_m\Big\} = ||\tau||$$

In a similar way we find that $|\lambda_{\overline{a}}| \leq ||\tau||$ for all $\overline{a} \in \mathbb{Z}_m$ and hence

$$max\Big\{|\lambda_{\overline{a}}| \;\Big|\; \overline{a} \in \mathbb{Z}_m\Big\} \leq ||\tau||$$

On the other hand we have

$$||\tau|| \leq max\Big\{||\lambda_{\overline{a}}\chi_{\overline{a}}|| \;\Big|\; \overline{a} \in \mathbb{Z}_m\Big\} = max\Big\{|\lambda_{\overline{a}}| \;\Big|\; \overline{a} \in \mathbb{Z}_m\Big\}$$

It follows that the elements $\chi_{\overline{a}}$ are orthonormal. ∎

Let $F_m : V(\mathbb{Z}_m) \to V(\widehat{\mathbb{Z}_m})$ be the finite Fourier transform.

Proposition 1.6 $F_m(f)(\tau) = (1/\sqrt{m})\sum_{\overline{a}\in\mathbb{Z}_m} f(\overline{a})\tau(\overline{a})$.

Proof Since $f = \sum_{\overline{a}\in\mathbb{Z}_m} f(\overline{a})\delta_{\overline{a}}$ we have

$$\begin{aligned}F_m(f)(\tau) &= \textstyle\sum_{\overline{a}\in\mathbb{Z}_m} f(\overline{a})F_m(\delta_{\overline{a}})(\tau)\\ &= (1/\sqrt{m})\textstyle\sum_{\overline{a}\in\mathbb{Z}_m} f(\overline{a})\Big(\sum_{\chi\in\widehat{\mathbb{Z}_m}} \chi(\overline{a})\delta_\chi(\tau)\Big)\\ &= (1/\sqrt{m})\textstyle\sum_{\overline{a}\in\mathbb{Z}_m} f(\overline{a})\tau(\overline{a})\end{aligned}$$

∎

The isomorphism $\chi : \mathbb{Z}_m \to \widehat{\mathbb{Z}_m}$ induces an isomorphism $\psi : V(\widehat{\mathbb{Z}_m}) \to V(\mathbb{Z}_m)$. The composition

$$\psi \circ F_m : V(\mathbb{Z}_m) \to V(\widehat{\mathbb{Z}_m}) \to V(\mathbb{Z}_m)$$

is still denoted as F_m.

Proposition 1.7 $F_m(\delta_{\overline{a}}) = (1/\sqrt{m})\chi_{\overline{a}}$ *and* $F_m(\chi_{\overline{a}}) = \sqrt{m}\delta_{-\overline{a}}$

Proof For all $\overline{a} \in \mathbb{Z}_m$ is

$$\begin{aligned} F_m(\delta_{\overline{a}})(\overline{b}) &= F_m(\delta_{\overline{a}})(\chi_{\overline{b}}) \\ &= (1/\sqrt{m}) \textstyle\sum_{\overline{u}\in\mathbb{Z}_m} \chi_{\overline{b}}(\overline{u})\delta_{\overline{a}}(\overline{u}) \\ &= (1/\sqrt{m})\chi_{\overline{b}} = 1\sqrt{m}\chi_{\overline{a}}(\overline{b}) \end{aligned}$$

We also have

$$\begin{aligned} F_m(\chi_{\overline{a}})(\overline{b}) &= (1/\sqrt{m}) \textstyle\sum_{\overline{u}\in\mathbb{Z}_m} \chi_{\overline{b}}(\overline{u})\chi_{\overline{a}}(\overline{u}) \\ &= (1/\sqrt{m}) \textstyle\sum_{\overline{u}\in\mathbb{Z}_m} \chi_{\overline{a}+\overline{b}}(\overline{u}) \end{aligned}$$

This last sum equals 0 if $\overline{a} + \overline{b} \neq \overline{0}$ and equals m if $\overline{a} + \overline{b} = \overline{0}$.
Hence $\sum_{\overline{u}\in\mathbb{Z}_m} \chi_{\overline{a}+\overline{b}}(\overline{u}) = \delta_{-\overline{a}}(\overline{b})$. ∎

Corollary 1.8 $F_m^2(\delta_{\overline{a}}) = \delta_{-\overline{a}}$

As a consequence of Prop 1.2 the results for $\mathbb{Z}_m$ can be generalized for arbitrary abelian groups.

Let A be an abelian group which is isomorphic with the product $\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r}$.(Where $char(\overline{k}) \not| m_i$.)
Let ξ_i be a generator for the group $k_{m_i}^*$ and let $\chi^{(i)} : \mathbb{Z}_{m_i} \to \widehat{\mathbb{Z}_{m_i}}$ be the isomorphism defined by $\chi^{(i)}(\overline{a})(\overline{b}) = \xi_i^{ab}$. Let $\chi_{\overline{a}}^{(i)} = \chi^{(i)}(\overline{a})$.
We have the finite Fourier transform

$$F : V(\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r}) \to V(\widehat{\mathbb{Z}_{m_1}} \times \ldots \times \widehat{\mathbb{Z}_{m_r}})$$

defined by

$$F(\delta_{(\overline{a_1},\ldots,\overline{a_r})}) = \sum_{\chi_i \in \widehat{\mathbb{Z}_{m_i}}} \chi_1(\overline{a_1}) \ldots \chi_r(\overline{a_r})\delta_{(\chi_1,\ldots,\chi_r)}$$

The isomorphism $\chi : \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r} \to \widehat{\mathbb{Z}_{m_1}} \times \ldots \times \widehat{\mathbb{Z}_{m_r}}$, defined by

$$\chi(\overline{a_1}, \ldots, \overline{a_r}) = (\chi_{\overline{a_1}}^{(1)}, \ldots, \chi_{\overline{a_r}}^{(r)})$$

induces an isomorphism

$$\psi : V(\widehat{\mathbb{Z}_{m_1}} \times \ldots \times \widehat{\mathbb{Z}_{m_r}}) \to V(\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r})$$

The composition

$$\psi \circ F : V(\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r}) \to V(\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r})$$

is still denoted as F.
If $\overline{a} = (\overline{a_1}, \ldots, \overline{a_r}) \in \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r}$, define then $\chi_{\overline{a}} \in V(\widehat{\mathbb{Z}_{m_1}} \times \ldots \times \widehat{\mathbb{Z}_{m_r}})$ by

$$\chi_{\overline{a}}(\overline{b}) = \chi_{\overline{a_1}}^{(1)}(\overline{b}_1) \ldots \chi_{\overline{a_r}}^{(r)}(\overline{b}_r)$$

Proposition 1.9

1. $\left\{\delta_{\overline{a}} \;\middle|\; \overline{a} \in \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r}\right\}$ *and* $\left\{\chi_{\overline{a}} \;\middle|\; \overline{a} \in \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r}\right\}$ *are both orthonormal bases for* $V(\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_r})$.
2. $F(\delta_{\overline{a}}) = (1/\sqrt{m})\chi_{\overline{a}}$ *and* $F(\chi_{\overline{a}}) = (\sqrt{m})\delta_{-\overline{a}}$ *where* $m = m_1 \ldots m_r$.

Proof Similar calculation as for the $\mathbb{Z}_m$ case. ∎

2 The action of the theta group

Let A be a finite abelian group with order m such that $char(\overline{k}) \not| m$.
The theta group $\mathcal{G}(A)$ is defined as $\mathcal{G}(A) = k^* \times A \times \widehat{A}$. The multiplication on $\mathcal{G}(A)$ is defined by

$$(\lambda, x, \chi).(\mu, y, \tau) = (\lambda\mu\tau(x), xy, \chi\tau)$$

Proposition 2.1 *The sequence*

$$1 \to k^* \xrightarrow{\nu} \mathcal{G}(A) \xrightarrow{\mu} A \to 1$$

with $\nu(\lambda) = (\lambda, 1, 1)$ *and* $\mu(\lambda, x, \chi) = x$ *is exact.*

Proof See [2]. ∎

The theta group acts on $V(A)$ in the following way

$$f^{(\lambda,x,\chi)}(z) = \lambda.\chi(z)f(xz)$$

In a similar way we have the theta group $\mathcal{G}(\widehat{A}) = k^* \times \widehat{A} \times A$ which acts on $V(\widehat{A})$ by

$$g^{(\lambda,\chi,x)}(\tau) = \lambda\tau(x)g(\chi\tau)$$

(The bidual $\widehat{\widehat{A}}$ is canonically identified with A.)

Lemma 2.2 $\delta_b^{(\lambda,a,\chi)} = \lambda\chi(a^{-1}b)\delta_{a^{-1}b}$

Proof It is clear that $\delta_b(ax) = \delta_{a^{-1}b}(x)$ and hence

$$\delta_b^{(\lambda,a,\chi)}(x) = \lambda\chi(x)\delta_b(a^{-1}x) = \lambda\chi(a^{-1}b)\delta_{a^{-1}b}(x)$$

∎

Proposition 2.3 *The map* $\alpha : \mathcal{G}(A) \to \mathcal{G}(\widehat{A})$ *defined by*

$$\alpha(\lambda, x, \chi) = (\lambda\chi^{-1}(x), \chi, x^{-1})$$

is an isomorphism and for all $f \in V(A)$ *and* $(\lambda, a, \chi) \in \mathcal{G}(A)$ *we have*

$$F(f^{(\lambda,a,\chi)}) = F(f)^{\alpha\left((\lambda,a,\chi)\right)}$$

Proof It is clear that α is bijective and

$$\begin{aligned}\alpha\Big((\lambda,x,\chi)(\mu,y,\tau)\Big) &= \alpha(\lambda\mu\tau(x),xy,\chi\tau)\\ &= (\lambda\mu\tau(x)\tau^{-1}(x)\tau^{-1}(y)\chi^{-1}(xy),\chi\tau),xu)\\ &= (\lambda\chi^{-1}(x),\chi,x^{-1})(\mu\tau^{-1}(y),\tau,y)\\ &= \alpha(\lambda,x,\chi)\alpha(\mu,y,\tau)\end{aligned}$$

We have to prove the second assertion only for the basis functions δ_b, $(b \in A)$.

$$\begin{aligned}F\Big(\delta_b^{(\lambda,a,\chi)}\Big) &= F\Big(\lambda\chi(a^{-1}b)\delta_{a^{-1}b}\Big)\\ &= \lambda\chi(a^{-1}b)\textstyle\sum_{\tau\in\widehat{A}}\tau(a^{-1}b)\delta_\tau\end{aligned}$$

On the other hand we have

$$\begin{aligned}F(\delta_b)^{\alpha(\lambda,a,\chi)}(\nu) &= \textstyle\sum_{\tau\in\widehat{A}}\delta_\tau^{(\lambda\chi^{-1}(a),\chi,a^{-1})}(\nu)\\ &= \textstyle\sum_{\tau\in\widehat{A}}\tau(b)\lambda\chi^{-1}(a)\nu(a^{-1}\delta_\tau(\chi\nu)\\ &= \textstyle\sum_{\tau\in\widehat{A}}\tau(b)\lambda\chi^{-1}(a)\chi^{-1}(a^{-1})\tau(a^{-1})\delta_{\chi^{-1}\tau}(\nu)\\ &= \textstyle\sum_{\tau'\in\widehat{A}}\lambda\chi(b)\tau'(b)\chi(a^{-1})\tau'(a^{-1})\delta_{\tau'}(\nu)\\ &= \lambda\chi(a^{-1}b)\textstyle\sum_{\tau'\in\widehat{A}}\tau(a^{-1}b)\delta_\tau(\nu)\end{aligned}$$

This proves the second assertion. ∎

The following lemma will be used in the next section.

Lemma 2.4 *Let $\nu : V(A) \to V(A)$ be a $\mathcal{G}(A)$-automorphism of $V(A)$. Then there exists a constant element $\rho \in k^*$ such that $\nu(f) = \rho f$ for all $f \in V(A)$.*

Proof Let $\nu(\delta_a) = \sum_{b\in A}\gamma_{b,a}\delta_b$ with $\gamma_{b,a} \in k$.

$$\Rightarrow \nu\Big(\delta_a^{(\lambda,x,\chi)}\Big) = \lambda\chi(ax^{-1})\sum_{b\in A}\gamma_{b,ax^{-1}}\delta_b$$

Furthermore we have

$$\Big(\nu(\delta_a)\Big)^{(\lambda,x,\chi)} = \sum_{b\in A}\lambda\gamma_{bx,a}\chi(b)\delta_b$$

Hence $\chi(ax^{-1})\gamma_{b,ax^{-1}} = \gamma_{bx,a}\chi(b)$ for all $a, b \in A$ and $\chi \in \widehat{A}$. It follows that

- $\gamma_{b,b}\chi(b) = \gamma_{a,a}$ for all $a, b \in A$ and $\chi \in \widehat{A}$.

$$\Rightarrow \forall a \in A : \gamma_{a,a} = \gamma_{1,1}$$

- $\gamma_{b,a}\chi() = \gamma b, a\chi(b)$ for all $a \neq b \in A$ and $\chi \in \widehat{A}$.

$$\Rightarrow \forall a \neq b : \gamma_{b,a} = 0$$

Let $\rho = \gamma_{1,1}$. It follows that $\nu(f) = \rho f$ for all $f \in V(A)$. ∎

3 Theta functions on an analytic torus

Let $T = G/\Lambda$ be a g-dimensional analytic torus. So $G \cong (k^*)^g$ and $\Lambda \subset G$ is a free discrete subgroup of rank g.
Let H be the character group of G. So H is a free abelian group of rank g and each nowhere vanishing holomorphic function on G has a unique decomposition λu with $\lambda \in k^*$ and $u \in H$, (cf [1]).
The lattice Λ acts on $\mathcal{O}^*(G)$ in the following way :

$$\forall \gamma \in \Lambda, \alpha \in \mathcal{O}^*(G) : \alpha^\gamma(z) = \alpha(\gamma z)$$

A cocycle $\xi \in \mathcal{Z}^1\Big(\Lambda, \mathcal{O}^*(G)\Big)$ has a canonical decomposition

$$\xi_\gamma(z) = c(\gamma)p\Big(\gamma, \sigma(\gamma)\Big)\sigma(\gamma)(z), \quad \gamma \in \Lambda$$

with $c \in Hom(\Lambda, k^*)$, $\sigma \in Hom(\Lambda, H)$ and $p : \Lambda \times H \to k^*$ a bihomomorphism such that $p^2\Big(\gamma, \sigma(\delta)\Big) = \sigma(\delta)(\gamma)$ and $p\Big(\gamma, \sigma(\delta)\Big) = p\Big(\delta, \sigma(\gamma)\Big)$ for all $\gamma, \delta \in \Lambda$.
We assume that ξ is positive and non-degenerate. This means that σ is injective and $|p\Big(\gamma, \sigma(\gamma)\Big)| < 1$ for all $\gamma \neq 1$.
Remark The fact that such a cocycle exists implies that T is analytically isomorphic with an abelian variety, (see [1]).

The cocycle ξ induces an analytic morphism $\lambda_\xi : G \to Hom(G, k^*)$ which is defined by $\lambda_\xi(x)(\gamma) = \sigma(\gamma)(x)$.
Let $\hat{G} = Hom(G, k^*)$ and let $\hat{\Lambda} = \Big\{ u|_\Lambda \;\Big|\; u \in H \Big\}$. Then $\hat{G} \cong (k^*)^g$ and $\hat{\Lambda}$ is a lattice in $\hat{G}$.
The analytic torus $\hat{T} = \hat{G}/\hat{\Lambda}$ is called the dual torus of T and $\hat{T}$ isomorphic with the dual abelian variety of T.
The morphism λ_ξ induces an isogeny $\lambda_{\overline{\xi}} : T \to \hat{T}$ of degree $[H : \sigma(\Lambda)]^2$. We assume that $char(\overline{k}) \not| [H : \sigma(\Lambda)]$. This means that $\lambda_{\overline{\xi}}$ is a separable isogeny.
More details about $\lambda_{\overline{\xi}}$ can be found in [5] and [6].
If $x \in G$ determines an element $\overline{x} \in Ker(\lambda_{\overline{\xi}})$ then there exists a character $u_x \in H$ such that

$$\forall \gamma \in \Lambda : \sigma(\gamma)(x) = u_x(\gamma)$$

If $\gamma \in \Lambda$ then $\overline{\gamma} = 1$ and $u_\gamma = \sigma(\gamma)$.
The map $e : Ker(\lambda_{\overline{\xi}}) \times Ker(\lambda_{\overline{\xi}}) \to k^*$, defined by $e(\overline{x}, \overline{y}) = u_y(x)/u_x(y)$ is a non-degenerate, anti-symmetric pairing on $Ker(\lambda_{\overline{\xi}})$ and hence $Ker(\lambda_{\overline{\xi}}) = K_1 \oplus K_2$ where K_1 and K_2 are subgroups of order $[H : \sigma(\Lambda)]$ which are maximal with respect to the condition that e is trivial on K_i.
Let $\mathcal{L}(\xi)$ be the vectorspace of holomorphic theta functions of type ξ. An element $h \in \mathcal{L}(\xi)$ is a holomorphic function on G which satisfies the equation

$$\forall \gamma \in \Lambda : f(z) = \xi_\gamma(z) f(\gamma z)$$

The vectorspace $\mathcal{L}(\xi)$ has dimension $[H : \sigma(\Lambda)]$. Using the subgroups K_1 and K_2 it is possible to construct two bases for this vectorspace.

Let h_T be a fixed element in $\mathcal{L}(\xi)$ and let

$$\mathcal{G}(\xi) = \{(\overline{x}, f)| \quad \overline{x} \in \mathrm{Ker}\lambda_{\overline{\xi}}, f \in \mathcal{M}(T), \quad \mathrm{div}(f) = \mathrm{div}(\frac{h_T(xz)}{h_T(z)})\}$$

where $\mathcal{M}(T)$ is the space of meromorphic functions on T.
$\mathcal{G}(\xi)$ is a group for the multiplication defined by $(\overline{x}, f).(\overline{y}, g) = (\overline{xy}, g(xz)f(z))$. Moreover:

$$\forall (\overline{x}, f), (\overline{y}, g) \in \mathcal{G}(\xi) : [(\overline{x}, f), (\overline{y}, g)] = e(\overline{x}, \overline{y})$$

The sequence

$$1 \to k^* \xrightarrow{\nu} \mathcal{G}(\xi) \xrightarrow{\mu} Ker(\lambda_{\overline{\xi}}) \to 1$$

with $\nu(\lambda) = (1, \lambda)$ and $\mu(\overline{x}, f) = \overline{x}$ is exact. Furthermore there exist subgroups $\tilde{K}_1$ and $\tilde{K}_2$ in $\mathcal{G}(\xi)$ such that $\mu : \tilde{K}_i \to K_i$ is an isomorphism.
If $\overline{x} \in K_i$ then there exists a unique element $\tilde{x} \in \tilde{K}_i$ such that $\mu(\tilde{x}) = \overline{x}$. It follows that each element in $\mathcal{G}(\xi)$ has two decompositions $\lambda_1 \tilde{x}_2 \tilde{x}_1 = \lambda_2 \tilde{x}_1 \tilde{x}_2$ with $\lambda_i \in k^*$ and $\tilde{x}_i \in \tilde{K}_i$. The relation between λ_1 and λ_2 is given by

$$\lambda_1 = e(\overline{x_1}, \overline{x_2})\lambda_2$$

Proposition 3.1 *The maps $\alpha_i : \mathcal{G}(\xi) \to \mathcal{G}(K_i)$, $(i = 1, 2)$, defined by*

$$\alpha_1(\lambda_1 \tilde{x}_2 \tilde{x}_1) = \Big(\lambda_1, \overline{x_1}, e(\overline{x_2}, *)\Big)$$

$$\alpha_2(\lambda_2 \tilde{x}_1 \tilde{x}_2) = \Big(\lambda_2, \overline{x_2}, e(\overline{x_1}, *)\Big)$$

are isomorphisms of groups.

Proof Since the pairing e is non-degenerate the map $K_2 \to \hat{K}_1$ defined by $\overline{x_2} \mapsto e(x_2, *)$ is an isomorphism. Hence α_1 is bijective. An easy calculation shows that α_1 is a homomorphism.
A similar argument holds for α_2. ∎

Since K_2 and $\hat{K}_1$ are isomorphic we have an isomorphism $\alpha : \mathcal{G}(\hat{K}_1) \to \mathcal{G}(K_2)$, (cf lemma 2.3).

Lemma 3.2 $\alpha_2^{-1} \circ \alpha \circ \alpha_1 = Id$

Proof Straightforward calculation. ∎

Let $T_i = T/K_i, (i = 1, 2)$. Then T_i is isomorphic with an analytic torus G_i/Λ_i and the canonical map $T \to T_i$ is induced by a surjective morphism $\psi_i : G \to G_i$. Furthermore there exists a cocycle $\xi_i \in \mathcal{Z}^1\Big(\Lambda_i, \mathcal{O}^*(G_i)\Big)$ such that $\xi = \psi_i^*(\xi_i)$.
The vectorspace $\mathcal{L}(\xi_i)$ of holomorphic theta functions on G_i is 1-dimensional.
Let $h_i \in \mathcal{L}(\xi_i)$ be a fixed non-zero element.

For each $\overline{a} \in K_1$ and $\overline{b} \in K_2$ we can define theta functions $h_{\overline{a}}$ and $h_{\overline{b}}$ by

$$h_{\overline{a}} = \Big(h_2 \circ \psi_2\Big)^{(\tilde{a})} \text{ and } h_{\overline{b}} = \Big(h_1 \circ \psi_1\Big)^{(\tilde{b})}$$

We proved in [4] that $\Big(h_{\overline{a}}\Big)_{\overline{a} \in K_1}$ and $\Big(h_{\overline{b}}\Big)_{\overline{b} \in K_2}$ are bases for $\mathcal{L}(\xi)$.
Using the results about the finite Fourier transform it is possible to give the relation between these bases.

Proposition 3.3 *The maps $\beta_i : \mathcal{L}(\xi) \to V(K_i), (i = 1, 2)$, defined by $\beta_i(h_{\overline{x}}) = \delta_{\overline{x}^{-1}}$ are isomorphisms and*

$$\forall h \in \mathcal{L}(\xi) \text{ and } (\overline{x}, f) \in \mathcal{G}(\xi) : \beta_i\Big(h^{(\overline{x},f)}\Big) = \beta_i(h)^{\alpha_i(\overline{x},f)}$$

If $F : V(K_1) \to V(\hat{K_1}) \cong V(K_2)$ is the finite Fourier transform then we have

$$\mathcal{L}(\xi) \overset{\beta_1}{\to} V(K_1) \overset{F}{\to} V(K_2) \overset{\beta_2^{-1}}{\to} \mathcal{L}(\xi)$$

and

$$\beta_2^{-1} \circ F \circ \beta_1(h_{\overline{a}}) = \Big(1/\sqrt{[H : \sigma(\Lambda)]}\Big) \sum_{\overline{b} \in K_2} e(\overline{b}, \overline{a}) h_{\overline{b}}$$

Proof For the proof of the first part we refer to [6] Furthermore we have

$$\begin{aligned} \beta_2^{-1} \circ F \circ \beta_1(h_{\overline{a}}) &= \beta_2^{-1} \circ F(\delta_{\overline{a}^{-1}}) = \Big(1/\sqrt{[H : \sigma(\Lambda)]}\Big)\beta_2^{-1}\Big(\textstyle\sum_{\overline{b} \in K_2} e(\overline{b}, \overline{a}^{-1})\delta_{\overline{b}}\Big) \\ &= \Big(1/\sqrt{[H : \sigma(\Lambda)]}\Big) \textstyle\sum_{\overline{b} \in K_2} e(\overline{b}, \overline{a}^{-1}) h_{\overline{b}^{-1}} \\ &= \Big(1/\sqrt{[H : \sigma(\Lambda)]}\Big) \textstyle\sum_{\overline{b} \in K_2} e(\overline{b}, \overline{a}) h_{\overline{b}} \end{aligned}$$

■

Since β_1, F and β_2 are compatible with the actions of the theta groups we find that $V(K_1) \overset{\beta_1 \circ \beta_2^{-1} \circ F}{\to} V(K_1)$ is a $\mathcal{G}(K_1)$-automorphism of $V(K_1)$ and hence there exists a constant element $\rho \in k^*$ such that $\beta_1 \circ \beta_2^{-1} \circ F(f) = \rho f$ for all $f \in V(K_1)$.
It follows that $\beta_2^{-1} \circ F \circ \beta_1(h_{\overline{a}}) = \rho h_{\overline{a}}$ for all $\overline{a} \in K_1$. We can conclude :

Theorem 3.4 *(Transformation formula)*

$$\forall \overline{a} \in K_1 : \rho h_{\overline{a}} = \sum_{\overline{b} \in K_2} e(\overline{b}, \overline{a}) h_{\overline{b}}$$

Remark The bases $\big(h_{\overline{x}}\big)_{\overline{x} \in K_i}, (i = 1, 2)$, depend on the choices of the theta functions h_1 and h_2. These theta functions are unique up to multiplication with a non-zero constant. It follows that it is not possible to get rid of the constant ρ in the transformation formula.

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G. Van Steen
Universiteit Antwerpen
Departement Wiskunde en Informatica
Groenenborgerlaan 171
2020 Antwerpen, Belgie