Very ample line bundles on blown - up projective varieties

Edoardo Ballico Marc Coppens

Abstract

Let X be the blowing - up of the smooth projective variety V. Here we study when a line bundle M on X is very ample and, if very ample, the k-very ampleness of the induced embedding of X.

Introduction.

In the last few years several mathematicians studied (from many points of view and with quite different aims and techniques) the following situation. Let $\pi : X \mapsto V$ be the blowing - up of the variety V at finitely many points P_1,...,P_s; study the geometric and cohomological properties of X. Many authors (see e.g. [1, 4, 6, 7, 8, 9] and the references quoted there) were interested in the projective embeddings of X, e.g. to determine the very ample line bundles on X. Every $M \in Pic(X)$ is of the form $M \simeq \pi^*(F) - \sum_{1 \le i \le s} a_i E_i$ with $F \in Pic(V)$, $E_i = \pi^{-1}(P_i)$ the exceptional divisors and a_i integers. The very ampleness of M is studied in terms of F and the integers a_i. The conditions on F and on the integers a_i are usually both numerical (often obvious necessary conditions) and "positivity properties" of F. Often we are interested in the very ampleness of a line bundle M whose associated $F \in Pic(V)$ is of the form $F \simeq L \otimes R$ with $L, R \in Pic(V)$. It seems both natural and technically very useful to split the conditions on $F \simeq L \otimes R$ into conditions on L and conditions on R. Our main result on this topic is the following one. In all this paper we will use the notations (without the assumptions) introduced in its statement (as the following one). If P is a smooth point of a variety T, aP (with $a > 0$, a integer) will

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denote the $(a-1)^{\text{th}}$- infinitesimal neighborhood of P in T, i.e. $\mathbf{I}_{aP/T} := (\mathbf{I}_{P/T})^a$; aP is often called a fat point; $\sum_j n_j P_j$ denotes the scheme which is the disjoint union of the fat points $n_j P_j$.

Theorem 0.1. *Let V be a smooth n-dimensional complete variety, $Y = \{P_1, ..., P_s\} \subseteq V$ a set of s distinct points and $\pi : X \mapsto V$ the blowing - up of V at Y; let $E_i := \pi^{-1}(P_i), 1 \le i \le s$, be the exceptional divisors. E will denote both $E_1 + \ldots + E_s$ and its support. Fix integers $m_i \ge 1, 1 \le i \le s$, and line bundles R and L on V. Set $M := \pi^*(R \otimes L) \otimes \mathbf{O}_X(-m_1E_1 - \ldots - m_sE_s) \in Pic(X)$. Set $\mathbf{m} := m_1P_1 + \ldots m_sP_s$. Assume R spanned, L very ample and $H^1(V, \mathbf{I}_m \otimes R) = 0$. Assume the following condition :*

Property (C1): If $Y' \subseteq Y$ imposes at most 2 conditions on L and Z is a 0-dimensional subscheme containing every m_iP_i with $P_i \in Y'$ and with length $l(Z) \le l(\sum_{j\in y'} m_jP_j) + 2$ (where $y' := \{j \in \{1, ..., s\} : P_j \in Y'\}$) then Z imposes $l(Z)$ conditions on $L \otimes R$.

Then M is very ample.

In case $V = \mathbf{P}^n$, property (C1) becomes: If $Y' \subseteq Y$ is a subset of a line in $\mathbf{P}^n$ and $L \otimes R = \mathbf{O}(a)$, then $l(\sum_{j\in y'} m_jP_j) \le a - 1$. This condition is clearly necessary otherwise the proper transform of that line on X will be contracted to a point. Property (C1) is the natural generalization to our general situation of this condition.

We think it is interesting to study the higher order geometric properties of the embeddings of X. We recall two very natural definitions introduced in [3]. Let Y be a complete integral variety and $L \in Pic(Y)$; fix an integer $k \ge 0$; L is called k-very ample if for every subscheme Z of Y with length $l(Z) \le k+1$, the restriction map $H^0(Y, L) \mapsto H^0(Z, L|Z)$ is surjective, i.e. Z imposes independent conditions on L. If Y is smooth and $L \in Pic(Y)$, L was called k-jet ample in [3] if for all sets of integers $\{b_j\}$ with $b_j > 0$ and $\Sigma_j b_j \le k+1$, and for all choices of different $P_j \in Y$, the restriction map $H^0(Y, L) \mapsto H^0(Z, L|Z)$ is surjective, where $Z := \cup_j b_jP_j$. For instance L is 0-very ample if and only if it is spanned, while very ampleness, 1-very ampleness and 1-jet ampleness are equivalent. On this topic our main results are Theorem 0.2 and its variation Proposition 4.1.

Theorem 0.2. *Let V be a smooth n-dimensional complete variety, $Y = \{P_1, \ldots, P_s\}$ a finite subset of V. Let $\pi : X \mapsto V$ be the blowing-up of V at Y. Let $E_i = \pi^{-1}(P_i)$, $1 \le i \le s$, be the exceptional divisors. Fix positive integers $k, m_1, \ldots, m_s$. Set $Y(\mathbf{m}) := \sum_{1\le i\le s} m_iP_i$ and $Y(j,k) = (\sum_{i\ne j} m_iP_i) + (m_j + k)P_j$. Fix line bundles L and R on V and set $M := \pi^*(L \otimes R) \otimes \mathbf{O}_X(-m_1E_1 - \ldots - m_sE_s) \in Pic(X)$. Assume L k-very ample, R spanned, $H^1(V, L \otimes R) = 0, m_i \ge k$ for every i, the following condition (\$) and the following Property (C2) :*

For every integer j with $1 \le j \le s, Y(j,k)$ imposes independent conditions to R, i.e. we have $H^1(V, \mathbf{I}_{Y(j,k)} \otimes R) = 0$. (\$)

Property (C2) : for every integer j with $1 \le j \le s$, for every scheme Z with $l(Z) = k+1$ and $Z \cap E_j = \emptyset$, set $|L(-Z)| := \{x \in H^0(V, L), x \ne 0$ with 0-locus $Dx \supset Z\}$, $F(-Z) = \cap_{x\in|L(-Z)|} Dx$, $Y' := Y'_Z := Y \cap F(-Z)$. Set $n_{ij} := m_i$ if $i \ne j$ and $i \in Y'_Z$, $n_{jj} := m_j + k$ if $j \in Y'_Z$, $n_{ij} = 0$ if $P_i \notin Y'_Z$. Then for every such Z and every j, $1 \le j \le k$, $Z + (\sum_{1\le i\le s} n_{ij}P_i\}$ imposes independent conditions on $L \otimes R$.

Then M is k-very ample.

In section 2 we give another criterion for k-very ampleness (see Theorem 2.1).

We prove in that section only the particular (but very important) case $V = \mathbf{P}^n$ (see Proposition 2.2) because the statement of 2.2 shows the geometric significance of the assumptions Property (C1), (C2), (C3), (C3'), (C4) and the proof of the general case needs only notational changes. Furthermore, in the case $V = \mathbf{P}n$ the geometric assumptions are necessary ones and most of the cohomological ones are true for all line bundles.

The starting point of this paper was [5], Th. 2, whose statement is generalized by Theorem 0.1. Theorem 0.1 will be proved in section 1 with a long case by case checking. On this proof we stress the Local Computations (see Section 1) which are very useful (at least as inspiration) for the proofs of Theorem 2.1, Proposition 2.2 and in a key subcase of part (a) of the proof of 0.2.

We work always over an algebraically closed base field. A key difference between the statements of 0.1, 0.2, 2.1, 2.2, 4.1 and previous work in this area is given by the use of the very natural conditions called "Property (C1), (C2), (C3), (C3') and (C4)".

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1 Proof of Theorem 0.1.

In this paper we work always over an algebraically closed field $\mathbf{K}$ with arbitrary characteristic. If Z is a 0-dimensional scheme, $l(Z)$ will denote its length. If x is a section of a line bundle, $x \neq 0, D_x$ or Dx will denote the associated effective Cartier divisor; sometimes, if the line bundle is on a blown - up variety to avoid misunderstanding we will often write D'_x instead of D_x. In the setting of 0.1 often we will start with $x \in H^0(V, \mathbf{I}_m \otimes L \otimes R), x \neq 0$, and it as a section, x', of M on X; in this case D'_x will denote the divisor $D_{x'}$ on X.

For the proof of Theorems 0.1, 0.2 and of many of the results in this paper we need the following lemma.

Lemma 1.1. *Let $Z \subset V$ be a 0-dimensional scheme. Assume L very ample, R spanned and $H^1(V, \mathbf{I}_Z \otimes R) = 0$. Then for every $x \in H^0(V, L), x \neq 0$, the map $H^0(V, \mathbf{I}_Z \otimes L \otimes R) \mapsto H^0(D_x, \mathbf{I}_{(Z \cap Dx)/Dx} \otimes L \otimes R)$ is surjective.*

Proof of 1.1. Let Z' be the residual of Z with respect to D_x. Since $H^1(V, \mathbf{I}_Z \otimes R) = 0$, we have $H^1(V, \mathbf{I}_{Z'} \otimes R) = 0$. Consider the residual exact sequence

$$0 \mapsto \mathbf{I}_{Z'} \otimes L^{-1} \mapsto \mathbf{I}_Z \mapsto \mathbf{I}_{(Z \cap Dx)/Dx} \mapsto 0 \qquad (1)$$

We obtain the lemma from the cohomology exact sequence of (1) tensored by $L \otimes R$.

For the proof of 0.1 and (at least as language and inspiration) of 2.1, 2.2 and the key subcase "$l(Z') = 1$" in part (a) of the proof of 0.2, we will need the following local computations.

LOCAL COMPUTATIONS: Fix a very ample $J \in Pic(V)$, integers $n_i \geq 0, 1 \leq i \leq s$ and $u \in H^0(V, J)$, u $\neq 0$, vanishing at least of order n_i at each P_i. Hence u induces a global section u' of $\pi^*(J) - \sum_{1 \leq i \leq s} n_i E_i$; call $D_u \subset V$ and $D'_u \subset X$ the corresponding effective Cartier divisors. Set $P := P_1, m := n_1$ and $E' := E_1$. Fix $c' \in E' \cap D'_u$ and let c be the corresponding line in $T_P V$. Choose formal coordinates $z_1, ..., z_n$ for V at P such that c corresponds to $z_2 = \ldots = z_n = 0$. The Taylor expansion of u at P is $u = u_n + u_{n+1} + \ldots$. Then $z_1, w_2, \ldots, w_n$ are local coordinates for X at c' with $\pi^{-1}(z_i) = z_1 w_i$ for every $i > 1$. The local equation for D'_u at c' is given by $u_m(1, w_2, \ldots, w_n) + u_{m+1}(1, w_2, \ldots, w_n) z_1 + \ldots$. Hence $c' \in D'_u$ if and only if $u_m(1, 0, \ldots, 0) = 0$. This motivates the introduction of the following 0-dimensional scheme supported by P and denoted (with an abuse of notations) by $mP + c'$. The scheme $mP + c'$ is defined by the ideal sheaf of all $f \in \mathbf{K}[[z_1, \ldots, z_n]]$, with Taylor exspansion $f = f_0 + f_1 + f_2 + \ldots$ with $f_j = 0$ if $j < m$ and $f_m(1, 0, ..., 0) = 0$. Now take $v' \in T_{c'}(E)$ and let v be the corresponding plane of $T_P V$. We assume that the coordinates are chosen so that $v = \{z_3 = \ldots = z_n = 0\}$. Assume $u_m(1, 0, \ldots, 0) = 0$. Then $v' \in T_{c'}(D'_u)$ if and only if $u_m(1, z_2, \ldots, z_n)$ has not order 1 with respect to z_2. This motivates the introduction of the following 0-dimensional scheme supported by P and denoted (with an abuse of notations) by $mP + v'$. The scheme $mP + v'$ is defined by the ideal sheaf of all $f \in \mathbf{K}[[z_1, \ldots, z_n]]$, with Taylor expansion $f = f_0 + f_1 + f_2 + \ldots$ with $f_j = 0$ if $j < m$, $f_m(1, 0, ..., 0) = 0$ and $f_m(1, z_2, \ldots, z_n)$ has not order 1 with respect to z_2.

Proof of Theorem 0.1. The proof is a case by case long analysis to show first that M is spanned, then that $|M|$ separates distinct points and at the end that $|M|$ separates tangent directions.

(a) Proof that M is spanned.

(a1) Fix $a \in X \setminus E$ and see it as a point of V, too. Since L is very ample there is $x \in H^0(V, L)$ with $a \in D_x$ and $D_x \cap Y = 0$. Since $(L \otimes R)|D_x$ is spanned, we can find $u \in H^0(D_x, (L \otimes R)|D_x)$ with $a \notin D_u$. By Lemma 1.1 we obtain a section of M not vanishing at a.

(a2) Now take $a \in E$, say $a \in E_1$ and set $P := P_1, m := m_1, E' := E_1$. Set $|L(-a)| := \{x \in H^0(V, L) : P + a \subset D_x\}, F(-a) := \cap\{D_x : x \in |L(-a)|\}$ and $Y' := Y \cap F(-a)$. Hence Y' is as in the Property (C1). Take a general $x \in |L(-a)|$. By Property (C1), the scheme $((m_1 P_1 + a) + m_2 P_2 + \ldots + m_s P_s) \cap D_x$ imposes independent conditions to $(L \otimes R)|D_x$. Hence it imposes one more condition than $(m_1 P_1 + m_2 P_2 + \ldots + m_s P_s) \cap D_x$. From Lemma 1.1 we obtain $u \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$ with D_u not containing $m_1 P_1 + a$. Hence $a \notin D'_u$, proving that M is spanned at a.

(b) Proof that $|M|$ separates points.Take two different points a, b of X.

(b1) First assume $\{a, b\} \cap E = \emptyset$ and see a and b as points on V, too. Set $|L(-a-b)| := \{x \in H^0(V, L) : \{a, b\} \subset D_s\}, F(-a-b) := \cap\{D_x : x \in |L(-a-b)|\}$ and $Y' := Y \cap F(-a-b)$. Hence Y' is as in the Property (C1). By Property (C1) we find $u' \in H^0(D_x, \mathbf{I}_{\mathbf{m} \cap Dx/Dx} \otimes L \otimes R)$ with $a \in D_{u'}$ and $b \notin D_{u'}$. By Lemma 1.1 we find $u \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$ with $a \in D_u$ and $b \notin D_u$.

(b2) Now assume $b \notin E$ and $a \in E$, say $a \in E' := E_1$. Set again $P := P_1$. Write also $c' := a$ and use the corresponding notations c, and so on, as in the Local Computations. Set $|L(-P-b)| := \{x \in H^0(V, L) : \{P, b\} \subset D_s\}, F(-P-b) := \cap\{D_x : x \in |L(-P-b|\}$ and $Y' := Y \cap F(-P-b)$. Hence Y' is as in the Property

(C1). First assume $c \subset T_P(F(-P-b))$. Take a general $x \in |L(-P-b)|$. By Property (C1) we obtain $u' \in H^0(D_x, \mathbf{I}_{\mathbf{m}\cap Dx/Dx} \otimes L \otimes R)$ with $b \in D_{u'}$ and c not contained in $T_P(D_{u'})$. By Lemma 1.1 there is $u \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$ with $b \in D_u$, c not contained in $T_P(D_u)$. Hence $b \in D_u$ and $a \notin D_u$, as wanted.

(b3) Now assume c not contained in $T_P(F(-P-b))$. Since $\mathbf{m}$ imposes independent conditions to R, there is $s' \in H^0(V,R)$ vanishing on $(m_1-1)P_1 + m_2P_2 + \ldots + m_sP_s$ but not on $(m_1-1)P_1 + c'$. Take $x \in |L(-P-b)|$ with c not contained in $T_P(Dx)$ and set $u := s'x \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$. We have $b \in D_u$ but $m_1P_1 + c$ not contained in D_u. Hence $b \in D'_u$ and $a \notin D'_u$.

(b4) Now assume $a, b \in E$ with $\pi(a) \neq \pi(b)$ (say $\pi(a) = P_1$ and $\pi(b) = P_2$). Set $|L(-P_1-P_2)| := \{x \in H^0(V,L) : \{P_1,P_2\} \subset D_x\}, F(-P_1-P_2) := \cap\{D_x : x \in |L(-P_1-P_2)|\}$ and $Y' := Y \cap F(-P_1-P_2)$. Then Y' is as in the Property (C1). Use again the notations $P, c' = a, c, E'$. First assume c not contained in $T_P(F(-P_1-P_2))$. Take $y \in H^0(V,R)$ vanishing on $(m_1-1)P_1 + m_2P_2 + \ldots + m_sP_s$ but not on $(m_1-1)P_1 + c$. Take $x \in |L(-P_1-P_2)|$ with c not contained in $T_P(D_x)$ and set $u := yx \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$. Then D_u contains $(m_2+1)P_2$, hence $m_2P_2 + b$, but not $m_1P_1 + c'$, i.e. D'_u separates a and b. A similar argument works if the line of $T_{P_2}(V)$ corresponding to b is not contained in $T_{P_2}(F(-P_1-P_2))$.

(b5) Now we assume that the lines c and c" are contained respectively in $T_P(F(-P_1-P_2))$ and in $T_{P_2}(F(-P_1-P_2))$. Set $Y' := F(-P_1-P_2) \cap Y$; it is as in Property (C1). Take a general $x \in |L(-P_1-P2)|$. By Property (C1) we can find $y \in H^0(D_x, \mathbf{I}_{\mathbf{m}\cap Dx/Dx} \otimes R \otimes L)$ with D_y containing $(m_1P_1 \cap D_x) + a$ but not $(m_2P_2 \cap D_x) + b$. By Property (C1) there exists $u \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$ containing $(m_1P_1 \cap D_x) + a$ but not $(m_2P_2 \cap D_x) + b$. Hence $a \in D'_u$ and $b \notin D'_u$.

(b6) Finally assume $\{a,b\} \subset E$ with $\pi(a) = \pi(b)$, say $\pi(a) = P := P_1$. Take $y \in H^0(V,R)$ vanishing on m $_2P_2 + \ldots + m_sP_s$ and on $(m_1-1)P_1 + a$ but not on $(m_1-1)P_1 + b$. Take $x \in H^0(V, L \otimes \mathbf{I}_P)$ such that, if c" is the line corresponding to b, then $c" \cap T_P(Dx) = 0$. Set $u := yx \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$. Then u vanishes on $m_1P_1 + a$ but not on $m_1P_1 + b$. Hence D'_u separates a and b.

(c) Proof that $|M|$ separates tangent directions. Fix $b \in X$ and $v \in T_b(X)$, $v \neq 0$.

(c1) First assume $b \notin E$ and consider b and v as $ab \in V, v \in T_b(V)$. Set $|L(-v)| := \{x \in H^0(V,L) : b \in D_x$ and $v \in T_b(Dx)\}$ and $F(-v) := \cap\{D_x : x \in |L(-v)|\}$. Then $Y' := Y \cap F(-v)$ is as in Property (C1); set $y' := \{j \in \{1,\ldots,s\} : P_j \in Y'\}$. We apply Property (C1) to $(\sum_{i\in y'} m_iP_i) + v$ seeing v as a length 2 subscheme of V. By (C1) for a general $x \in |L(-v)|$ we obtain $y \in H^0(D_x, \mathbf{I}_{\mathbf{m}\cap Dx/Dx} \otimes L \otimes R)$ with $b \in D_y$ and $v \notin T_b(D_y)$. By Lemma 1.1 we find $u \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$ with $b \in D'_{u'}$ and $v \notin T_b(D'_{u'})$.

(c2) From now on we assume $b \in E$, say $b \in E_1 = E'$. First we assume $v \in T_b(E')$. However, the local computations we will make for this case will settle also a part of the case $v \notin T_b(E')$. Take $y \in H^0(V,R)$ with y vanishing on $(m-1)P_1 + m_2P_2 + \ldots + m_sP_s$ but not on $(m-1)P_1 + b$. Take $x \in H^0(V,L)$ with $P_1 \in D_x$, the line $\lambda \subset T_P(V)$ corresponding to b contained in $T_P(D_x)$ but the plane V_v corresponding to v not contained in $T_P(D_x)$. Set $u := xy \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$. We want to study u in terms of the Local Computation; since L is very ample, we may use global sections of L to obtain local coordinates around$P$. We have $y = y_{m-1} + y_m + \ldots$ with $c := y_{m-1}(1,0,\ldots,0) \neq 0$ and $x = x_1$ with $x_1(1,0,\ldots,0) = 0$ but $x_1(0,v_2,\ldots,v_n) \neq 0$.

Thus $u = u_m + u_{m+1} + \dots$ with $u_m = y_{m-1}x_1$ satisfying $u_m(1,0,\dots,0) = 0$ and $u_m \equiv z_1^{m-1}x_1 \mod (z_2,\dots,z_n)^2$. Hence the linear term of $u_m(1,z_2,\dots,z_n)$ does not vanishes at $(v_2,\dots,v_n)$. This gives u and u' with $b \in D_{u'}$ but $v \notin T_b(Du')$. On the other hand, consider $\lambda' := (1,0,\dots,0) \in T_b(X)$. For $u' \in H^0(X,M)$ we have $u'(z_1,w_2,\dots,w_n) \equiv cx_1(1,w_2,\dots,w_n) \mod (z_1,w_2,\dots,w_n)^2$ and so $\lambda' \in T_b(D'u')$. We conclude that for each $v \in T_b(E)$ we find $u \in H^0(X,M)$ with $b \in D_{u'}, \lambda' \in T_b(D_{u'}), v \notin T_b(D_{u'})$.

Now, take $|L(-b)|, F(-b)$ and $Y' := Y \cap F(-b)$ as before in step (a2). Y' is as in the Property (C1). First, we make a further local computation concerning $\lambda' = (1,0,\dots,0) \in T_b(X)$. Take u vanishing on mP; in $\mathbf{K}[[z_1,\dots,z_n]]$ write $u = u_m + u_{m+1} + \dots$; hence $\pi^*(u) = z^m u_m(1,w_2,\dots,w_n) + z^{m+1}u_{m+1}(1,w_2,\dots,w_n) + \dots$. One has $b \in D_u$ if and only if $u_m(1,0,\dots,0) = 0$. In that case $\lambda' \in T_b(Du)$ if and only if $u_{m+1}(1,0,\dots,0) = 0$. This is the motivation for the introduction of the subscheme $mP + \lambda'$ defined by the ideal $\{f \in \mathbf{K}[[z_1,\dots,zn]] : f = f_0 + f_1 + \dots$ with $f_j = 0$ for $j < m, f_m(1,0,\dots,0) = f_{m+1}(1,0,\dots,0) = 0\}$. First suppose $mP+\lambda'$ not contained in $F(-b)$. Take $s' \in H^0(V,R)$ vanishing on $(m-1)P + m_2P_2 + \dots + m_sP_s$ but not on $(m-1)P + b$ and take $x \in |L(-b)|$ non vanishing on $mP + \lambda'$. Set $y := s'x \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$. Since we have

$$\pi^*(s') = z^{m-1}s'_{m-1}(1,w_2,\dots,w_n) + z^m s'_m(1,w_2,\dots,w_n) + \dots \\ \text{with } s'_{m-1}(1,0,\dots,0) \neq 0$$

and

$$\pi^*(x) = zx_1(1,w_2,\dots,w_n) + z^2x_2(1,w_2,\dots,w_n) + \dots \text{ with } x_1(1,0,...,0) = 0$$

and

$$x_2(1,0,...,0) \neq 0,$$

it follows that

$$\begin{aligned}\pi^*(y) = z^m s'_{m-1}(1,w_2,\dots,w_n)x_1(1,w_2,...,w_n) \\ + z^{m+1}(s'_{m-1}(1,w_2,...,w_n)x_2(1,w_2,\dots,w_n) \\ + s'_m(1,w_2,\dots,w_n)x_1(1,w_2,\dots,w_n)) + \dots\end{aligned}$$

$$\begin{aligned}(\text{ i.e. } y_m(1,w_2,\dots,w_n) = s'_{m-1}(1,w_2,\dots,w_n)x_1(1,w_2,\dots,w_n) \\ \text{and } y_{m+1}(1,w_2,\dots,w_n) = s'_{m-1}(1,w_2,\dots,w_n)x_2(1,w_2,\dots,w_n) \\ + s'_m(1,w_2,\dots,w_n)x_1(1,w_2,\dots,w_n))\end{aligned}$$

with

$$y_m(1,0,\dots,0) = 0(\text{ i.e. } b \in D'_y),$$

$$y_{m+1}(1,0,\dots,0) = s'_{m-1}(1,0,\dots,0)x_2(1,0,\dots,0) \neq 0$$

(hence λ' not contained in T $_b(D'y)$). Thus in this case we can find $q \in H^0(X,M)$ with $b \in D_q$ and $\lambda' \notin T_b(Dq)$.

Now we consider the case $mP + \lambda' \subset F(-b)$. We take a general $x \in |L(-b)|$. The Property (C1) implies the existence of $s' \in H^0(D_x, \mathbf{I}_{\mathbf{m} \cap Dx/Dx} \otimes L \otimes R)$ vanishing on $mP + b$ but not on $mP + \lambda'$. By Lemma 1.1 we find $y \in H^0(V, \mathbf{I_m} \otimes L \otimes R)$ vanishing on $mP + b$ but not on $mP + \lambda'$. Again we find $b \in D'_y$ but $\lambda' \notin T_b(D'_y)$.

Now assume the existence of $v \in T_b(X)$ such that $v \in T_b(Dq)$ for every $q \in H^0(X, M)$ vanishing at b. It follows that $v \notin T_b(E')$ and $v \notin \mathbf{K}\lambda'$. Let W be the plane spanned by v and λ' and $v' \in W \cap T_P(E'), v' \neq 0$. We would find that for each $x \in H^0(X, M)$ satisfying $b \in D_{x'}$ and $\lambda' \in T_b(Dx)$ we have $W \subset T_b(Dx)$; in particular $v' \in T_b(Dx)$. We proved that this is not the case.

Now the proof of Theorem 0.1 is over.

2 Other very ampleness results.

We can prove the following general result.

Theorem 2.1. *Fix integers $k \geq 1, s \geq 1, m_i, 1 \leq i \leq s$, with $m_i \geq k$ for every i, a smooth complete n- dimensional variety V, a set $Y = \{P_1 \ldots, P_s\}$ on V and $k+1$ line bundlesR, $L_1, \ldots, L_k$ on V. Let $\pi : X \to V$ be the blowing - up of V at Y. Set $\mathbf{m} = m_1P_1 + \ldots + m_sP_s$. Assume $H^1(V, R \otimes \mathbf{I_m}) = 0$ and L_i very ample for every $i, 1 \leq i \leq s$. Assume*

$$H^1(V, R) = H^2(V, R) = H^1(V, R \otimes L_1 \otimes \ldots \otimes L_j) = H^2(V, R \otimes L_1 \otimes \ldots \otimes L_j = 0$$

for every $j, 1 \leq j \leq k$. Assume the following Property (C3):

Property (C3): For every j with $1 \leq j < k$, for every $W \subseteq \mathbf{m}$ with supp(W) imposing at most 2 conditions on L_{k-j}, every scheme $Z \subset V$ with $l(Z) - l(W) \leq k + 1 - j$ imposes $l(Z)$ conditions to $R \otimes L_1 \otimes \ldots \otimes L_{k-j}$.

Let $E_i := \pi^{-1}(P_i), 1 \leq i \leq s$, be the exceptional divisors; set $M := \pi^*(R \otimes L_1 \otimes \ldots \otimes L_k) \otimes \mathbf{O}_X(-m_1E_1 - \ldots - m_sE_s) \in Pic(X)$. Then M is k-very ample.

A key motivation for the proof of Theorem 2.1 was the introduction of schemes $aP_j + Z$ as in the Local Calculations in the proof of Theorem 0.1 and the corresponding analysis. The use of k very ample line bundles in the statement of 2.1 does not look optimal. It would be better if we could prove a similar statement with a unique k-very ample line bundle L instead of a tensor product $L_1 \otimes \ldots \otimes L_k$. This will be Theorem 0.2, but there we impose stronger condition on Y. In the important special case $V = \mathbf{P}^n$ however a line bundle $\mathbf{O}_V(m)$ is k-very ample if and only if $m \geq k$ and then $\mathbf{O}_V(m) = \mathbf{O}_V(1)^{\otimes m}$ is the tensor product of k very ample line bundles. Furthermore, all the higher order cohomology groups of the relevant line bundles vanish when $V = \mathbf{P}^n$. Hence Theorem 2.1 gives a sharp result in this case and we want to restate and prove it in this important case. The proof for the general case is an immediate translation of this proof. Therefore we leave it to the reader.

Proposition 2.2. *Fix integers $k \geq 1, t \geq 1, s \geq 1, m_i, 1 \leq i \leq s$, with $m_i \geq k$ and a set $Y = \{P_1, \ldots, P_s\}$ on $\mathbf{P}^n$. Let $\pi : X \to \mathbf{P}^n$ be the blowing - up of $\mathbf{P} := \mathbf{P}^n$ at Y. Set $\mathbf{m} := m_1P_1 + \ldots + m_sP_s$. Assume $H^1(\mathbf{P}^n, \mathbf{I_m}(t)) = 0$ and assume the following Property (C3'):*

Property (C3'): No subscheme Z of $\mathbf{m}$ with $l(Z) \geq t + 1$ has support contained in a line of $\mathbf{P}^n$.

Set $E_i := \pi^1(P_i), 1 \leq i \leq s$. Then $M := \pi^*(\mathbf{O_P}(k+t)) \otimes \mathbf{O}_X(-m_1E_1 - \ldots - m_sE_s)$ is $k-$very ample.

Proof. The case $k = 1$ is Theorem 0.1 with $V = \mathbf{P}^n$. We use induction on k. So, suppose $k = f > 1$ and suppose the proposition holds for $k \leq f - 1$. Let Z be a 0-dimensional subscheme of X of length $f + 1$. If the set $\pi(Z_{\text{red}})$ is not contained in Y, then take a hyperplane H with $H \cap Y = \emptyset$ and $\pi(Z_{\text{red}}) \cap H \neq \emptyset$. Set $w = 0$ in this case. If $\pi(Z_{\text{red}}) \subseteq Y$, then choose $a \in Z_{\text{red}}$ with, say, $P := P_1 = \pi(a)$. Let G be the line in $\mathbf{P}^n$ through P defined by a. Say $G \cap Y = \{P_1, \ldots, P_w\}$. Take a hyperplane H in $\mathbf{P}^n$ with $G \subseteq H$ and $H \cap Y = H \cap G$. Let H' be the proper transform of H in X. Then $\pi' := \pi|_{H'}: H' \to H$ is the blowing - up of $H \simeq \mathbf{P}^n$ at $P_1, \ldots, P_w$. Let Z' be the residual subscheme of Z with respect to H. From the exact residual sequence

$$0 \to \mathbf{I}_{Z'}(-H') \to \mathbf{I}_Z \to \mathbf{I}_{Z \cap H/H} \to 0$$

we obtain the exact sequence

$$H^1(X, M \otimes \mathbf{I}_{Z'}(-H')) \to H^1(X, M \otimes \mathbf{I}_Z) \to H^1(H', M \otimes \mathbf{O}_{H'} \otimes \mathbf{I}_{Z \cap H'/H'}) \qquad (2)$$

By Property (C3') we have $l(H' \cap Z) + \sum_{1 \leq i \leq w} m_i \leq k + t + 1$. On the other hand $M \otimes \mathbf{O}_{H'} = \pi'^*(\mathbf{O}_H(k+t)) \otimes \mathbf{O}_{H'}(-\Sigma_{1 \leq i \leq w} m_i(E_i \cap H'))$. This implies that $H' \cap Z$ imposes independent conditions on $M \otimes \mathbf{O}_H$, so we have an inclusion of $H^1(H', M \otimes \mathbf{O}_{H'} \otimes \mathbf{I}_{Z \cap H'/H'})$ into $H^1(H', M \otimes \mathbf{O}_{H'}) \simeq H^1(H, \mathbf{I}_{\mathbf{m}}'(k+t))$ with $\mathbf{m}' = \Sigma_{1 \leq i \leq w} m_i Pi$. Since $\sum_{1 \leq i \leq w} m_i \leq t$, this implies $H^1(H', M \otimes \mathbf{O}_{H'}) = 0$, hence $H^1(H', M \otimes \mathbf{O}_{H'} \otimes \mathbf{I}_{Z \cap H'/H'})$. On the other hand $M(-H') = \pi^*(\mathbf{O}_{\mathbf{P}}(k + t - 1)) \otimes \mathbf{O}_X(-\sum_{1 \leq i \leq w}(m_i - 1)E_i - \sum_{j > w} m_j E_j)$. We may apply the inductive assumption to this line bundle and obtain that $M(-H')$ is $(f - 1)-$very ample. Since $l(Z') < l(Z) = f + 1$ we find that Z' imposes independent conditions on $M(-H')$. Since $H^1(X, M(-H')) = 0$, it follows that $H^1(X, M \otimes \mathbf{I}_{Z'}(-H')) = 0$. By the exact sequence (2) we find $H^1(X, M \otimes \mathbf{I}_Z) = 0$. This implies that M is $f-$ very ample.

3 Proof of Theorem 0.2.

Here we prove Theorem 0.2. The proof will give easily other results (see section 4).

Proof of Theorem 0.2. The proof is by induction on k. It is divided into two parts, the first one being the initial case $k = 1$.

(a) First assume $k = 1$. Let Z be a length 2 subscheme of X. Fix $j, 1 \leq j \leq s$, such that if $E \cap Z \neq \emptyset$, then $Z \cap E_j \neq \emptyset$. Let Z' be the residual subscheme of Z with respect to E_j. We have the exact residual sequence

$$0 \to \mathbf{I}_{Z'}(-Ej) \to \mathbf{I}_Z \to \mathbf{I}_{Z \cap E_j/E_j} \to 0 \qquad (3)$$

and the corresponding exact cohomology sequence. Since $m_j \geq 1, l(Z \cap E_j) \leq 2$, we find $H^1(E_j, M \otimes \mathbf{I}_{Z \cap E_j/E_j}) = 0$. Note that $H^1(X, M(-E_j)) \simeq H^1(V, \mathbf{I}_{Y(j,1)} \otimes L \otimes R)$. Since $Y(j, 1)$ imposes independent conditions to R, it imposes independent conditions on $L \otimes R$. Hence $h^1(V, \mathbf{I}_{Y(j,1)} \otimes L \otimes R) \leq h^1(V, L \otimes R) = 0$. If we are able to prove that Z' imposes independent conditions to $M(-E_j)$, then we also find $H^1(X, M \otimes \mathbf{I}_Z) = 0$ by (3). In case $Z' = \emptyset$, there is nothing to prove. Suppose $l(Z') = 1$.

First assume $Q := \pi(Z') \notin Y$. Take $x \in H^0(V, L)$ with $Q \in D_x$ and $D_x \cap Y = \emptyset$. Since R and L are spanned there is $s' \in H^0(D_x, L \otimes R|D_x)$ with $Q \notin D_{s'}$. By Lemma

1.1 this lifts to $w \in H^0(V, \mathbf{I}_{Y(j,1)} \otimes L \otimes R)$ with $Q \notin D_w$. It follows that Z' imposes one condition to $M(-E_j)$.

Now assume $\pi(Z') \cap Y \neq \emptyset$. Since $l(Z') = 1$ by the choice of j the scheme Z is unreduced, not in E $_j$ but with support on E_j. Hence $\pi(Z') = P_j$. With the notations of the Local Computations of section 1 we consider $yP_j + \mathbf{t}$ with y $= m_j$ or 1 or $m_j + 1$ and $\mathbf{t}$ the tangent vector to V at P_j corresponding to the point $Z \cap E_j$. Take $s' \in H^0(V, R)$ vanishing with order at least m_r at each $P_r \in Y$ but not on $m_j Pj + \mathbf{t}$; take $x \in H^0(V, L)$ vanishing at P_j but not on $P_j + \mathbf{t}$. Thus $s := s'x \in H^0(V, \mathbf{I}_{Y(j,1)} \otimes L \otimes R)$ vanishes at $(m_j + 1)P_j$ but not on $(mj + 1)Pj + \mathbf{t}$. This implies thatZ' imposes one condition to $M(-E_j)$.

Finally assume $l(Z') = 2$ and $\pi(Z) \cap Y = \emptyset$. We consider Z as subscheme of V. Since L is very ample, Z imposes independent conditions to $H^0(V, L)$. Take a general $x \in H^0(V, L \otimes \mathbf{I}_Z)$. From the Property (C2) we know that $((m_1P_1 + \cdots + m_sP_s) \cap D_x) + Z$ imposes independent conditions on $L \otimes R \otimes \mathbf{O}_{Dx}$. Hence by the lemma 1.1 $m_1P_1 + \cdots + m_sP_s + Z$ imposes independent conditions on $L \otimes R$, concluding the proof of the case $k = 1$.

(b) Now assume $k = f \geq 2$ and the result true for $k < f$. Now Z is a subscheme of X with length $f + 1$. Fix again $j, 1 \leq j \leq s$, such that if $E \cap Z \neq \emptyset$, then $Z \cap E_j \neq \emptyset$. Let again Z' be the residual scheme of Z with respect to E_j. Since $m_j \geq f$, the proof given in part (a) shows that $H^1(E_j, \mathbf{I}_{Z \cap E_j / E_j} \otimes M) = 0$. As in part (a) we have $H^1(X, M(-E_j)) = 0$. Hence by Lemma 1.1 it is sufficient to prove that Z' imposes independent conditions to $M(-E_j)$. If $l(Z') = f + 1$ (i.e. $Z \cap E = \emptyset$), this is done as in part (a). Assume $l(Z') < f + 1$; note that $\deg(M(-E_j) \cap E_j) + l(Z') = m_j + 1 + l(Z') \leq m_j + k$; we may use the inductive assumption by the particular shape of condition (\$) and property (C2).

Now the proof of Theorem 0.2 is over.

4 Easy generalizations.

Here we list two results whose proof is exactly the same as the one of Theorem 0.2.

Proposition 4.1. *Let V be a smooth n-dimensional complete variety, $Y = \{P_1, \ldots, P_s\}$ a finite subset of V. Let $\pi : X \to V$ be the blowing-up of V at Y. Let $E_i = \pi^{-1}(P_i), 1 \leq i \leq s$, be the exceptional divisors. Fix positive integers $k, m_1, \ldots, m_s$. Set $Y(\mathbf{m}) := \sum_{1 \leq i \leq s} m_i P_i$ and $Y(j,k) = (\sum_{i \neq j} m_i P_i) + (m_j + k)P_j$. Fix line bundles L and R on V and set $M := \pi^*(L \otimes R) \otimes \mathbf{O}_X(-m_1E_1 - \ldots - m_sE_s) \in Pic(X)$. Assume L $k-$jet ample, R spanned, $H^1(V, L \otimes R) = 0, m_i \geq k$ for every i, the following condition (\$) and the following property (C4):*

For every integer j with $1 \leq j \leq s$, $Y(j,k)$ imposes independent conditions to R, i.e. we have $H^1(V, \mathbf{I}_{Y(j,k)} \otimes R) = 0$. (\$)

Property (C4) : for every integer j with $1 \leq j \leq s$, for every choice of integers $b_r > 0$ with $\sum_r b_r \leq k + 1$ and of points $A_r \in V$, and for every scheme $Z$with $Z \subseteq \sum_r b_r A_r$ and $Z \cap E_j = \emptyset$, set $|L(-Z)| := \{x \in H^0(V, L), x \neq 0$ with 0-locus$Dx \supset Z\}$, $F(-Z) = \cap_{x \in |L(-Z)|} Dx, Y' := Y'_Z := Y \cap F(-Z)$. Set $n_{ij} := m_i$ if $i \neq j$ and $i \in Y'$, $n_{jj} := m_j + k$ if $j \in Y'_Z$, $n_{ij} = 0$ if $P_i \notin Y'_Z$. Then for every such Z and every $j, 1 \leq j \leq k$, $Z + (\sum_{1 \leq i \leq s} n_{ij} P_i)$ imposes independent conditions on $L \otimes R$.

Then M is $k-$ jet ample.

Proof. This result is proven by the proof of Theorem 0.2 with no modification because :

(i) if $P \in E_j \subset X$ the residual scheme of aP (as fat point of X) is $(a-1)P$ and its restriction to E_j is aP (as fat point on $E_j \simeq \mathbf{P}^{n-1}$);

(ii) $\mathbf{O}_{E_j}(m_j)$ is m_j-jet ample (hence $k-$jet ample).

The main definitions related to $k-$spannedness ($k-$very ampleness, $k-$jet ampleness) using the surjectivity of maps work verbatim for vector bundles of arbitrary rank (and were introduced and discussed previously (see e.g. [2])). Note that for a higher rank bundle R the condition "R induces an embedding in a Grassmannian" is weaker than the condition "R is very ample", which in turn is weaker than the condition "R is 1-very ample" because traditionally the second concept means "$\mathbf{O}_{\mathbf{P}}(E)(1)$ is very ample". With these definitions we obtain trivially the following result.

Proposition 4.2. *Take all the assumption of Theorem 0.2 (resp. of Proposition 4.1), except that R and L are vector bundles of arbitrary rank and R is not assumed very ample but $1-$very ample (resp. $1-$jet ample). Then M is $k-$very ample (resp. $k-$jet ample).*

Now we discuss some of the generalized definitions around the concept of k-spannedness made in [2]. We will write "k^* spanned" as a shorthand for $k-$very ample (resp. $k-$jet ample, resp. $k-$spanned).

(4.3) "Generically k^* spanned" means: there is a Zariski open dense subset U of the base scheme, Y, such that the surjectivity of the restriction map holds for $0-$dimensional schemes Z with $Z_{\text{red}} \subseteq U$.

(4.4) If T is closed in Y, "k^* spannedness outside T" means that the testing $0-$dimensional schemes have support in $Y \setminus T$.

(4.5) Let D be an effective Cartier divisor of Y; "k^* spanned outside B and b^* ample along D" means the surjectivity condition of the restriction map for every cycle, say $Z = Z' \cup Z$", with $Z'_{\text{red}} \cap D_{\text{red}} = \emptyset, Z"_{\text{red}} \subseteq D_{\text{ red}}$ and $Z" \subset bD$".

Remark 4. 6. If we drop the condition "$m_i \geq k$" in the statements of 0.2 or 4.1 or 4.2 we obtain the same result except that along each E_i the line bundle M is shown to be only $\min(k, m_i)^*$ spanned.

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E. Ballico
Dept. of Mathematics,
University of Trento, 38050 Povo (TN),
Italy
e-mail: (bitnet) ballico@itncisca or ballico@science.unitn.it
fax: italy + 461881624

M. Coppens
Katholieke Industriele Hogeschool der Kempen, Campus H.I. Kempen
Kleinhoefstraat 4 B - 2440 Geel Belgium
e-mail: marc.coppens@khk.be