On the Γ-stability of systems of differential equations in the Routh-Hurwitz and the Schur-Cohn cases

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Abstract

If Γ is a pathwise-connected region in the complex plane, the problem of Γ-stability consists of establishing necessary and sufficient conditions on a set S of n-th degree polynomials to have their zeros inside Γ. In the present work, this problem is settled in the special case when Γ is either the imaginary axis or the unit circle and S is the set of all convex combinations of real or complex polynomials which are stable in the Routh-Hurwitz or the Schur-Cohn sense. Such conditions are required in problems related to the stability of complex systems of differential equations whose coefficients are subject to perturbations.

1 Introduction

The problem of eigenvalue distribution of systems of differential equation with respect to a given curve has been studied for quite a long time. The most commonly used curves are the imaginary axis and the unit circle due to the importance of the problem of stability of linear systems. Very efficient solutions have been devised to handle these two special cases, but they are mainly restricted to the case of real coefficients. Among these solutions, the Routh-Hurwitz test for the imaginary axis case and the Schur-Cohn test for the unit circle case are the most celebrated. In this respect the list of references is long indeed, and we confine it to [1], [2] and [4]

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where the above problems have been widely discussed. Recently, the complex counterpart of the Routh array for real systems was developed based on the theory of positive para-odd functions, see [6] and [7]. In [5], relationships between both types of stability- Hurwitz and Schur- were established in both the real and the complex case. Consider the eigenvalue distribution of systems of differential equations with respect to a general curve. More precisely, given a pathwise-connected region Γ in the complex plane, and a set S of n-th degree polynomials. We propose the following problem : under what conditions do all polynomials of S have their zeros inside Γ. When this condition holds, we say that S is Γ-stable. This problem is much too difficult to be handled all at once and no necessary and sufficient conditions for Γ-stability of complex systems or polynomials are known. In the present work, a version of the above problem is going to be considered, where the set S will be the set of all convex combinations of real or complex polynomials which are stable in the Routh-Hurwitz or the Schur-Cohn sense, and the question we answer here is : What are the necessary and sufficient conditions for the set S to be Γ-stable where Γ is either the imaginary axis or the unit circle. Such criteria are required in a variety of problems especially those dealing with the stability of complex systems whose coefficients are subject to perturbations. In section 2, we give our notations and we recall some basic results related to stability theory. In section 3 we obtain necessary and sufficient conditions for the stability of a convex combination of two complex polynomials, each stable in the Routh-Hurwitz sense. The Schur-Cohn counterpart is dealt with in section 4.

2 Notations and Basic Results

Consider two complex systems of differential equations with characteristic polynomials given respectively by :

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n, \quad a_0 \neq 0$$

and

$$g(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m, \quad b_0 \neq 0$$

Without loss of generality, we may assume that both polynomials are monic, i.e. $a_0 = b_0 = 1$. It is obvious that the coefficients of f and g are complex numbers.

Define the resultant matrix $R(f,g)$ as follows :

$$\begin{bmatrix} 1 & a_1 & a_2 & \cdot & \cdot & \cdot & a_n & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & a_1 & \cdot & \cdot & \cdot & a_{n-1} & a_n & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} & a_n \\ 0 & \cdot & \cdot & 0 & 1 & b_1 & \cdot & \cdot & \cdot & b_{m-1} & b_m \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & b_1 & b_2 & \cdot & \cdot & b_m & 0 & \cdot & \cdot & 0 \\ 1 & b_1 & \cdot & \cdot & \cdot & b_m & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

The resultant of $R(f,g)$ denoted $|R(f,g)|$ is the determinant of the $(m+n) \times (m+n)$ resultant matrix $R(f,g)$. We recall the following two facts on resultants which may be found in [4].

Lemma 2.1. The polynomials $f(z)$ and $g(z)$ have a common nonconstant factor if and only if $|R(f,g)| = 0$.

Lemma 2.2. If $f(z) = \prod_{j=1}^{n}(z-\alpha_j)$ and $g(z) = \prod_{k=1}^{m}(z-\beta_k)$ are the factorized forms of f and g, then the resultant of $R(f,g)$ is given by

$$|R(f,g)| = (-1)^{n(n-1)/2}\prod_{j=1}^{n}\prod_{k=1}^{m}(\alpha_j-\beta_k).$$

The paraconjugate f^* of f was defined in [6] by

$$f^*(z) = \overline{f(-\bar{z})}$$

where $\bar{z}$ denotes the complex conjugate of z. So

$$f^*(z) = (-1)^n z^n + (-1)^{n-1}\overline{a_1}z^{n-1} + \cdots \overline{a_{n-2}}z^2 - \overline{a_{n-1}}z + \overline{a_n}$$

Following the procedure of [6], define

$$h(z) = \begin{cases} \dfrac{f-f^*}{f+f^*}, & n \text{ odd} \\ \dfrac{f+f^*}{f-f^*}, & n \text{ even} \end{cases}$$

then h can be written in the form

$$h(z) = \frac{z^n + i\mathrm{Im}a_1 z^{n-1} + \mathrm{Re}a_2 z^{n-2} + i\mathrm{Im}a_3 z^{n-3} + \mathrm{Re}a_4 z^{n-4} + \cdots}{\mathrm{Re}a_1 z^{n-1} + i\mathrm{Im}a_2 z^{n-2} + \mathrm{Re}a_3 z^{n-3} + i\mathrm{Im}a_4 z^{n-4} + \cdots}.$$

$\frac{1}{h}$ is sometimes referred to as the test fraction corresponding to the function f [7]. If $N(z)$ is the numerator and $D(z)$ the denominator of h, then

$$f(z) = \tfrac{1}{2}[N(z) + D(z)] \qquad (1).$$

In what follows, we shall refer to N and D as the elements of the decomposition of f. Lemmas 2.3 and 2.4 below will be needed in the proof of the main theorem.

Lemma 2.3. $N(z_0) = 0$ if and only if $N(-\overline{z_0}) = 0$ and $D(z_0) = 0$ if and only if $D(-\overline{z_0}) = 0$.

The proof follows from the observation that if n is odd, then $N = f - f^*$ and $D = f + f^*$ and if n even, then $N = f + f^*$ and $D = f - f^*$.

Lemma 2.4. [3]. The system of differential equations with characteristic polynomial $f(z)$ is stable in the Routh-Hurwitz sense if and only if $\mathrm{Re}a_1 > 0$, and the zeros of $N(z)$ and $D(z)$ are simple, lie on the imaginary axis and interlace.

3 The Hurwitz Case

Let $f_1(z)$ and $f_2(z)$ be two monic n-th degree polynomials. Form the convex combination of f_1 and f_2

$$f(z,\lambda) = \lambda f_1(z) + (1-\lambda)f_2(z) \quad \text{for } \lambda \in (0,1) \qquad (2).$$

It is to be noted that $f(z,\lambda)$ is also a monic polynomial. We may write

$$f(z,\lambda) = N(z,\lambda) + D(z,\lambda),$$

where $N(z, \lambda)$ and $D(z, \lambda)$ are the elements of the decomposition of $f(z, \lambda)$ as in (1).

Let $R_\lambda = R[N(z, \lambda), D(z, \lambda)]$ be the resultant matrix of the polynomials $N(z, \lambda)$ and $D(z, \lambda)$ where each is treated as a complex polynomial in z parametrized by the real variable λ belonging to $(0, 1)$. R_λ is a $(2n-1) \times (2n-1)$ square matrix. Since the elements of R_λ are convex combinations of the appropriate coefficients of $f_1(z)$ and $f_2(z)$, then by lemma 2.2, the resultant $|R_\lambda|$ (determinant of R_λ) is a polynomial of degree at most $(2n-1)$ in λ with complex coefficients.

The following theorem characterizes the stability in the Routh-Hurwitz sense of a convex combination of two polynomials.

Theorem 3.1. Suppose $f_1(z)$ and $f_2(z)$ are both stable in the Routh-Hurwitz sense, then $f(z, \lambda)$ as defined in (2) is stable in the Routh-Hurwitz sense if and only if the resultant $|R_\lambda| \neq 0$, for all $\lambda \in (0, 1)$.

Proof. Let $f_j(z) = N_j(z) + D_j(z)$ for $j = 1, 2$ where N_j and D_j are the elements of the decomposition of f_j as in (1). Since $f_1(z)$ is stable in the Routh-Hurwitz sense, then by lemma 2.4, the zeros of N_1 and D_1 are simple and interlacing on the imaginary axis. The same is true for the zeros of N_2 and D_2.

To prove the necessity, suppose $|R_{\lambda_0}| = 0$ for some $\lambda_0 \in (0, 1)$, then by lemma 2.1, $N(z, \lambda_0)$ and $D(z, \lambda_0)$ must have a common zero, where $N(z, \lambda_0)$ and $D(z, \lambda_0)$ are the elements of the decomposition of $f(z, \lambda_0)$ in

$$f(z, \lambda_0) = N(z, \lambda_0) + D(z, \lambda_0).$$

But this violates the simple alternating property of the zeros of $N(z, \lambda_0)$ and $D(z, \lambda_0)$ which must hold by lemma 2.4 because $f(z, \lambda)$ is stable for all $\lambda \in (0, 1)$.

For the sufficiency, suppose $|R_\lambda| \neq 0$ for all $\lambda \in (0, 1)$. We note first that the zeros of $N(z, \lambda)$ and $D(z, \lambda)$ vary continuously as λ varies continuously in $(0, 1)$. We claim that $f(z, \lambda)$ is stable for all $\lambda \in (0, 1)$. Otherwise the simple interlacing property of the zeros will be violated for some $\lambda_0 \in (0, 1)$, and this could happen in one of two cases only :

case1 : at least one zero of $N(z, \lambda_0)$ or $D(z, \lambda_0)$ leaves the imaginary axis, or

case2 : $N(z, \lambda_0)$ and $D(z, \lambda_0)$ have a common zero on the imaginary axis.

Now case1 is impossible by applying lemma 2.3 to $N(z, \lambda_0)$ and $D(z, \lambda_0)$ after realizing that $N(z, 0)$ and $D(z, 0)$ have simple interlacing zeros on the imaginary axis because $f_2(z)$ is Hurwitz stable and $f_2(z) = N(z, 0) + D(z, 0)$.

By lemma 2.1, case2 implies that $|R_{\lambda_0}| = 0$, and therefore case2 is also impossible.

4 The Schur Case

To deal with this case, we shall decompose the monic polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

in a different manner. Define the circular symmetric $h(z)$ and antisymmetric $k(z)$ of $f(z)$as follows

$$h(z) = \frac{1}{2}[f(z) + z^n \overline{f(\frac{1}{\overline{z}})}]$$

and

$$k(z) = \frac{1}{2}[f(z) - z^n \overline{f(\frac{1}{\bar{z}})}]$$

Clearly

$$f(z) = h(z) + k(z) \tag{3}$$

and we call $h(z)$ and $k(z)$ the elements of the decomposition of $f(z)$. The next lemma is the Schur counterpart to lemma 2.3.

Lemma 4.1. $h(z_0) = 0$ if and only if $h(\frac{1}{\bar{z}_0}) = 0$, and $k(z_0) = 0$ if and only if $k(\frac{1}{\bar{z}_0}) = 0$.

The Schur counterpart to lemma 2.4 is given bellow,

Lemma 4.2. $f(z)$ is stable in the Schur-Cohn sense if and only if $|a_n| < 1$ and the zeros of $h(z)$ and $k(z)$ are simple, lie on the unit circle $|z| = 1$ and interlace.

Suppose now that $g_1(z)$ and $g_2(z)$ are two monic n-th degree polynomials. Form their convex combination as follows :

$$g(z, \lambda) = \lambda g_1(z) + (1 - \lambda) g_2(z)$$

for $\lambda \in (0, 1)$. Again $g(z, \lambda)$ is a monic polynomial. Denote by $h(z, \lambda)$ and $k(z, \lambda)$ the elements of the decomposition of $g(z, \lambda)$ as defined in (3),

$$g(z, \lambda) = h(z, \lambda) + k(z, \lambda).$$

By looking at $h(z, \lambda)$ and $k(z, \lambda)$ as polynomials in z with coefficients which are 1-st degree polynomials in λ, the resultant matrix $R_\lambda = R[h(z, \lambda), k(z, \lambda)]$ is a square $2n \times 2n$ matrix, and by lemma 2.2 the resultant $|R_\lambda|$ is a polynomial in λ of degree at most $2n$. The Schur counterpart to theorem 3.1 may now be given.

Theorem 4.1. Suppose g_1 and g_2 are both stable in the Schur-Cohn sense, then

$$g(z, \lambda) = \lambda g_1(z) + (1 - \lambda) g_2(z)$$

is stable in the Schur-Cohn sense for all λ in $(0, 1)$ if and only if $|R_\lambda| \neq 0$ for all $\lambda \in (0, 1)$.

The proof is identical to that of theorem 3.1. One has to employ lemma 4.1 instead of lemma 2.3, and lemma 4.2 should replace lemma 2.4 concerning the simplicity of the zeros and their interlacing property on the unit circle.

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