On Two Combination Rules for 0-1-Sequences

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1 Statement of problems and results

Consider two players who compete with each other by successively taking part in a series of 'win-or-lose' games for one person, henceforth called trials. Two *termination rules* are taken into consideration:

(i) Stop as soon as one player has reached a prespecified number N of successful trials;

(ii) stop after the Nth trial.

The player having more successes at the time of termination will be declared the final winner; under rule (ii) there is of course also the possibility of a tie. For determining who, at any given time, will carry out the next trial, two *switching rules* are deliberated:

Rule 1. Always alternate between the players;

Rule 2. the current player continues if and only if he/she was successful in the last trial.

We first assume that the outcomes scored by the two players form two deterministic sequences of 1's (for successes) and 0's (for failures), say $\mathcal{U} = (U_1, U_2, \ldots)$ and $\mathcal{V} = (V_1, V_2, \ldots)$. To present the problem in a formal manner, let us take either U_1 or V_1 to start with. Call the resulting sequence $X^{(i)} = (X_1^{(i)}, X_2^{(i)}, \ldots)$ if switching rule $i \in \{1, 2\}$ is used. Let $M, N \in \mathbb{N}$ be arbitrary positive integers. We consider, for the two rules, the set of all pairs $(\mathcal{U}, \mathcal{V})$ of 0-1-sequences for which there are more 1's from the U-sequence than from the V-sequence among the first N components of $X^{(i)}$ for $i = 1, 2$, and also the set of all $(\mathcal{U}, \mathcal{V})$ for which there are N 1's from the U's before there are M 1's from the V's. The gist of this paper is to show that some surprisingly simple inclusions and even equalities hold between these sets. In

Received by the editors June 1995.
Communicated by M. Hallin.

the case when the sequences $\mathcal{U}$ and $\mathcal{V}$ are random this leads to some nonintuitive relations between the corresponding winning probabilities.

2 Deterministic sequences

In this section let $(\mathcal{U}, \mathcal{V}) \in \{0,1\}^\infty \times \{0,1\}^\infty$ be an arbitrary pair of 0-1-sequences. For $i = 1, 2$ we denote by $S_{U,N}^{(i)}$ ($S_{V,N}^{(i)}$) the number of 1's among the U's (V's) in $(X_1^{(i)}, \ldots, X_N^{(i)})$.

Theorem 1.

(a) *If N is even and one starts with V_1, or if N is odd and one starts with U_1, then the relations $S_{U,N}^{(1)} > S_{V,N}^{(1)}$ and $S_{U,N}^{(2)} > S_{V,N}^{(2)}$ are equivalent.*

(b) *If N is even and one starts with U_1, or if N is odd and one starts with V_1, then $S_{U,N}^{(2)} = S_{V,N}^{(2)}$ implies that $S_{U,N}^{(1)} = S_{V,N}^{(1)}$. The converse implication does not hold.*

Proof. Assume that $N = 2n$ for some $n \in \mathbb{N}$; the case of an odd N is treated similarly.

(a) Let $S_j = U_1 + \ldots + U_j$, $S'_j = V_1 + \ldots + V_j$. Since obviously $S_{U,N}^{(1)} > S_{V,N}^{(1)}$ if and only if $S_n > S'_n$, it remains to show that $S_n > S'_n$ is also equivalent to $S_{U,N}^{(2)} > S_{V,N}^{(2)}$. Consider the moment when the $(n+1)$th U_j or V_k, whichever comes first, is placed in $X^{(2)}$. If this value is a V-value (*case* 1), there are $n - n_1$ U-values among the previous $X_j^{(2)}$ for some index $n_1 \geq 0$. As the number of 0's coming so far from the U-sequence is equal to that coming so far from the V-sequence (note that $X^{(2)}$ starts with V_1), and is equal to $n - S'_n$, the number of 1's from the U_j's so far is equal to $n - n_1 - (n - S'_n) = S'_n - n_1$. In order to achieve $S_{U,N}^{(2)} > S_{V,N}^{(2)}$, there are thus at least $n_1 + 1$ more 1's required from the U-sequence. But there are at most $\max[0, N - (n+1) - (n - n_1)] = \max[0, n_1 - 1]$ components left among $(X_1^{(2)}, \ldots, X_N^{(2)})$. Hence $S_{U,N}^{(2)} > S_{V,N}^{(2)}$ is impossible in case 1.

On the other hand, if the $(n+1)$th U precedes the $(n+1)$th V (*case* 2), prior to its placement there have been $n - n_2$ V-values before for some index $n_2 \geq 0$, the number of 0's among them exceeding that of 0's from the U-sequence by 1. Therefore, the number of V's so far is equal to $n - n_2 - (n - S_n + 1) = S_n - n_2 - 1$. Further, there are $\max[0, N - (n+1) - (n - n_2)] = \max[0, n_2 - 1]$ positions in $(X_1^{(2)}, \ldots, X_N^{(2)})$ still to be taken by future U's or V's. Thus, $S_{V,N}^{(2)} \leq (S_n - n_2 - 1) + \max[0, n_2 - 1] \leq S_n - 1 < S_{U,N}^{(2)}$.

Hence, in case 1 (2) we have $S_{U,N}^{(2)} \leq S_{V,N}^{(2)}$ ($S_{U,N}^{(2)} > S_{V,N}^{(2)}$). In case 1, the number of 1's among $U_1, \ldots, U_{n-n_1}$ is $S'_n - n_1$, so that $S_n \leq (S'_n - n_1) + n_1 = S'_n$. In case 2, the number of 1's among $V_1, \ldots, V_{n-n_2}$ is $S_n - n_2 - 1$, so that $S'_n \leq (S_n - n_2 - 1) + n_2 < S_n$. This completes the proof of (a).

(b) If $S_{U,N}^{(2)} = S_{V,N}^{(2)}$, then the numbers of U's and V's in $(X_1^{(2)}, \ldots, X_N^{(2)})$ are the same, because the number of U's equal to 1 is equal to that of V's which are 1, and the 0's from both sequences alternate so that their numbers must also coincide, as N is even. In this case a typical sequence is of the form

$$1_U, \ldots, 1_U, 0_U, 1_V, \ldots, 1_V, 0_V, 1_U, \ldots, 1_V, 0_V, 1_U, \ldots, 1_U \tag{1}$$

or

$$1_U, \ldots, 1_U, 0_U, 1_V, \ldots, 1_V, 0_V, 1_U, \ldots, 1_V, 0_V \tag{2}$$

($N/2$ U's and $N/2$ V's in (2.1) and (2.2)), where the notation is self-explanatory. (Of course it is possible that the sequence starts with 0_U.) The corresponding sequence generated under switching rule 1, i.e., with alternating U's and V's, is easily obtained: Proceeding from left to right, shift each V-value as far as possible to the left such that the sequence to its left corresponds to $U, V, U, V, \ldots$. After all V's have been shifted accordingly, shift the 1_U's on the right, if there are any, to the left such that finally one arrives at a completely alternating sequence. It follows that $S^{(2)}_{U,N} = S^{(2)}_{V,N}$ implies $S^{(1)}_{U,N} = S^{(1)}_{V,N}$.

Finally, consider sequences $\mathcal{U} = (1, \ldots, 1, 0, \ldots)$ and $\mathcal{V} = (1, \ldots, 1, \ldots)$ each starting with $N/2$ 1's and $\mathcal{U}$ having a 0 in the $(1 + N/2)$th position. Then $S^{(1)}_{U,N} = S^{(1)}_{V,N} = N/2$, but $S^{(2)}_{U,N} = N > N - 1 = S^{(2)}_{V,N}$.

Next we consider the reaching of certain levels under the two switching rules. Let $\sigma^{(i)}_N$ ($\tau^{(i)}_N$) be the index of the Nth 1 coming from the U-sequence (V-sequence) in $(X^{(i)}_1, X^{(i)}_2, \ldots)$.

Theorem 2. *If $N \geq M$, the relation $\sigma^{(1)}_N < \tau^{(1)}_M$ implies that $\sigma^{(2)}_N < \tau^{(2)}_M$. If $N = M$, the two inequalities are equivalent.*

Proof. We assume that one starts with U_1, the other case being treated similarly. Let $\alpha_N = \inf\{j \geq 1 \mid U_1 + \ldots + U_j = N\}$ and assume that $\alpha_N = N + j$ and $\sigma^{(1)}_N < \tau^{(1)}_M$. If $U_1 + \ldots + U_k$ equals N for the first time for $k = N + j$, then $\sigma^{(1)}_N < \tau^{(1)}_M$ iff among the first $N + j - 1$ V-values of $X^{(1)}$ there are less than M 1's (recall that in $X^{(1)}$ U-values and V-values alternate). Therefore, $\alpha_N = N + j$, $\sigma^{(1)}_N < \tau^{(1)}_M$ is equivalent with $\alpha_N = N + j$, $V_1 + \ldots + V_{N+j-1} \leq M - 1$.

On the other hand, the case $\alpha_N = N + j, \sigma^{(2)}_N < \tau^{(2)}_M$ occurs iff the Nth 1 of $U_1, U_2, \ldots$ is the $(N + j)$th element of that sequence and (since the 0's of the U-sequence and of the V-sequence alternate under rule 2) in the V-sequence the jth 0 appears before the Mth 1, i.e. there are less than M 1's among the first $M + j - 1$ V-values in $X^{(1)}$. Consequently, $\alpha_N = N + j$, $\sigma^{(2)}_N < \tau^{(2)}_M$ is satisfied if and only if $\alpha_N = N + j$, $V_1 + \ldots + V_{M+j-1} \leq M - 1$. Next note that trivially,

$$V_1 + \ldots + V_{N+j-1} \leq M - 1 \Longrightarrow V_1 + \ldots + V_{M+j-1} \leq M - 1,$$

as $N \geq M$. It follows that

$$\alpha_N = N + j,\ \sigma^{(1)}_N < \tau^{(1)}_M \Longrightarrow \alpha_N = N + j,\ \sigma^{(2)}_N < \tau^{(2)}_M.$$

with equivalence holding if $N = M$. Since j was arbitrary, Theorem 2 is proved.

3 The probabilistic setting

Let us now assume that $\mathcal{U}$ and $\mathcal{V}$ are two random sequences with no restrictions whatsoever on their joint probability distribution. Thus, each player may show an arbitrary random performance in the trials; any conceivable kind of stochastic dynamics between the players' outcomes is permissible, reflecting e.g. effects caused by learning from experience or other sources, stubborn sticking to unsuccessful strategies, fatigue, internal or external changes in the trials etc., as well as any reactions on the opponent's successes or lack of these.

The results in Section 2 have immediate consequences for the winning probabilities of the players under both termination rules and switching rules. These probabilities are given by

$$p_N^{(i)} = P\left(S_{U,N}^{(i)} > S_{V,N}^{(i)}\right), \qquad \text{and} \qquad q_{N,M}^{(i)} = P\left(\sigma_N^{(i)} < \tau_M^{(i)}\right).$$

By Theorem 2, we can conclude that

$$q_{N,M}^{(1)} \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} q_{N,M}^{(2)} \qquad \text{if} \quad N \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} M. \tag{3}$$

Using Theorem 1, it is seen that if the sequence starts with U_1,

$$p_N^{(1)} \left\{ \begin{array}{c} \leq \\ = \end{array} \right\} p_N^{(2)} \qquad \text{if} \quad N \text{ is} \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right., \tag{4}$$

while if it starts with V_1, we obtain

$$p_N^{(1)} \left\{ \begin{array}{c} \leq \\ = \end{array} \right\} p_n^{(2)} \quad \text{if} \quad N \text{ is} \left\{ \begin{array}{c} \text{odd} \\ \text{even} \end{array} \right.. \tag{5}$$

To see (3.2) for even N, note that by Theorem 1 we can conclude that

$$P\left(S_{U,N}^{(2)} = S_{V,N}^{(2)}\right) \leq P\left(S_{U,N}^{(1)} = S_{V,N}^{(1)}\right), \quad P\left(S_{U,N}^{(2)} < S_{V,N}^{(2)}\right) = P\left(S_{U,N}^{(1)} < S_{V,N}^{(1)}\right)$$

(reversing the roles of U and V to obtain the equation). Hence,

$$\begin{aligned} p_N^{(2)} &= 1 - P\left(S_{U,N}^{(2)} = S_{V,N}^{(2)}\right) - P\left(S_{U,N}^{(2)} < S_{V,N}^{(2)}\right) \\ &\geq 1 - P\left(S_{U,N}^{(1)} = S_{V,N}^{(1)}\right) - P\left(S_{U,N}^{(1)} < S_{V,N}^{(1)}\right) \\ &= p_N^{(1)}. \end{aligned}$$

It is surprising that the relations (3.1)-(3.3), in particular the equalities, hold independently of the distribution of $(\mathcal{U}, \mathcal{V})$. The results are not intuitive (at least to the author) even in the case of i.i.d. Bernoulli trials U_j and V_j. Actually this note originated from an effort to prove the false claim that in the Bernoulli case $P(U_j = 1) > P(V_j = 1)$ implies that $q_{N,N}^{(1)} < q_{N,N}^{(2)}$.

We have not found references to switching problems like the ones considered here in the voluminous literature on random 0-1-sequences. The models of paired comparisons and their ramifications, as investigated e.g. by Maisel [3], Uppuluri and Blot [9], Menon and Indira [4], Groeneveld and Arnold [2], Nagaraya and Chan

[5], Stadje [7], and Sigrist [6], seem to be the most closely related ones. However, in these models a trial is a kind of match of the two 'players' against each other in which each one of them can score the point, while in the situation studied here every trial is carried out by only one player so that some switching rule is required. However, suppose that in every match only one player (called the 'attacker', say) can score, while the other takes the role of a 'defender'. This modification leads to our model. There is also a connection to Banach's matchbox problem in which items are successively removed from two piles following some selection rule for the piles (see Goczyla [1] and Stirzaker [8] and the references given there). If additionally to the pile selection rule we introduce the possibility that removals may fail with some probability (possibly variable and depending on the other trials), our model also applies.

Acknowledgement. I am grateful to the referee who in particular noticed that $\mathcal{U}$ and $\mathcal{V}$ do not have to be independent of each other, as I had originally assumed.

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