The prescribed mean curvature equation for a revolution surface with Dirichlet condition

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Abstract

We give conditions on a continuous and bounded function H in R^2 to obtain at least two weak solutions of the mean curvature equation with Dirichlet condition for revolution surfaces with boundary, using variational methods.

Introduction

The prescribed mean curvature equation with Dirichlet condition for a vector function $X : B \longrightarrow R^3$ is the system of non linear partial equations

$$(1) \begin{cases} \triangle X = 2H(X) X_u \wedge X_v & in \quad B \\ X = X_0 & in \quad \partial B \end{cases}$$

where B is the unit disk in R^2, $\wedge$ denotes the exterior product in R^3 and $H : R^3 \longrightarrow R$ is a given continuous function.

When H is bounded and X_0 is in the Sobolev space $H^1(B, R^3)$, we call $X \in H^1(B, R^3)$ a weak solution of (1) if $X \in X_0 + H^1_0(B, R^3)$ and for every $\phi \in C^1_0(B, R^3)$

$$\int_B \nabla X \cdot \nabla \phi + 2H(X) X_u \wedge X_v \cdot \phi = 0.$$

In certain cases, weak solutions are obtained as critical points in $X_0 + H^1_0(B, R^3)$ of the functional

$$D_H(X) = D(X) + 2V(X)$$

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with

$$D(X) = \frac{1}{2}\int_B |\nabla X|^2$$

the Dirichlet integral and

$$V(X) = \frac{1}{2}\int_B Q(X) \cdot X_u \wedge X_v$$

the Hildebrandt volume, and Q is the associated function to H which satisfies $divQ = 3H$, $Q(0) = 0$, [H2].

For X_0 non constant and H constant, verifying that $0 < |H|\|X_0\|_\infty < 1$, there are two weak solutions: a local minimum of D_H in $X_0 + H_0^1(B, R^3)$, [H1], [S1], and a second weak solution which is not a local minimum of D_H, called an unstable weak solution, [B-C], [S1], [S2].

When H is not constant, in certain cases there are also two weak solutions, [LD-M], [S3].

For X a revolution surface, $X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, $f, g \in C^2(I)$, $I = [a,b]$, the problem (1) becomes

$$(\text{Dir}) \begin{cases} f'' - f = -2H(f,g)fg' & \text{in } I \\ g'' = 2H(f,g)ff' & \text{in } I \\ f(a) = \alpha_1 \quad f(b) = \beta_1 \\ g(a) = \alpha_2 \quad g(b) = \beta_2 \end{cases}$$

with $H : R^2 \longrightarrow R$ a given continuous and bounded function, and α_1, α_2, β_1, β_2 positive numbers.

In 1. we see that also, in this case, there exists an associated functional to H.

In 2. we prove that this functional has a global minimum in a convex subset of $H^1(I, R^2)$, which provides a weak solution of (Dir).

In 3., we use a variant of the Mountain Pass Lemma to find, under certain conditions a second weak solution of (Dir), corresponding to an unstable critical point of the functional. We can apply the Mountain Pass Lemma without considering bounded convex subsets of H^1, as in the general case. So, it is simpler to obtain another solution. Finally we show a family of functions H, for which the corresponding system (Dir) admits, at least, two weak solutions.

We denote $W^{1,p}(\Omega, R^n)$ the usual Sobolev spaces, [A], and $H^1(\Omega, R^n) = W^{1,2}(\Omega, R^n)$. Finally, if $X \in H^1(\Omega, R^n)$, we denote $\|X\|_{L^2(\partial\Omega, R^n)} = (\int_{\partial\Omega} |TrX|^2)^{\frac{1}{2}}$, where $Tr : H^1(\Omega, R^n) \longrightarrow L^2(\partial\Omega, R^n)$ is the usual trace operator, [A].

1 The associated variational problem.

Consider two real valued functions $f,\, g \in C^2[I]$, with fixed positive boundary values

$$f(a) = \alpha_1 \quad f(b) = \beta_1, \qquad g(a) = \alpha_2 \quad g(b) = \beta_2.$$

When f is positive and g is non decreasing the generated revolution surface in parametric form, associated to these functions, is

$$X(u,v) = (f(u)\cos v,\ f(u)\sin v,\ g(u)).$$

The mean curvature of this surface is

$$H(f,g) = \frac{1}{2}\left(\frac{g\prime}{f\sqrt{f\prime^2 + g\prime^2}} + \frac{f\prime g\prime\prime - f\prime\prime g\prime}{(f\prime^2 + g\prime^2)^{\frac{3}{2}}}\right),$$

see [D], and [O].

The H-surface system $\triangle X = 2H(X)X_u \wedge X_v$ is, in this case, equivalent to the system

$$(2) \begin{cases} f\prime\prime - f = -2H(f,g)fg\prime \\ g\prime\prime = 2H(f,g)ff\prime. \end{cases}$$

From now on, we consider the system (2). We see that there exists a functional D_H corresponding to (2), i.e., (2) are the Euler Lagrange equations of D_H.

THEOREM 1: *Let $D_H : C^2([a,b],R^2) \longrightarrow R$ be the functional defined by*

$$D_H(f,g) = \int_a^b \frac{f\prime^2 + g\prime^2 + f^2}{2} + \int_0^1 t^2 H(tf,tg)dt(-f^2 g\prime + ff\prime g)dx.$$

Then if $\frac{d}{d\varepsilon} D_H(f+\varepsilon\phi_1, g+\varepsilon\phi_2)\Big|_{\varepsilon=0} = 0$ *for* $(\phi_1,\phi_2) \in C_0^1([a,b],R^2)$, (f,g) *is a solution of (2).*

REMARK: *We say that $(f,g) \in H^1(I,R^2)$ is a weak solution of (2) if (f,g) is a critical point of D_H.*

<u>*Proof:*</u> $D_H = D_1 + D_2$, with

$$D_1(f,g) = \int_a^b \frac{f\prime^2 + g\prime^2 + f^2}{2} dx$$

and

$$D_2(f,g) = \int_a^b \int_0^1 t^2 H(tf,tg)dt(-f^2 g\prime + ff\prime g)dx.$$

Then

$$\frac{d}{d\varepsilon} D_1(f+\varepsilon\phi_1, g+\varepsilon\phi_2)\Big|_{\varepsilon=0} = \int_a^b (-f\prime\prime + f)\phi_1 - g\prime\prime\phi_2 dx$$

and

$$\frac{d}{d\varepsilon}D_2(f+\varepsilon\phi_1,g+\varepsilon\phi_2)\Big|_{\varepsilon=0}=$$
$$=\int_a^b\int_0^1\Big(t^3\Big(\frac{\partial H}{\partial x_1}(tf,tg)\phi_1+\frac{\partial H}{\partial x_2}(tf,tg)\phi_2\Big)(-f^2g\prime+ff\prime g)+$$
$$+t^2H(tf,tg)[(-2fg\prime+f\prime g)\phi_1+ff\prime\phi_2+fg\phi_1\prime-f^2\phi_2\prime]\Big)dtdx.$$

By partial integration in

$$\int_a^b\int_0^1 t^2H(tf,tg)dtf^2\phi_2\prime \quad\text{and}\quad \int_a^b\int_0^1 t^2H(tf,tg)fg\phi_1\prime$$

we get

$$\frac{d}{d\varepsilon}D_2(f+\varepsilon\phi_1,g+\varepsilon\phi_2)\Big|_{\varepsilon=0}=$$
$$\int_a^b\Big[\int_0^1\Big(t^3\frac{\partial H}{\partial x_1}(tf,tg)f^2g\prime-3t^2H(tf,tg)fg\prime-t^3\frac{\partial H}{\partial x_2}(tf,tg)fgg\prime\Big)dt\Big]\phi_1+$$
$$+\Big[\int_0^1\Big(t^3\frac{\partial H}{\partial x_2}(tf,tg)ff\prime g+t^3\frac{\partial H}{\partial x_1}(tf,tg)f^2f\prime+3t^2H(tf,tg)ff\prime\Big)dt\Big]\phi_2dx.$$

By partial integration of the terms

$$\int_a^b\int_0^1 t^2H(tf,tg)fg\prime dtdx \quad\text{and}\quad \int_a^b\int_0^1 t^2H(tf,tg)ff\prime dtdx$$

we obtain

$$\frac{d}{d\varepsilon}D_2(f+\varepsilon\phi_1,g+\varepsilon\phi_2)\Big|_{\varepsilon=0}=-\int_a^b\Big(H(f,g)fg\prime\phi_1+H(f,g)ff\prime\phi_2\Big)dx.$$

Then

$$\frac{d}{d\varepsilon}D_H(f+\varepsilon\phi_1,g+\varepsilon\phi_2)\Big|_{\varepsilon=0}=\int_a^b(-f\prime\prime+f-H(f,g)fg\prime)\phi_1+(-g\prime\prime+H(f,g)ff\prime)\phi_2dx.$$

Finally if $\frac{d}{d\varepsilon}D_H(f+\varepsilon\phi_1,g+\varepsilon\phi_2)\Big|_{\varepsilon=0}=0,\quad(\phi_1,\phi_2)\in C_0^1(I,R^2)$, it follows that (f,g) verifies (2).

Remark: *We call* $dD_H(f,g)(\phi_1,\phi_2)=\frac{d}{d\varepsilon}D_H(f+\varepsilon\phi_1,g+\varepsilon\phi_2)\Big|_{\varepsilon=0}$.

2 The Dirichlet problem associated to H.

Consider the Dirichlet problem in I, associated to the mean curvature equation (2), for a revolution surface given by

$$\text{(Dir)}\begin{cases} f'' - f = -2H(f,g)fg' & \text{in } I \\ g'' = 2H(f,g)ff' & \text{in } I \\ f(a) = \alpha_1 \quad f(b) = \beta_1 \\ g(a) = \alpha_2 \quad g(b) = \beta_2 \end{cases}$$

where $H : R^2 \longrightarrow R$ is continuous.

As we saw in 1. a critical point of D_H is a weak solution of (2). In the following theorem we give conditions to have local minima of D_H in a convenient subset of H^1, which provide weak solutions of (Dir).

THEOREM 2: *Let $H : R^2 \longrightarrow R$ be a continuous function verifying $|H(X_1, X_2)X_1(X_1, X_2)| \leq c$, and $D_H : H^1(I, R^2) \longrightarrow R$ the functional associated to H. Let $T = (f_0, g_0) + H_0^1(I, R^2)$ with f_0, $g_0 \in H^1(I)$ and $f_0(a) = \alpha_1$, $f_0(b) = \beta_1$, $g_0(a) = \alpha_2$, $g_0(b) = \beta_2$. Then D_H has a minimum $(\tilde{f}, \tilde{g})$ in T and therefore $(\tilde{f}, \tilde{g})$ is a solution of (Dir).*

Proof: We prove that D_H is weakly lower semicontinuous in H^1 and coercive in T. As T is an affine subspace of H^1, and hence weakly closed, D_H has a minimum $(\tilde{f}, \tilde{g})$ in T.

From

$$D_H(f,g) \geq \int_a^b \frac{f'^2 + g'^2 + f^2}{2} - c\sqrt{f'^2 + g'^2}dx$$

we deduce that D_H is coercive.

Suppose (f_n, g_n) is a sequence in T such that (f_n, g_n) weakly converges to $(f, g) \in T$ in H^1.

Then a subsequence (f_n, g_n) converges to (f, g) in L^2 and again a subsequence $(f_n, g_n) \longrightarrow (f, g)$ a.e. in I.

Given $\delta > 0$, by Egorov's theorem there exists $I_\delta \subset I$, with $|I_\delta| < \delta$ and $Q(f_n, g_n)f_n \longrightarrow Q(f, g)f$ uniformly in $I - I_\delta$.

For $\varepsilon > 0$ fixed, and $Q(f,g) = \int_0^1 t^2 H(tf, tg)dt(f, g)$,

$$D_H(f_n, g_n) = \int_I \frac{f'^2_n + g'^2_n + f_n^2}{2} + \int_{I-I_\delta} \Big(Q(f_n, g_n)f_n - Q(f,g)f\Big)(-g'_n, f'_n) +$$

$$+ \int_{I-I_\delta} Q(f,g)f(-g'_n, f'_n) + \int_{I_\delta} Q(f_n, g_n)f_n(-g'_n, f'_n).$$

But

$$\int_{I-I_\delta} |\Big(Q(f_n, g_n)f_n - Q(f,g)f\Big)(-g'_n, f'_n)|dx \leq \varepsilon \|(g'_n, f'_n)\|_2$$

and $\|(g'_n, f'_n)\|_2$ is bounded since the sequence is weakly convergent in H^1.

Also, as $\int_{I-I_\delta} Q(f,g)f(-g\prime_n, f\prime_n)$ is linear, it is weakly lower semicontinuous in H^1.

Finally,

$$|\int_{I_\delta} Q(f_n,g_n)f_n(-g\prime_n, f\prime_n)| \leq \int_{I_\delta} |Q(f_n,g_n)f_n||(-g\prime_n, f\prime_n)| \leq$$

$$\leq c\int_{I_\delta} |(-g\prime_n, f\prime_n)| \leq c|I_\delta|^{\frac{1}{2}}\|\sqrt{g\prime_n^2 + f\prime_n^2}\|_2.$$

So

$$D_H(f_n,g_n) \geq \int_I \frac{f\prime^2 + g\prime^2 + f^2}{2} + Q(f,g)f(-g\prime, f\prime) - 3\varepsilon.$$

3 Weak solutions via the Mountain Pass Lemma.

Under certain conditions it is possible to find other weak solutions of (Dir), using the Mountain Pass Lemma, [A-R], corresponding to critical points of D_H. These points are known as unstable H-surfaces, [S1]. First, we give some technical lemmas.

LEMMA 3: *Consider $D_H : H^1 \longrightarrow R$ the associated functional to (2), suppose that $|H(X_1,X_2)X_1(X_1,X_2)| \leq c$ in R^2 then D_H is continuous and $dD_H : H^1 \longrightarrow (H^1_0)^*$ is continuous.*

Proof: Let X_n be a sequence in H^1, $X_n \longrightarrow X$, $X \in H^1$. We prove that every subsequence of $\{X_n\}$ has a subsequence $\{X_n\}$ such that $D_H(X_n) \longrightarrow D_H(X)$.

As $X_n \longrightarrow X$ in H^1 there exists a subsequence $\{X_n\}$, $X_n \longrightarrow X$ a. e. in I. From Egorov's theorem there exists a subset $I_\delta \subset I$ with $|I_\delta| \leq \delta$ verifying $X_n \longrightarrow X$ and $Q(X_n) \longrightarrow Q(X)$ uniformly in $I - I_\delta$.

Setting $X_n = (f_n, g_n)$ and $X = (f,g)$ we have

$$|D_H(X_n) - D_H(X)| \leq$$
$$|D(X_n) - D(X)| + |\int_I Q(f_n,g_n)f_n(-g\prime_n, f\prime_n) - Q(f,g)f(-g\prime, f\prime)|.$$

But

$$\Big|\int_I Q(f_n,g_n)f_n(-g\prime_n, f\prime_n) - Q(f,g)f(-g\prime, f\prime)\Big| =$$
$$\Big|\int_{I-I_\delta} Q(f_n,g_n)f_n(-g\prime_n, f\prime_n) - Q(f,g)f(-g\prime_n, f\prime_n) + Q(f,g)f(-g\prime_n, f\prime_n) -$$
$$- Q(f,g)f(-g\prime, f\prime) + \int_{I_\delta} Q(f_n,g_n)f_n(-g\prime_n, f\prime_n) - Q(f,g)f(-g\prime, f\prime)\Big|.$$

Now

$$\Big|\int_{I-I_\delta} Q(f_n,g_n)f_n(-g\prime_n, f\prime_n) - Q(f,g)f(-g\prime_n, f\prime_n)\Big| \leq \varepsilon\int_{I-I_\delta} |(-g\prime_n, f\prime_n)|,$$

$$\Big|\int_{I-I_\delta} Q(f,g)f(-g\prime_n, f\prime_n) - Q(f,g)f(-g\prime, f\prime)\Big| \leq c\int_I |(-g\prime_n, f\prime_n) - (-g\prime, f\prime)|,$$

$$\int_{I_\delta} \left| Q(f_n, g_n) f_n(-g\prime_n, f\prime_n) \right| \le c \int_{I_\delta} |(-g\prime_n, f\prime_n)| \le c|I_\delta|^{\frac{1}{2}} \Big(\int_I g\prime_n^2 + f\prime_n^2 \Big).$$

To see that dD_H is continuous consider D_1 and D_2 as in Theorem 1 and $\phi = (\phi_1, \phi_2) \in H_0^1$,

$$\begin{aligned} &|dD_2(X_n)(\phi) - dD_2(X)(\phi)| = \\ &\quad = \Big| \int_I (-H(f_n, g_n) f_n g\prime_n + H(f,g) f g\prime) \phi_1 + (H(f_n, g_n) f_n f\prime_n - H(f,g) f f\prime) \phi_2 \Big| \le \\ &\quad \le \int_I | - H(f_n, g_n) f_n (g\prime_n - g\prime) \phi_1 | + |(-H(f_n, g_n) f_n + H(f,g) f) g\prime \phi_1 | + \\ &\quad + |H(f_n, g_n) f_n (f\prime_n - f\prime) \phi_2) + |(H(f_n, g_n,) f_n - H(f,g) f) f\prime \phi_2 |. \end{aligned}$$

Using Egorov's theorem again the proof is complete.

Lemma 4: *Consider H as in Lemma 3. Then D_H satisfies a Palais-Smale condition in T: any sequence $\{X_n\}$ in T such that $D_H(X_n)$ is bounded and $dD_H(X_n) \longrightarrow 0$ is relatively compact.*

Proof: Let $X_n = (f_n, g_n)$, from

$$k \ge D_H(X_n) \ge \int_I \frac{f\prime_n^2 + g\prime_n^2}{2} - k_1 \Big(\int_I f\prime_n^2 + g\prime_n^2 \Big)^{\frac{1}{2}}$$

we obtain that $\{X_n\}$ is bounded in H^1 and $X_n \longrightarrow X \in T$ weakly in H^1.

Consider $Y_n = X_n - X$ in H_0^1 $dD_H(X_n)(Y_n) \longrightarrow 0$ since $\{Y_n\}$ is bounded. But

$$\begin{aligned} &dD_H(X_n)(Y_n) = \\ &= \int_I f\prime_n (f\prime_n - f\prime) + g\prime_n (g\prime_n - g\prime) + f_n (f_n - f) - H(f_n, g_n) f_n g\prime_n (f_n - f) \\ &+ H(f_n, g_n) f_n f\prime_n (g_n - g) = \int_I (f\prime_n - f\prime)^2 + (g\prime_n - g\prime)^2 + (f_n - f)^2 + f\prime (f\prime_n - f\prime) \\ &+ g\prime (g\prime_n - g\prime) + f(f_n - f) - H(f_n, g_n) f_n g\prime_n (f_n - f) + H(f_n, g_n) f_n f\prime_n (g_n - g). \end{aligned}$$

Now, notice that

$$\Big| \int_I H(f_n, g_n) f_n g\prime_n (f_n - f) \Big| \le c \|g\prime_n\|_2 \|f_n - f\|_2.$$

In the same way

$$\int_I H(f_n, g_n) f_n f\prime_n (g_n - g) \longrightarrow 0$$

for (f_n, g_n) a subsequence of the initial sequence.

We conclude that there exists a subsequence $X_n \longrightarrow X$ in H^1.

Remark: *Notice that in this case the Palais Smale condition holds in T and it is not necessary to consider bounded subsets of H_1.*

LEMMA 5: *For H as in Lemma 3 dD_H is the Frèchet derivative of D_H.*
Proof: For $X \in H^1$ the map $T_X : H^1 \longrightarrow R$ given by $T_X(h) = dD_H(X)(h)$ is linear and bounded and verifies

$$\frac{|D_H(X+h) - D_H(X) - T_X(h)|}{\|h\|_{H^1}} = |(dD_H(X+\delta h) - dD_H(X))(h^*)|$$

where $h^* = \dfrac{h}{\|h\|_{H^1}}$ and $0 \leq \delta \leq 1$, and the last expresion goes to zero, by Lemma 3.

As in [S1], we have the following result:

THEOREM 6: *Let $H : R^2 \longrightarrow R$ be as in Lemma 3, and suppose that X_0 is a local minimun of D_H in T and that there exists X_1 such that $D_H(X_1) < D_H(X_0)$. If*

$$\beta = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} D_H(\gamma(t))$$

where $\Gamma = \{\gamma : [0,1] \longrightarrow T, \gamma$ continuous, $\gamma(0) = X_0, \quad \gamma(1) = X_1\}$, then D_H admits an unstable critical point X_2 with $D_H(X_2) = \beta$.

REMARK: *Consider D_H as in Lemma 3, and suppose that there exists X_0, a local minimum of D_H in T, and $X_1 \in T$, with $D_H(X_1) < D_H(X_0)$. Then there exists at least three weak solutions of (Dir).*

Now we give a family of functions H, verifying the conditions above. Consider any continuous function $H : R^2 \longrightarrow R$ such that

$$H(x_1, x_2) = \begin{cases} \dfrac{1}{2} & x_1^2 + x_2^2 \leq 5 \\ \dfrac{c}{x_1\sqrt{x_1^2 + x_2^2}} & x_1^2 + x_2^2 \geq 5 + \varepsilon \end{cases}$$

with $\varepsilon > 0$, $c > 0$. Then H verifies the condition in Lemma 3. If $I = [0,1]$, then $(1, x)$ is a critical point of D_H, with boundary conditions $f(0) = f(1) = 1$ and $g(0) = 0$, $g(1) = 1$. The point $(f, g) = (1 + k(x^2 - x), x)$, with $0 < k < \dfrac{10}{21}$, verifies the same conditions and

$$D_H(1 + k(x^2 - x), x) < D_H(1, x) = \frac{5}{6}.$$

Then we conclude that there exist at least two weak solutions to the Dirichlet problem.

References

[A] Adams, R. A., Sobolev Spaces, Academic Press,1975.

[A-R] Ambrosetti, A., Rabinowitz, Ph., Dual variational methods in critical point theory and applications, J. Func. Anal., 14, 1973, (349-381).

[B-C] Brezis, H., Coron, J. M., Multiple solutions of H-systems and Rellich's conjecture, Com. Pure Appl. Math., 37, 1984, (149-187).

[D] Do Carmo, M., Differential geometry of curves and surfaces, Prentice-Hall, 1976.

[H1] Hildebrandt, S., On the Plateau problem for surfaces of constant mean curvature, Com. Pure Appl.Math., 23, 1970, (97-114).

[H2] Hildebrandt, S., Randwertprobleme für Flächen mit vorgeschriebener mittlerer Krümmung und Anwendungen auf die Kapillaritätstheorie. Math.Z., 112, 1969, (205-213).

[LD-M] Lami Dozo, E., Mariani, M. C., A Dirichlet problem for an H-system with variable H, Manuscripta Math., 81, 1993, (1-14).

[O] Osserman, R., A survey of minimal surfaces, Van Nostrand Reinhold Company, 1969.

[S1] Struwe, M., Plateau's problem and the calculus of variations, Princeton Univ. Press, 1988.

[S2] Struwe, M., Non-uniqueness in the Plateau problem for surfaces of constant mean curvature, Arch. Rat. Mech. Anal. 93, 1986, (135-157).

[S3] Struwe, M., Multiple solutions to the Dirichlet problem for the equation of prescribed mean curvature, Preprint.

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