# Generalized quadrangles with a thick hyperbolic line weakly embedded in projective space

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**Abstract**

Let $\Gamma$ be a generalized quadrangle weakly embedded in projective space such that $\{a,b\}^{\perp\perp}$ contains a point different from $a$ and $b$, where $a$ and $b$ are opposite points of $\Gamma$. We prove that $\Gamma$ admits non-trivial central elations. Further, each central elation of $\Gamma$ is induced by a special linear transformation of the underlying vector space. This generalizes a result of Lefèvre-Percsy [3, Th. 1]. Furthermore, we show that $\Gamma$ is a Moufang quadrangle.

## 1 Introduction

A point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is called a *(thick) generalized quadrangle* if the incidence graph of $\Gamma$ has diameter 4 and girth 8 (i.e., the length of a shortest circuit is 8) and each element is incident with at least three elements. We always identify a line of $\Gamma$ with the set of points incident with it. Generalized quadrangles have been introduced by Tits. They have the following property: If $l$ is a line and $p$ is a point not on $l$, then $p$ is collinear with a unique point of $l$; called the *projection* of $p$ onto $l$. Examples of generalized quadrangles are polar spaces of rank 2 associated to a non-degenerate pseudo-quadratic or $(\sigma,\epsilon)$-hermitian form, see Tits [10, §8].

Let $\Gamma$ be a generalized quadrangle and $p$ a point of $\Gamma$. An automorphism of $\Gamma$ which fixes every point collinear with $p$, is called a *central elation* with center $p$. If $x$ is a point of $\Gamma$ collinear with $p$, then we write $x \in p^\perp$. If $x_1, x_2, x_3, x_4$ are points of $\Gamma$ such that $x_2 \in x_1^\perp, x_3 \in x_2^\perp$, $x_4 \in x_3^\perp$ and $x_4 \in x_1^\perp$, then $(x_1, x_2, x_3, x_4)$ is an

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*apartment* of $\Gamma$. Non-collinear points are called *opposite*. For opposite points $a$ and $b$, the set $\{a, b\}^{\perp\perp}$ is called a *hyperbolic line* on $a$ and $b$. For more information on generalized quadrangles, we refer to the monograph of Payne & Thas [4], to Thas [8], or (also for the infinite case) to Van Maldeghem [12].

Let $V$ be a vector space over some skew field $K$. By $\langle M\rangle$ we denote the subspace of $V$ generated by $M$. The 1-dimensional subspaces of $V$ are called points and the 2-dimensional subspaces lines. A linear mapping $t : V \to V$ is a *transvection*, if $H := \{v \in V \mid vt = v\}$ is a hyperplane of $V$ and $P := \{vt - v \mid v \in V\}$ is a point contained in $H$. We call $H$ the hyperplane and $P$ the point (or center) associated to $t$. By $\mathrm{SL}(V)$ we denote the subgroup of the group $\mathrm{GL}(V)$ of all invertible linear transformations from $V$ in $V$, which is generated by the transvections. The elements of $\mathrm{SL}(V)$ are also called *special linear transformations*.

Let $\Gamma$ be a generalized quadrangle. We say that $\Gamma$ is *weakly embedded* in the projective space $\mathbf{P}(V)$, if there exists an injective map $\pi$ from the set of points of $\Gamma$ to the set of points of $\mathbf{P}(V)$ such that

($a$) the set $\{\pi(x) \mid x \text{ point of } \Gamma\}$ generates $\mathbf{P}(V)$,

($b$) for each line $l$ of $\Gamma$, the subspace of $\mathbf{P}(V)$ spanned by $\{\pi(x) \mid x \in l\}$ is a line,

($c$) if $x, y$ are points of $\Gamma$ such that $\pi(y)$ is contained in the subspace of $\mathbf{P}(V)$ generated by the set $\{\pi(z) \mid z \in x^\perp\}$, then $y \in x^\perp$.

The map $\pi$ is called the *weak embedding*. Weakly embedded polar spaces have been introduced by Lefèvre-Percsy [3]. Recently, they have been studied by Steinbach, Thas and Van Maldeghem, see [5], [6], [9]. For each point $p$ of $\Gamma$, we denote by $H_p := \langle \pi(p^\perp)\rangle$ the hyperplane of $\mathbf{P}(V)$ spanned by $\pi(p^\perp)$, see Lemma 2.1. An equivalent formulation of Condition ($c$) is that for each point $p$ of $\Gamma$, the set $\pi(p^\perp)$ does not generate $\mathbf{P}(V)$.

In [6] Steinbach & Van Maldeghem classify the generalized quadrangles weakly embedded in projective space under the assumption that the *degree* of the weak embedding is $> 2$. This means that each secant line (that is a line of $\mathbf{P}(V)$ which is spanned by two non-collinear points of $\Gamma$) contains a third point of $\Gamma$. The first step is to show that $\Gamma$ is a Moufang quadrangle. Then the several classes of Moufang quadrangles are treated separately; some of them without the assumption on the degree. The proof of the Moufang condition in Steinbach & Van Maldeghem [6] relies on the fact that $\Gamma$ admits central elations (induced by transvections on $V$), according to a result due to Lefèvre-Percsy [3, Th. 1].

Let $\Gamma$ be a generalized quadrangle weakly embedded in $\mathbf{P}(V)$ with $a, b$ opposite points of $\Gamma$. Under the assumption, that the hyperbolic line $\{a, b\}^{\perp\perp}$ contains a third point, it is possible (with one exception) to construct transvections on $V$ leaving $\Gamma$ invariant (see Theorem 4.1). The example of a generalized quadrangle arising from an ordinary quadratic form with non-trivial radical of the bilinear form (in characteristic 2, see Section 3) shows, that this assumption is weaker than assuming that a secant line contains a third point. Hence we obtain a generalization of the result of Lefèvre-Percsy mentioned above. But in general we may not conclude that a central elation of $\Gamma$ with center $p$ is induced by a transvection associated to the point $\pi(p)$, see Lemma 3.3. In characteristic $\neq 2$, this conclusion remains valid, except for the *universal weak embedding* of the symplectic quadrangle $W(2)$ over GF(2) (see Section 5). For this exceptional weak embedding, where $W(2)$ is weakly embedded of degree 2 in a 5-dimensional vector space in characteristic $\neq 2$, see Van Maldeghem [12, Section 8.6]. The central elations are induced by linear transformations; not by transvections, but by homologies.

In the proof of Theorem 4.1 we need the result (see Proposition 2.1) that if $p, q, r$ are different collinear points of $\Gamma$, then $H_p \cap H_q \subseteq H_r$ or $(\Gamma, \pi)$ is the universal weak embedding of $W(2)$. Proposition 2.1 is an important tool in the classification of weakly embedded generalized quadrangles of degree 2 in Steinbach & Van Maldeghem [7], since it makes it possible to construct non-trivial axial elations of $\Gamma$.

Theorem 4.1 yields that $\Gamma$ is a Moufang quadrangle (see Theorem 6.1), similarly as in Steinbach & Van Maldeghem [6] with arguments depending on degree $> 2$ replaced by the existence of a third point in $\{a, b\}^{\perp\perp}$ and Proposition 2.1.

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## 2 A property of weak embeddings

In this section, $\Gamma$ is a generalized quadrangle weakly embedded in the projective space $\mathbf{P}(V)$ (via $\pi$), where $V$ is a vector space over the skew field $K$. We show that $\pi$ has the following important property: If $p, q, r$ are different collinear points of $\Gamma$, then $H_p \cap H_q \subseteq H_r$ or $(\Gamma, \pi)$ is the universal weak embedding of $W(2)$ (see Proposition 2.1). This resembles the fact that a vector is perpendicular to all vectors of a line, if it is perpendicular to two vectors spanning the line (read $v$ is perpendicular to $p$ instead of $v$ in $H_p$); compare the one or all-axiom in polar spaces due to Buekenhout and Shult.

**Lemma 2.1**
*For each point $a$ of $\Gamma$, the subspace $H_a = \langle \pi(a^\perp) \rangle$ is a hyperplane of $\mathbf{P}(V)$. If $b$ is a point opposite $a$, then $H_a \cap H_b = \langle \pi(a^\perp \cap b^\perp) \rangle$.*

**Proof.** If $b$ is a point of $\Gamma$ with $b \notin a^\perp$, then the subspace of $\Gamma$ generated by $a^\perp$ and $b$ is $\Gamma$ itself, see Cohen & Shult [2, (1.1)i)]. Hence $H_a$ is properly contained in $\langle H_a, \pi(b) \rangle = \mathbf{P}(V)$, and $H_a$ is a hyperplane of $\mathbf{P}(V)$.

Every line of $\Gamma$ through $a$ contains a point in $a^\perp \cap b^\perp$. Hence the subspace of $\Gamma$ generated by $a^\perp \cap b^\perp$ and $a$ is $a^\perp$ itself. This shows that $H_a = \langle \pi(a^\perp \cap b^\perp), \pi(a) \rangle$ and $H_a \cap H_b = \langle \pi(a^\perp \cap b^\perp) \rangle$. ∎

#### Remark

For weak embeddings $\pi$ of degree 2, we use the following method to calculate image points under $\pi$. Let $(x_1, x_2, x_3, x_4)$ be an apartment in $\Gamma$. Then $U := \langle \pi(x_1), \pi(x_2), \pi(x_3), \pi(x_4) \rangle$ is a 4-dimensional subspace of $V$. The set of all points $x$ of $\Gamma$ with $\pi(x) \subseteq U$ together with the lines of $\Gamma$ through these points yields a (not necessarily thick) generalized quadrangle $\Gamma'$, which is weakly embedded in $\mathbf{P}(U)$.

Let $t+1$ be the number of lines of $\Gamma'$ through a point of $\Gamma'$. Considering $x_1{}^\perp \cap x_3{}^\perp$, we obtain a line of $\mathbf{P}(U)$, which is not a line of $\Gamma'$ and meets $\Gamma'$ in exactly $t+1$ points. If the degree of the weak embedding is 2, then $t+1=2$. This means that $\Gamma'$ is a grid (and any line of $\Gamma$ is a so-called *regular line*). There are exactly two lines of $\Gamma'$ through each point of $\Gamma'$.

Let $x$ be a point on $x_1x_2\backslash\{x_1,x_2\}$ and set $y := x_3x_4 \cap x^\perp$. Let $a$ be a point on $x_1x_4\backslash\{x_1,x_4\}$ and set $b_1 := xy \cap a^\perp$ and $b_2 := x_2x_3 \cap a^\perp$. Then $a, b_1, b_2$ are collinear, since there are only two lines of $\Gamma'$ through $a$. We have $\pi(b_2) \subseteq \langle \pi(x_2), \pi(x_3)\rangle \cap \langle \pi(a), \pi(x), \pi(y)\rangle$. Since this intersection is a point, we obtain equality. We will use this argument with the $3\times 3$-grid several times in the following.

**Proposition 2.1**
*Let $\Gamma$ be a (thick) generalized quadrangle weakly embedded in $\mathbf{P}(V)$. For different collinear points $p, q, r$ of $\Gamma$, we have $H_p \cap H_q \subseteq H_r$, except for the case where $(\Gamma, \pi)$ is the universal weak embedding of the symplectic quadrangle over* GF(2)*.*

We first prove some special cases of Proposition 2.1 in separate lemmas.

**Lemma 2.2**
*Proposition* 2.1 *holds when lines of $\Gamma$ have three points.*

**Proof.** If $\Gamma$ is a (thick) generalized quadrangle with three points per line, then there are exactly $t+1$ lines through each point where $t \in \{2,4\}$. For each $t$, there is only one quadrangle, namely the orthogonal quadrangle over GF(2) in vector space dimension 5 or 6, respectively. The weak embeddings of these quadrangles have been determined in Steinbach [5] and Steinbach & Van Maldeghem [6, (5.1.1)]. They are induced by a semi-linear mapping (and $H_p \cap H_q \subseteq H_r$ holds) or we have the universal weak embedding of $W(2)$ (which is an exception for Proposition 2.1, as we may deduce from Van Maldeghem [12, Section 8.6]). ∎

**Lemma 2.3**
*Proposition* 2.1 *holds when $V$ has vector space dimension* 5 *and $\pi$ is of degree* 2*.*

**Proof.** Because of Lemma 2.2, we may assume that lines of $\Gamma$ have more than three points. We prove $H_p \cap H_q \subseteq H_r$. Let $(p, q, t, z)$ be an apartment in $\Gamma$ and set $s := zt \cap r^\perp$. Then $U := \langle \pi(p), \pi(q), \pi(t), \pi(z)\rangle$ is a 4-dimensional subspace of $\Gamma$. There exists $a \in q^\perp \cap z^\perp$ with $\pi(a) \not\subseteq U$. (Otherwise $H_q \cap H_z = \langle \pi(q^\perp \cap z^\perp)\rangle \subseteq U$ and $H_q \cap H_z = U \cap H_q \cap H_z = \langle \pi(p), \pi(t)\rangle$. But then $V$ is 4-dimensional.) Then $V = U \oplus \pi(a)$. We set

$$\begin{aligned} x &:= rs \cap a^\perp, \quad b_1 := xa \cap p^\perp, \quad b_2 := za \cap r^\perp, \\ y_1 &:= pz \cap x^\perp, \quad y_2 := qa \cap y_1{}^\perp. \end{aligned}$$

We choose $p', q', z', t', a' \in V$ such that

$$\begin{array}{rclrclrcl} \pi(p) &=& \langle p'\rangle, & \pi(q) &=& \langle q'\rangle, & \pi(r) &=& \langle p'+q'\rangle, \\ & & & \pi(z) &=& \langle z'\rangle, & \pi(y_1) &=& \langle p'-z'\rangle, \\ & & & \pi(t) &=& \langle t'\rangle, & \pi(s) &=& \langle t'-z'\rangle, \\ & & & \pi(a) &=& \langle a'\rangle, & \pi(b_2) &=& \langle z'+a'\rangle. \end{array}$$

(For any point $b$ of $\Gamma$, we denote by $b'$ a vector in $V$ such that $\pi(b) = \langle b'\rangle$.)

Since the set of all points $d$ of $\Gamma$ with $\pi(d) \subseteq U$ is a grid by the remark on page 449, we see that $y_1, x$ and $c := qt \cap {y_1}^\perp$ are collinear. Hence

$$\pi(x) \subseteq \langle \pi(r), \pi(s) \rangle \cap \langle \pi(y_1), \pi(q), \pi(t) \rangle = \langle p' + q' + t' - z' \rangle$$

and $\pi(c) = \langle q' + t' \rangle$. Similarly, using the apartment $(p, q, a, z)$, we obtain that

$$\pi(y_2) \subseteq \langle \pi(q), \pi(a) \rangle \cap \langle \pi(y_1), \pi(r), \pi(b_2) \rangle = \langle a' - q' \rangle.$$

We are left with calculating $\pi(b_1)$. Set $n_1 := zt \cap {b_1}^\perp$. Then there exists $\gamma \in K$ such that $\pi(n_1) = \langle t' - \gamma z' \rangle$. We have $\pi(b_1) \subseteq \langle \pi(x), \pi(a) \rangle \cap \langle \pi(n_1), \pi(r), \pi(b_2) \rangle = \langle p' + q' + t' - z' + (\gamma - 1)a' \rangle$. Set $n_2 := qt \cap {b_1}^\perp$. Then $\pi(n_2) \subseteq \langle \pi(c), \pi(q) \rangle \cap \langle \pi(y_1), \pi(y_2), \pi(b_1) \rangle = \langle \gamma q' + t' \rangle$.

We first assume that $\gamma \neq 0$. Then $H_{b_1} = \langle \pi(n_1), \pi(n_2), \pi(p), \pi(a) \rangle$. Because of $\pi(x) \subseteq H_{b_1}$, we may compare coefficients. This yields $\gamma = 2$ and $\pi(b_1) = \langle p' + q' + t' - z' + a' \rangle$. Since

$$\begin{aligned} H_p &= \langle \pi(p), \pi(q), \pi(z), \pi(b_1) \rangle = \langle p', q', z', t' + a' \rangle, \\ H_q &= \langle \pi(p), \pi(q), \pi(t), \pi(a) \rangle = \langle p', q', t', a' \rangle, \end{aligned}$$

we have $H_p \cap H_q = \langle p', q', t' + a' \rangle \subseteq \langle p', q', t' - z', z' + a' \rangle = \langle \pi(p), \pi(q), \pi(s), \pi(b_2) \rangle = H_r$.

We are thus left with the case $\gamma = 0$. Then

$$\pi(b_1) = \langle {b_1}' \rangle, \text{ where } {b_1}' = p' + q' + t' - z' - a',$$

and $t \in {b_1}^\perp$. Because of $H_{b_1} \cap H_q = \langle p', t', a' \rangle \subseteq H_z$, we see that ${b_1}^\perp \cap q^\perp \subseteq z^\perp$.

Let $r_1 \in pq$ with $\pi(r_1) = \langle \lambda p' + q' \rangle$, $0 \neq \lambda \in K$. For $s_1 := zt \cap {r_1}^\perp$, we obtain

$$\pi(s_1) \subseteq \langle z', t' \rangle \cap \langle {r_1}', {y_1}', c' \rangle = \langle t' - \lambda z' \rangle.$$

Using the apartment $(c, x, a, q)$, we calculate that $\pi(m) = \langle p' + q' - z' - a' \rangle$, where $m := tb_1 \cap y_1 y_2$. Further for $f := pb_1 \cap {s_1}^\perp$, we see

$$\pi(f) \subseteq \langle p', {b_1}' \rangle \cap \langle {s_1}', {y_1}', m' \rangle = \langle -\lambda p' + {b_1}' \rangle.$$

Set $g_0 := r_1 s_1 \cap {b_1}^\perp$ and $g := b_1 g_0 \cap q^\perp$. Then $g \in {b_1}^\perp \cap q^\perp \subseteq z^\perp$. Hence

$$\pi(g) \subseteq \langle {r_1}', {s_1}', {b_1}' \rangle \cap H_q \cap H_z = \langle \lambda(\lambda - 1)p' - (\lambda - 1)t' + \lambda a' \rangle.$$

Hence, for $g_0 = b_1 g \cap r_1 s_1$, we obtain $\pi(g_0) = \langle \lambda {r_1}' + {s_1}' \rangle$. For $i := qt \cap f^\perp$, we see $\pi(i) \subseteq \langle q', t' \rangle \cap \langle f', g', z' \rangle = \langle \lambda q' + t' \rangle$. Let $w := rs \cap i^\perp$. Then $\pi(w) \subseteq \langle r', s' \rangle \cap \langle p', z', i' \rangle = \langle \lambda r' + s' \rangle$. Similarly, for $w_1 := r_1 s_1 \cap i^\perp$, we calculate $\pi(w_1) = \langle \lambda {r_1}' + {s_1}' \rangle = \pi(g_0)$. Hence $g_0 = w_1$. We set $k := pb_1 \cap w^\perp$. Then

$$\pi(k) \subseteq \langle p', {b_1}' \rangle \cap \langle w', z', g' \rangle = \langle (1 - \lambda)p' + {b_1}' \rangle.$$

The calculation of $\pi(k)$ uses that $\lambda \neq 0$. On the other hand

$$\pi(k) \subseteq \langle p', {b_1}' \rangle \cap \langle w', q', a' \rangle = \langle (\lambda - 1)p' + {b_1}' \rangle.$$

This yields $1 - \lambda = \lambda - 1$. Since we assume that the lines of $\Gamma$ have more than three points, there exists $r_1 \in pq$ such that $\pi(r_1) = \langle \lambda p' + q' \rangle$, where $0, 1 \neq \lambda \in K$. Hence $\operatorname{char} K = 2$ and $\pi(b_1) = \langle p' + q' + t' - z' + a' \rangle$. The result now follows as above. ■

**Lemma 2.4**
*Proposition* 2.1 *holds when* $V$ *has vector space dimension* 5 *and* $\pi$ *is of degree* $> 2$.

**Proof.** The complete list of examples in Steinbach & Van Maldeghem [6] yields that $\Gamma$ is an orthogonal, a hermitian or a mixed quadrangle and $\pi$ is induced by a semi-linear mapping. Hence $H_p \cap H_q \subseteq H_r$ holds. ∎

**Proof of Proposition 2.1:** By Lemma 2.2 we may assume that the lines of $\Gamma$ have more than three points. We first consider the case where $V$ is finite-dimensional. We show that $H_p \cap H_q \subseteq H_r$ holds by induction on $\dim V$. The intersection $H_p \cap H_q$ has codimension 2 in $V$. Hence if $V$ is 4-dimensional, we obtain $H_p \cap H_q = \langle \pi(p), \pi(q) \rangle \subseteq H_r$. The case where $V$ is 5-dimensional is Lemma 2.3 and Lemma 2.4.

Let $V$ be at least 6-dimensional. Then there exists $0 \neq w \in H_p \cap H_q \cap H_r$, $w \notin \langle \pi(p), \pi(q) \rangle$. For any point $b$ of $\Gamma$, we denote by $b'$ a vector in $V$ such that $\pi(b) = \langle b' \rangle$. Let $(r, q, t, s)$ be an apartment in $\Gamma$. We extend $w, r', q', t', s'$ to a basis of $V$, in a way that each new basis vector is of the form $z'$ for some point $z$ of $\Gamma$ (note that $w \notin \langle r', q', t', s' \rangle$, since otherwise $w \in \langle \pi(p), \pi(q) \rangle$). We denote the resulting basis by $\{w\} \cup \mathcal{B}$.

Let $v \in H_p \cap H_q$. Then $v \in H_r$, when $v - \lambda w \in H_r$ where $\lambda \in K$. Since $w \in H_p \cap H_q$, we may hence assume that $v$ is contained in the hyperplane $H := \langle \mathcal{B} \rangle$ of $V$. Let $H_0$ be the set of all points $x$ of $\Gamma$ with $\pi(x) \subseteq H$. Then $H_0$ is a subspace of $\Gamma$ and a generalized quadrangle (containing an ordinary quadrangle), weakly embedded in $\mathbf{P}(H)$. Since $p, q, r$ are points of $H_0$, we may apply induction to $H_0$. This yields $W := \langle \pi(p^\perp \cap H_0) \rangle \cap \langle \pi(q^\perp \cap H_0) \rangle \subseteq \langle \pi(r^\perp \cap H_0) \rangle \subseteq H_r$. Since $\langle \pi(p^\perp \cap H_0) \rangle$ is a hyperplane of $\mathbf{P}(H)$ by Lemma 2.1, we see that $\langle \pi(p^\perp \cap H_0) \rangle = H_p \cap H$. Hence $v \in H_p \cap H_q \cap H = W \subseteq H_r$. This proves the claim in the finite-dimensional case.

Since in general $v$ is a finite linear combination of the above basis vectors, we may extend the result to the infinite-dimensional case. (Note that $v$ is contained in a finite-dimensional subspace $U$ of $V$, spanned by points of $\Gamma$ such that $U$ contains $r', q', t', s'$.) ∎

**Lemma 2.5**
*Let* $\mathcal{S}$ *be a (thick) non-degenerate polar space of rank at least* 3 *weakly embedded in* $\mathbf{P}(V)$. *For different collinear points* $p, q, r$ *of* $\Gamma$, *we have* $H_p \cap H_q \subseteq H_r$.

**Proof.** Let $\pi : \mathcal{S} \to \mathbf{P}(V)$ be a weak embedding of the non-degenerate polar space $\mathcal{S}$ of rank at least 3. If $\mathcal{S}$ is classical, then the result follows as in Lemma 2.4. Using the classification of non-degenerate polar spaces of rank at least 3, see Tits [10, §8, §9], Cohen [1, 3.34], we may hence assume that $\mathcal{S}$ has rank 3. As in Lemma 2.2 we may assume that the lines of $\mathcal{S}$ have more than three points. Let $p, q$ be different collinear points of $\mathcal{S}$ and choose $a \in p^\perp \cap q^\perp$ with $a$ not on $pq$. For $b \in p^\perp \cap q^\perp$ with $b \notin a^\perp$, the set of points in $a^\perp \cap b^\perp$ together with the lines of $\mathcal{S}$ through these points yields a generalized quadrangle $\Gamma$, weakly embedded in $\mathbf{P}(V')$, where $V' = \langle \pi(x) \mid x \in \Gamma \rangle$. For each point $z$ of $\Gamma$, we set $H'_z = \langle \pi(x) \mid x$ point of $\Gamma$, $x$ collinear with $z$ in $\Gamma \rangle$. Then $H'_p \cap H'_q \subseteq H'_r \subseteq H_r$ by Proposition 2.1. Further, $H_p = \langle H'_p, \pi(a), \pi(b) \rangle$ and similarly for $H_q$. Hence $H_p \cap H_q = \langle H'_p \cap H'_q, \pi(a), \pi(b) \rangle \subseteq H_r$. ∎

## 3 Central elations in generalized quadrangles arising from forms

Let $L$ be a skew field with involutory anti-automorphism $\sigma$. For $\epsilon \in \{1, -1\}$, we set

$$\Lambda_{min} := \{c - \epsilon c^\sigma \mid c \in L\}, \quad \Lambda_{max} := \{c \in L \mid \epsilon c^\sigma = -c\}.$$

Let $W$ be a (left) vector space over $L$ and $q : W \to L/\Lambda_{min}$ be a non-degenerate pseudo-quadratic form with associated trace-valued $(\sigma, \epsilon)$-hermitian form $f : W \times W \to L$ in the sense of Tits [10, (8.2.1)]. If $q$ is not an ordinary quadratic form, we may (and will) assume $\epsilon = -1$ and $1 \in \Lambda_{min}$ by Tits [10, (8.2.2)]. (In the remaining case $(\sigma, \epsilon) = (\mathrm{id}, 1)$, hence $L$ commutative and $\Lambda_{min} = 0$.) For $U \subseteq W$, we set $U^\perp := \{w \in W \mid f(w, u) = 0 \text{ for all } u \in U\}$. The radical of $f$ is $\mathrm{Rad}(W, f) := W^\perp$. Since $q$ is non-degenerate, we have $q(r) \neq 0$ for all $0 \neq r \in \mathrm{Rad}(W, f)$. An isometry of $W$ is a linear mapping $\varphi : W \to W$ with $q(w\varphi) = q(w)$ for $w \in W$.

If $q$ has Witt index 2, than the set of all singular points and lines of $\mathbf{P}(W)$ (points and lines, where the pseudo-quadratic form $q$ vanishes) yields a generalized quadrangle, which is thick, except for the case that $q$ is an ordinary quadratic form and $\dim W = 4$.

In Section 3, let $\Gamma$ be a thick generalized quadrangle arising from some vector space $W$ (over $L$) endowed with a non-degenerate pseudo-quadratic form $q$ (with associated $(\sigma, \epsilon)$-hermitian form $f$). We write points of $\Gamma$ as $\langle p \rangle$ with a singular vector $p$ and we refer with the $\perp$-symbol to the form $f$. In particular, $p^\perp$ is a hyperplane of $W$. Our aim is to describe all central elations (see Section 1) of $\Gamma$.

**Lemma 3.1**
*Any central elation of $\Gamma$ with center $\langle p \rangle$ is induced by an isometry $t$ of $W$ which satisfies $t|_{p^\perp} = \mathrm{id}$.*

**Proof.** If $\tau$ is a central elation of $\Gamma$ with center $\langle p \rangle$, then $\tau : \Gamma \to \mathbf{P}(W)$ is a weak embedding. From Steinbach [5] and Steinbach & Van Maldeghem [6, (5.1.1)], we may deduce that $\tau$ is induced by a semi-linear mapping $\varphi : W \to W$ (with respect to an automorphism $\alpha : L \to L$), see also Tits [10, (8.6)]. Since $\langle w \rangle \varphi = \langle w \rangle$ for all $w \in p^\perp$, $w$ singular, there exists $c \in L$ such that $x\varphi = cx$ for all $x \in p^\perp$ and $d^\alpha = cdc^{-1}$ for $d \in L$. Then $t : W \to W$, defined by $w \mapsto c^{-1}(w\varphi)$ for $w \in W$, is the desired isometry of $W$. ∎

**Lemma 3.2**
*Let $0 \neq p \in W$ be singular and let $t$ be an isometry of $W$ with $t|_{p^\perp} = \mathrm{id}$. Then there exist $a \in L$ and $r_a \in \mathrm{Rad}(W, f)$ with $q(r_a) = a + \Lambda_{min}$ such that*

$$wt = w + f(w, p)(ap + r_a) \quad \textit{for } w \in W.$$

**Proof.** For $w \in W$, we have $wt - w \in p^{\perp\perp} = \langle p \rangle \oplus \mathrm{Rad}(W, f)$. Choose $x \in W$ with $f(x, p) = 1$ and $a \in L$, $r_a \in \mathrm{Rad}(W, f)$ with $xt = x + ap + r_a$. Since each vector of $W$ is of the form $s + \lambda x$, where $s \in p^\perp$ and $\lambda \in L$, we obtain $wt = w + f(w, p)(ap + r_a)$ for $w \in W$. Further, $q(x) = q(xt) = q(x) + q(r_a) + (a^\sigma + \Lambda_{min})$. Hence $q(r_a) = -a^\sigma + \Lambda_{min}$. If $q$ is a quadratic form with $\mathrm{Rad}(W, f) \neq 0$, then $\mathrm{char}\, L = 2$. Thus in any case $q(r_a) = a + \Lambda_{min}$. ∎

Combining Lemma 3.1 and Lemma 3.2, we see:

**Lemma 3.3**
*Any central elation of* $\Gamma$ *with center* $\langle p\rangle$ *is induced by a transvection with point-hyperplane pair* $(R, p^\perp)$, *where* $R$ *is a (not necessarily singular) point in* $p^{\perp\perp} = \langle p\rangle \oplus \mathrm{Rad}(W, f)$. *In particular,* $R \subseteq x^\perp$ *for* $x \in p^\perp$. ■

If $r_a = 0$ in Lemma 3.2, then $a \in \Lambda_{min}$ and $t$ is a transvection with center $\langle p\rangle$. Hence $\Gamma$ admits central elations unless $q$ is an ordinary quadratic form with $\mathrm{Rad}(W, f) = 0$. (Then for opposite points $a$ and $b$ of $\Gamma$, the hyperbolic line $\{a, b\}^{\perp\perp}$ has only two points.) We will generalize Lemma 3.3 to arbitrary weakly embedded generalized quadrangles in Section 4. Only in characteristic 2 it may happen that $R$ in Lemma 3.3 is different from $\langle p\rangle$ (since in characteristic $\neq 2$, we have $\mathrm{Rad}(W, f) = 0$). For a generalization to arbitrary weakly embedded generalized quadrangles, see Section 5.

### Remark

We may describe the group of all central elations of $\Gamma$ with center $\langle p\rangle$ as follows: We set $\Delta := \{a \in L \mid \text{ there exists } r_a \in \mathrm{Rad}(W, f) \text{ with } a + \Lambda_{min} = q(r_a)\}$. Then $c\Delta c^\sigma = \Delta$ for $0 \neq c \in L$. (If $\mathrm{Rad}(W, f) = 0$, in particular if $\mathrm{char}\, K \neq 2$, then $\Delta = \Lambda_{min}$.) For $a \in \Delta$, $r_a$ is unique and we define $t_a : w \mapsto w + f(w, p)(ap + r_a)$ for $w \in W$, where $0 \neq p \in W$ is singular. Then $t_a$ is an isometry of $W$ and $t_a t_b = t_{a+b}$ for $a, b \in \Delta$. We set $T_p := \{t_a \mid a \in \Delta\}$. Then $T_p \simeq (\Delta, +)$ is the group of central elations with center $\langle p\rangle$. If $q$ is a quadratic form with $\mathrm{Rad}(W, f) = 0$, then $T_p = 1$.

We close this section with a remark that for generalized quadrangles associated to $(\sigma, \epsilon)$-hermitian forms, we obtain similar results as for pseudo-quadratic forms.

### Remark

Let $\Gamma$ be a generalized quadrangle arising from a non-degenerate $(\sigma, \epsilon)$-hermitian form $f : W \times W \to L$ such that $\Lambda_{min} = \Lambda_{max}$ (e.g., a symplectic quadrangle in characteristic $\neq 2$). Without loss $\epsilon = \pm 1$. If $t$ is an isometry of $W$ with $t|_{p^\perp} = \mathrm{id}$, where $0 \neq p \in W$ with $f(p, p) = 0$, then, similarly as in Lemma 3.2, there exists $a \in \Lambda_{max}$ such that $wt = w + f(w, p)ap$ for $w \in W$, i.e., $t$ is a transvection.

## 4 The construction of central elations induced by transvections

Let $V$ be a vector space over the skew field $K$ and let $\Gamma$ be a generalized quadrangle weakly embedded in $\mathbf{P}(V)$ (with weak embedding $\pi$). For each point $p$ of $\Gamma$, we denote by $H_p$ the hyperplane of $\mathbf{P}(V)$ generated by $\pi(p^\perp)$.

Let $a, b$ be opposite points of $\Gamma$. If the hyperbolic line $\{a, b\}^{\perp\perp}$ contains a third point, then we prove that $\Gamma$ admits non-trivial central elations. Furthermore, we show that every central elation of a weakly embedded generalized quadrangle $\Gamma$ is induced by a transvection on $V$, except for the universal weak embedding of $W(2)$.

This generalizes a result of Lefèvre-Percsy [3, Th. 1]. For the case of polar spaces of rank at least 3, see at the end of Section 4.

**Theorem 4.1**
*Let $\Gamma$ be a (thick) generalized quadrangle weakly embedded in the projective space $\mathbf{P}(V)$ with $(\Gamma,\pi)$ not the universal weak embedding of $W(2)$. Let $a,b$ be opposite points of $\Gamma$ and $b' \neq a,b$ be a point of $\Gamma$ such that $a^\perp \cap b^\perp \subseteq b'^\perp$. Set $R := \langle \pi(b),\pi(b')\rangle \cap H_a$. Let $t$ be the transvection on $V$ with associated point-hyperplane pair $(R,H_a)$, which maps $\pi(b)$ to $\pi(b')$. Then for each point $x$ of $\Gamma$, there exists some point $x'$ of $\Gamma$ such that $\pi(x)t = \pi(x')$ (i.e., $\Gamma$ is invariant under $t$). Further, $a^\perp \cap x^\perp \subseteq x'^\perp$.*

**Proof.** First, we remark that $b' \notin a^\perp$. Since if $ab'$ is a line of $\Gamma$ and $x$ is the projection of $b$ onto $ab'$, we choose $x \neq y \in a^\perp \cap b^\perp$. By assumption $y \in b'^\perp$, hence $y \in ab' \cap b^\perp = x$, a contradiction. Similarly, $b' \notin b^\perp$.

Let $c$ be a point of $\Gamma$. We may assume $c \notin a^\perp$ and $c \neq b$.

(1) We assume $c \in b^\perp$. Let $e \neq b,c$ be the projection of $a$ onto $bc$. Then $e \in a^\perp \cap b^\perp$, hence $e \in b'^\perp$. Because of $c \in eb$, we have $\pi(c) \subseteq \langle \pi(e),\pi(b)\rangle$ and $\pi(c)t \subseteq \langle \pi(e),\pi(b')\rangle$. Further, $\pi(c) \subseteq \langle \pi(c),R\rangle$, hence $\pi(c)t \subseteq \langle \pi(c),R\rangle$. This shows that $\pi(c)t = \langle \pi(c),R\rangle \cap \langle \pi(e),\pi(b')\rangle$, since the two lines are different. (Otherwise $R \subseteq \langle \pi(e),\pi(b')\rangle \cap \langle \pi(a^\perp)\rangle = \pi(e)$. We choose $e \neq z \in b^\perp \cap b'^\perp$, then $\pi(e) = R \subseteq \langle \pi(b),\pi(b')\rangle \subseteq \langle \pi(z^\perp)\rangle$, a contradiction.)

We choose $e \neq x \in a^\perp \cap c^\perp$, and denote by $y$ the projection of $x$ onto $b'e$. Let $q$ be the projection of $b$ onto $ax$. Then $q \in a^\perp \cap b^\perp \subseteq b'^\perp$. Since $R \subseteq \langle \pi(b),\pi(b')\rangle \subseteq \langle \pi(q^\perp)\rangle = H_q$ and $R \subseteq \langle \pi(a^\perp)\rangle = H_a$, we obtain $R \subseteq H_a \cap H_q \subseteq H_x = \langle \pi(x^\perp)\rangle$ by Proposition 2.1.

We set $E := \langle \pi(e),\pi(b),\pi(b')\rangle$. Then $\pi(y) \subseteq E \cap \langle \pi(x^\perp)\rangle = \langle \pi(c),R\rangle$. We obtain $\pi(y) \subseteq \langle \pi(e),\pi(b')\rangle \cap \langle \pi(c),R\rangle = \pi(c)t$. We set $y =: c'$.

For $e \neq k \in a^\perp \cap c^\perp$, we denote by $l$ the projection of $k$ onto $b'e$. Then $\pi(l) = \pi(c)t = \pi(c')$ by the above argument. Hence $l = c'$ and $k \in c'^\perp$. This yields that $a^\perp \cap c^\perp \subseteq c'^\perp$.

(2) We assume that $c \notin b^\perp$ and that there is $f \in b^\perp \cap c^\perp$, $f \notin a^\perp$. By (1) there exists a point $f'$ with $\pi(f)t = \pi(f')$ and $a^\perp \cap f^\perp \subseteq f'^\perp$. We apply (1) again for the pair $(f,c)$, which yields the claim.

(3) We are left with the case $c \notin b^\perp$ and $b^\perp \cap c^\perp \subseteq a^\perp$. We choose different points $p,q \in b^\perp \cap c^\perp$. Since lines are thick, there is a point $g$ on $pc \backslash \{p,c\}$. We denote by $f$ the projection of $g$ onto $bq$. Then $f \in b^\perp$, $g \in f^\perp$, $c \in g^\perp$ and $f,g \notin a^\perp$. We apply (1) three times for the pairs $(b,f)$, $(f,g)$ and $(g,c)$. ■

If $\{a,b\}^{\perp\perp} \neq \{a,b\}$, where $a,b$ are opposite points of $\Gamma$, then we may choose $a,b \neq b' \in \{a,b\}^{\perp\perp}$ in Theorem 4.1. The inclusion $a^\perp \cap b^\perp \subseteq a^\perp \cap b'^\perp$ yields $a^\perp \cap b^\perp = a^\perp \cap b'^\perp$.

**Lemma 4.1**
*Let $\Gamma$ be as in Theorem 4.1. If $a,b$ are opposite points of $\Gamma$ such that the hyperbolic line $\{a,b\}^{\perp\perp}$ contains at least three points, then $\{p,q\}^{\perp\perp}$ contains at least three points for all opposite points $p,q$ of $\Gamma$.*

**Proof.** By Theorem 4.1 we know that the hyperbolic line $\{a,x\}^{\perp\perp}$ contains at least three points for all $x \notin a^\perp$. We use this argument repeatedly. If $b \notin q^\perp$, then we use the sequence $(a, b), (q, b), (q, p)$. We may hence assume that $p, q \in a^\perp \cap b^\perp$. We choose a third point $x$ on $bq$ and use the sequence $(a, b), (a, x), (p, x), (p, q)$. ∎

**Lemma 4.2**
*In the notation of Theorem 4.1, we have $y' \in x'^\perp$, for $y \in x^\perp$. The mapping $\theta$ defined by $x\theta = x'$, if $\pi(x)t = \pi(x')$ is a central elation of $\Gamma$ with center $a$.*

**Proof.** We may assume $x, y \notin a^\perp$. The first claim follows from the construction in Theorem 4.1(1) with $(x, y)$ instead of $(b, c)$. This yields that $\theta$ preserves collinearity. We see, that $\theta$ is bijective, using $t^{-1}$. ∎

**Lemma 4.3**
*Let $\Gamma$ be a generalized quadrangle weakly embedded in the projective space $\mathbf{P}(V)$ with $(\Gamma, \pi)$ not the universal weak embedding of $W(2)$. Let $a$, $b$ be opposite points of $\Gamma$ and let $b'$ be a third point with $a^\perp \cap b^\perp \subseteq b'^\perp$. Then there exists a central elation of $\Gamma$ with center $a$ mapping $b$ to $b'$. Further, each central elation $\tau$ of $\Gamma$ with center $p$ is induced by a transvection of $V$ with point-hyperplane pair $(R, H_p)$, where $R = \langle \pi(q), \pi(q\tau) \rangle \cap H_p$ for $q$ opposite $p$.*

**Proof.** By Theorem 4.1 and Lemma 4.2, the first claim is obvious. Next, let $\tau$ be a central elation of $\Gamma$ with center $p$ and let $q$ be some point opposite $p$. Then $p^\perp \cap q^\perp \subseteq p^\perp \cap (q\tau)^\perp$. (Since if $x \in p^\perp \cap q^\perp$, then $(qx)\tau = (q\tau)x$; i.e., $x \in (q\tau)^\perp$.) We have to show that there exists a transvection $t$ on $V$ with $\pi(x)t = \pi(x\tau)$ for all points $x$ of $\Gamma$. We set $R := \langle \pi(q), \pi(q\tau) \rangle \cap H_p$. Let $t \in \mathrm{SL}(V)$ be the transvection with point-hyperplane pair $(R, H_p)$ which maps $\pi(q)$ to $\pi(q\tau)$. If $x$ is a point of $\Gamma$, then Theorem 4.1 yields that $\pi(x)t = \pi(x')$ for some point $x'$ of $\Gamma$. By Lemma 4.2, the mapping $\theta$ defined by $x\theta = x'$ if $\pi(x)t = \pi(x')$ is a central elation of $\Gamma$ with center $p$ with $q\theta = q\tau$. Hence $\theta = \tau$ by Van Maldeghem [12, (4.4.2)(v)]. This yields $\pi(x)t = \pi(x\tau)$ for all points $x$ of $\Gamma$ and $t$ is unique with this property. We have thus extended $\tau$ to $\mathbf{P}(V)$. ∎

### Remark

In view of Lemma 2.5, Theorem 4.1 is also valid for weakly embedded polar spaces of rank at least 3; compare Cohen [1, p. 663].

## 5 The center of the inducing transvection in characteristic $\neq 2$

In this section, we show that any central elation of a weakly embedded generalized quadrangle in characteristic $\neq 2$ is induced by a transvection on $V$ with center $\pi(p)$, except for the universal weak embedding of $W(2)$. In characteristic 2, this is not valid, see Lemma 3.3.

**Lemma 5.1**
*Let $\Gamma$ be a generalized quadrangle weakly embedded in the projective space $\mathbf{P}(V)$ with $(\Gamma, \pi)$ not the universal weak embedding of $W(2)$. Let $\tau$ be a central elation of $\Gamma$ with center $p$, mapping the point $q$ opposite $p$ to $q'$. If $\operatorname{char} K \neq 2$, then*

$\pi(p) \subseteq \langle \pi(q), \pi(q') \rangle$. *In particular, the degree of* $\pi$ *is* $> 2$ *and* $\tau$ *is induced by a transvection of* $\mathrm{SL}(V)$ *with point-hyperplane pair* $(\pi(p), H_p)$.

**Proof.** By Lemma 4.3 $\tau$ is induced by a transvection $t$ of $\mathrm{SL}(V)$ with point-hyperplane pair $(R, H_p)$, where $R = \langle \pi(q), \pi(q') \rangle \cap H_p$. Our aim is to show that $R$ equals $\pi(p)$, provided that $\operatorname{char} K \neq 2$.

We write $\pi(q) = \langle v_q \rangle$ and $R = \langle r \rangle$ such that $v_q t = v_q + r \in \pi(q')$. Set $\pi(q'') := \pi(q')t$. Because of $(v_q + r)t = v_q + 2r$ and $\operatorname{char} K \neq 2$, we see that $q''$ is a third point of $\Gamma$ with $\pi(q'') \subseteq \langle \pi(q), \pi(q') \rangle$. Hence $q^\perp \cap q'^\perp \subseteq q''^\perp$. By Theorem 4.1, the transvection $\varphi$ of $\mathrm{SL}(V)$ with point-hyperplane pair $(\pi(q), H_q)$, mapping $\pi(q')$ to $\pi(q'')$, leaves $\Gamma$ invariant.

There exists $A \in K$ such that $(v_q + r)\varphi = v_q + r + Av_q \in \pi(q'')$. Comparing coefficients yields $A = -\frac{1}{2}$. Hence $r\varphi = r - \frac{1}{2}v_q$. Thus the matrices of $t$ and $\varphi$ with respect to the basis $\{r, v_q\}$ are

$$t \sim \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \varphi \sim \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix}.$$

The set $\{\pi(x) \mid x \in \Gamma, \pi(x) \subseteq \langle \pi(q), \pi(q') \rangle\}$ is invariant under $t$ and $\varphi$ and hence also under the group generated by $t$ and $\varphi$. Since this group contains the elements with matrix representation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we see that $\pi(q)$ may be mapped to $R$ under $\langle t, \varphi \rangle$. Hence there exists $x \in \Gamma$ with $R = \pi(x)$.

Next, we show that $x = p$. Let $y \in x^\perp \cap q^\perp$. Then $\pi(q') \subseteq \langle \pi(x), \pi(q) \rangle \subseteq H_y$ and $y \in q'^\perp$. Hence $x^\perp \cap q^\perp = x^\perp \cap q'^\perp = q^\perp \cap q'^\perp = q^\perp \cap p^\perp$, since $\tau$ is a central elation with center $p$. This yields $H_x = \pi(x) \oplus (H_x \cap H_q) \subseteq H_p$ and $H_x = H_p$. Thus $x^\perp = p^\perp$ and $x = p$. ■

## 6 Proof of the Moufang condition

In this section, $\Gamma$ is a generalized quadrangle weakly embedded in the projective space $\mathbf{P}(V)$ (via $\pi$), where $V$ is a vector space over the skew field $K$.

For different collinear points $p$ and $y$ of $\Gamma$, an automorphism of $\Gamma$ which fixes all points on $py$, all lines through $p$ and all lines through $y$, is called a $(p, py, y)$-*elation*. If for some line $pz$, the group of all $(p, py, y)$-elations acts transitively on the points of $pz$ different from $p$, we say that $(p, py, y)$ is a *Moufang path*. Dually we define when $(pz, p, py)$ is a Moufang path. If all paths $(p, py, y)$ and all paths $(pz, p, py)$ are Moufang paths, then $\Gamma$ is called a *Moufang quadrangle*. These definitions are due to Tits [11].

**Theorem 6.1**
*Let* $\Gamma$ *be a generalized quadrangle weakly embedded in the projective space* $\mathbf{P}(V)$. *If* $\Gamma$ *has a hyperbolic line with at least three points, then* $\Gamma$ *is a Moufang quadrangle.*

**Proof.** We may assume that the degree of the weak embedding is 2. Otherwise $\Gamma$ is a Moufang quadrangle by Steinbach & Van Maldeghem [6]. Furthermore, we may suppose that $\Gamma$ is not the symplectic quadrangle over $\mathrm{GF}(2)$.

Let $y, z$ be opposite points of $\Gamma$ and $p \in y^\perp \cap z^\perp$. Our aim is to show:

($*$) If $z'$ is a third point on $pz$, then there exists a $(p, py, y)$-elation which maps $z$ to $z'$.

By Lemma 4.1, we see that the hyperbolic line $\{y, z\}^{\perp\perp}$ contains at least three points. Hence there exists a point $a \neq y, z$ with $y^\perp \cap z^\perp \subseteq a^\perp$. By Lemma 4.3 there is a central elation $t_y$ of $\Gamma$ with center $y$ which maps $z$ to $a$. Because of $a^\perp \cap z^\perp \subseteq y^\perp$, Lemma 4.3 yields a central elation $t_a$ of $\Gamma$ which maps $z$ to $y$. Then $y' := z't_a \in py$ and $y'^\perp \cap a^\perp \subseteq z'^\perp$ by the definition of central elations. Let $t_{y'}$ be the central elation of $\Gamma$ with center $y'$ mapping $a$ to $z'$ by Lemma 4.3. We set $t := t_y t_{y'}$. Then $zt = z'$, all points on $py$ are fixed under $t$ and the line $pz$ is fixed under $t$.

We denote by the same names the extensions of $t_y, t_{y'}, t$ on the elements of $\mathbf{P}(V)$, see Lemma 4.3. Denote by $R_y := \langle \pi(a), \pi(z) \rangle \cap H_y$ the point of the extension of $t_y$, and similarly for $t_{y'}$. Since $R_y, R_{y'}$ are contained in $H_p$, we see that $H_p$ is invariant under $t_y$ and $t_{y'}$. The restriction of $t_y$ and $t_{y'}$ to $H_p$ is the identity on $H_p \cap H_y$ and $H_p \cap H_{y'}$, respectively. By Proposition 2.1 these two intersections coincide. Hence the restriction of $t$ to $H_p$ is a transvection. Its center is $\pi(p)$, since the line $\langle \pi(p), \pi(z) \rangle$ is fixed by $t$. This yields that every line of $\Gamma$ through $p$ is fixed by $t$.

Choose a point $q$ of $\Gamma$ such that $(p, y, q, z)$ is an apartment of $\Gamma$. By $q'$ we denote the projection of $z$ onto $(yq)t$. We show that $q^\perp \cap q'^\perp \subseteq p^\perp$. We have $V = \langle \pi(p), \pi(y), \pi(q), \pi(z) \rangle \oplus (H_p \cap H_y \cap H_q \cap H_z)$. Let $x \in q^\perp \cap q'^\perp$. We write $\pi(x) = \langle v_x \rangle$ with $v_x = Av_p + Bv_q + Cv_y + Dv_z + h$, where $A, B, C, D \in K$, $h \in H_p \cap H_y \cap H_q \cap H_z$. We have $h \in H_p \cap H_y \subseteq H_{y'}$ by Proposition 2.1. Hence $h = ht \subseteq (H_q)t = H_{qt}$. This yields $h \in H_y \cap H_{qt} \subseteq H_{q'}$, using Proposition 2.1. Because of $Av_p = v_x - Bv_q - Cv_y - Dv_z - h \in H_q$, we see $A = 0$. This yields $Bv_q = v_x - Cv_y - Dv_z - h \in H_{q'}$, hence $B = 0$. Thus $\pi(x) = \langle Cv_y + Dv_z + h \rangle \subseteq H_p$ and $x \in p^\perp$.

Hence there exists a central elation $t_p$ with center $p$ mapping $q'$ to $q$. The composition $\theta := tt_p$ maps $z$ to $z'$, fixes all points on $py$ and all lines through $p$. Moreover, the line $yq$ is fixed by $\theta$. Similarly as above, the restriction of $\theta$ to $H_y$ is a transvection with center $\pi(y)$. Hence $\theta$ fixes all lines through $y$, which proves ($*$).

Finally, $\Gamma$ is a Moufang quadrangle. The proof is the same as Steinbach & Van Maldeghem [6, (4.0.2)]. ∎

**Lemma 6.1**
*In the situation of Theorem* 6.1*, the subgroup of* $\mathrm{Aut}(\Gamma)$ *generated by all central elations is induced by* $\mathrm{PSL}(V)$.

**Proof.** For the universal weak embedding of $W(2)$, see Van Maldeghem [12, Section 8.6]. Let $G$ be the group generated by the central elations of $\Gamma$. For each central elation $\tau$ of $\Gamma$, we denote by $t$ the unique transvection of $V$ inducing $\tau$, see Lemma 4.3. We write each element of $G$ as a product $\tau_1 \dots \tau_r$ of central elations and define a mapping $\chi : G \to \mathrm{PSL}(V)$ by $(\tau_1 \dots \tau_r)\chi = t_1 \dots t_r$. Then $\chi$ is well-defined, since two automorphisms of $\mathbf{P}(V)$ are equal, if they coincide on all points $\pi(x)$, $x \in \Gamma$, which span $\mathbf{P}(V)$. Hence the mapping $\tau \mapsto t$ yields a homomorphism $\chi : G \to \mathrm{PSL}(V)$ with $\pi(x)^{\chi(g)} = \pi(xg)$ for all $g \in G$, $x \in \Gamma$. ∎

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