# Octonion hermitian quadrangles

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**Abstract**

We introduce hermitian generalized quadrangles over the octonions. These quadrangles extend the classical hermitian quadrangles over the reals, the complex numbers and the quaternions in a natural way. For the smallest quadrangle, $H_3\mathbb{O}$, we show that the group Spin(9) acts as a line-transitive automorphism group.

## Introduction

Octonions or Cayley division algebras are complex and beautiful objects which are, unfortunately, absent in finite geometry. They can be used to construct a family of particularly nice generalized quadrangles. The smallest of these quadrangles, $H_3\mathbb{O}$, has a line-transitive automorphism group. These quadrangles generalize and extend in a natural way the classical standard hermitian quadrangles over the reals, the complex numbers, or the quaternions. They where first described by Ferus-Karcher-Münzner [1] in connection with Clifford algebras and isoparametric hypersurfaces; later, Thorbergsson [7] proved by a topological argument that they are quadrangles.

We here take a different approach to these quadrangles: instead of real Clifford algebras we use the octonions, and we give an algebraic proof that the geometries are quadrangles. The approach via Clifford algebras can be found in [2]. It should be said that although the quadrangles originate from differential and topological geometry, the whole construction is purely algebraic and works whenever the field $\mathbb{R}$ of real numbers is replaced by a real closed field.

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The first section gives the definition of these geometries, and a proof that they are quadrangles. In the second section we examine the smallest example, $H_3\mathbb{O}$. It turns out that this quadrangle can be reconstructed from a group of automorphisms by Stroppel's method [5, 6], and that the subgroup lattice of this group contains all the information needed to recover the quadrangle.

## 1 Definition of the quadrangles

We denote the real numbers by $\mathbb{R}$, the complex numbers by $\mathbb{C}$, the (real) quaternion skew field by $\mathbb{H}$, and the (real) octonion division algebra by $\mathbb{O}$. All facts we need about these alternative fields can be found in the first chapter of the book by Salzmann *et al.* [4]. There are inclusions $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O}$. Let $\mathbb{F}$ be any of these four alternative fields. An element $x \in \mathbb{F}$ whose square is a nonpositive real number is called *pure*; accordingly, there is a direct sum decomposition

$$\mathbb{F} = \mathbb{R} \oplus \mathrm{Pu}(\mathbb{F})$$

of $\mathbb{F}$ into real and pure elements. The *standard involution* $x \longmapsto \bar{x}$ is the identity on real elements, and $-\mathrm{id}$ on the pure elements. Thus, it is an anti-automorphism of $\mathbb{F}$ of order two (if $\mathbb{F} = \mathbb{R}$, then $\bar{x} = x$). Put $\mathrm{Re}(a) = \frac{1}{2}(a + \bar{a})$. Then $\mathbb{F}$ becomes a real euclidean vector space with respect to the inner product $(a, b) \longmapsto \langle a, b\rangle = \mathrm{Re}(a\bar{b})$; the euclidean norm is $|a| = \sqrt{a\bar{a}}$. Put $V = \mathbb{F}^n$ and $d = \dim_{\mathbb{R}} \mathbb{F}$. Then $V$ is a real euclidean $dn$-dimensional vector space. Consider the $\mathbb{R}$-bilinear map

$$V \times V \longrightarrow \mathbb{F}, \qquad (x, y) \longmapsto (x|y) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n.$$

It has the following properties.

$$\mathrm{Re}(x|y) = \langle x, y\rangle \tag{1}$$
$$\langle ax, y\rangle = \langle x, \bar{a}y\rangle \tag{2}$$
$$\overline{(x|y)} = (y|x) \tag{3}$$
$$(ax, \bar{a}y) = a(x|y)a \tag{4}$$
$$(x|ax) = |x|^2\bar{a} \tag{5}$$
$$(ax|x) = a|x|^2 \tag{6}$$
$$|(x|y)| \leq |x| \cdot |y|. \tag{7}$$

These relations are well-known for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, so the interesting case is $\mathbb{F} = \mathbb{O}$. Equations (5), (6) hold because $\mathbb{O}$ is an alternative field; (4) is a Moufang identity, and (7) is proved below.

**Lemma 1 (Cauchy-Schwarz principle)**
*Suppose that $x \neq 0$. The equality*

$$|(x|y)| = |x| \cdot |y|$$

*holds if and only if $y = ax$ for some $a \in \mathbb{F}$. Moreover, this implies $(x|y) = |x|^2\bar{a}$.*

**Proof.** We have the following chain of inequalities.

$$\begin{aligned}|(x|y)| &= |x_1\bar{y}_1 + \cdots + x_n\bar{y}_n| \\ &\le |x_1\bar{y}_1| + \cdots + |x_n\bar{y}_n| = |x_1|\cdot|y_1| + \cdots + |x_n|\cdot|y_n| \\ &\le \sqrt{|x_1|^2 + \cdots + |x_n|^2}\sqrt{|y_1|^2 + \cdots + |y_n|^2} \\ &= |x|\cdot|y|\end{aligned}$$

This establishes (7). Suppose that equality holds. We may assume that $x \neq 0 \neq y$. If the first line and the second line are equal, then there exists an element $c \in \mathbb{F}$ of norm $|c| = 1$, such that $x_\nu\bar{y}_\nu = |x_\nu\bar{y}_\nu|c$ for $\nu = 1, \dots, n$. The next equality shows that there exists a real number $t$ such that $|y_\nu| = |x_\nu|t$, for $\nu = 1, \dots, n$. This yields $y_\nu = \bar{c}tx_\nu$ and thus $y = (\bar{c}t)x$. ■

**Definition 1**
Let $V = \mathbb{F}^n$. We call the elements of the set

$$\mathcal{P} = \Big\{(x, y) \in V \oplus V|\ |x|^2 + |y|^2 = 1,\ |(x|y)| = |x|\cdot|y|\Big\}$$

*points* and the elements of the set

$$\mathcal{L} = \Big\{(u, v) \in V \oplus V|\ |u|^2 = |v|^2 = 1/2,\ (u|v) = 0\Big\}$$

*lines*. Note that $\mathcal{L} \neq \emptyset$, provided that $n \ge 2$. Let $S = \{(c, s) \in \mathbb{R} \oplus \mathbb{F}|\ c^2 + |s|^2 = 1\} \cong \mathbb{S}^d$. By Cauchy-Schwarz, the point space can be rewritten as

$$\mathcal{P} = \{(cw, sw) \in V \oplus V|\ w \in V,\ |w| = 1,\ (c, s) \in S\}.$$

Note that

$$\begin{aligned}\langle(cw, sw), (u, v)\rangle &= \langle cw, u\rangle + \langle sw, v\rangle \\ &= \langle w, cu\rangle + \langle w, \bar{s}v\rangle \\ &= \langle w, cu + \bar{s}v\rangle \le 1/\sqrt{2}.\end{aligned}$$

By Cauchy-Schwarz, equality holds if and only if $w = \sqrt{2}(cu + \bar{s}w)$. We use this to define the *incidence:*

$$(cw, sw)\ \mathrm{I}\ (u, v) \qquad \text{if and only if} \qquad \langle(cw, sw), (u, v)\rangle = 1/\sqrt{2}.$$

We denote the resulting incidence structure by

$$H_{n+1}\mathbb{F} = (\mathcal{P}, \mathcal{L}, \mathrm{I}),$$

for $n \ge 2$.

It is clear that there are commuting inclusions

$$\begin{array}{ccccccc}
H_k\mathbb{R} & \longrightarrow & H_k\mathbb{C} & \longrightarrow & H_k\mathbb{H} & \longrightarrow & H_k\mathbb{O} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{k+1}\mathbb{R} & \longrightarrow & H_{k+1}\mathbb{C} & \longrightarrow & H_{k+1}\mathbb{H} & \longrightarrow & H_{k+1}\mathbb{O} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_\infty\mathbb{R} & \longrightarrow & H_\infty\mathbb{C} & \longrightarrow & H_\infty\mathbb{H} & \longrightarrow & H_\infty\mathbb{O}
\end{array}$$

for $k \geq 3$. The bottom line is the direct limit of these geometries. The condition for the incidence is equivalent to

$$(cw, sw) = 1/\sqrt{2}\left((1 + (c^2 - |s|^2))u + 2csv, 2c\bar{s}u + (1 - (c^2 - |s|^2))v\right).$$

Consider the map $\phi : \mathcal{P} \longrightarrow S$

$$(cw, sw) \longmapsto (|cw|^2 - |sw|^2, 2(cw|sw)) = (c^2 - |s|^2, 2c\bar{s}).$$

For $(c, s) \in S$ put

$$B_{c,s} = \begin{pmatrix} c & L_s \\ L_{\bar{s}} & -c \end{pmatrix},$$

where $L_s = (x \longmapsto sx)$. It is a straight-forward calculation that $B_{c,s}$ is an orthogonal involution on $V \oplus V$. Moreover, $B_{c,s}$ permutes $\mathcal{P}$ and $\mathcal{L}$; since $B_{c,s}$ preserves the inner product, it preserves the incidence. Note also the following:

$$B_{\phi(p)}p = p$$

for all $p \in \mathcal{P}$, and conversely, if $z \in V \oplus V$ is a unit vector which is invariant under some $B_{c,s}$, then $z \in \mathcal{P}$. Let $K$ denote the group of automorphisms generated by the $B_{c,s}$. Each $B_{c,s}$ acts on $\mathbb{R} \oplus \mathbb{F}$ as the orthogonal map

$$B_{c,s} : (r, t) \longmapsto -\left((r, t) - 2\langle (r, t), (c, s)\rangle (c, s)\right).$$

Equivalently, one can imbed $\mathbb{R} \oplus \mathbb{F}$ into $\mathrm{End}_{\mathbb{R}}(V)$ by the map $(r, t) \longmapsto B_{r,t}$; then $B_{c,s}$ acts by conjugation on this vector space of endomorphisms. If we endow $\mathrm{End}_{\mathbb{R}}(V)$ with the positive definite inner product

$$\langle X, Y\rangle = \mathrm{trace}(X \cdot Y^{\mathrm{trsp}})/dn,$$

then this is an isometric linear imbedding

$$\mathbb{R} \oplus \mathbb{F} \hookrightarrow \mathrm{End}_{\mathbb{R}}(V).$$

The map $\phi : \mathcal{P} \longrightarrow S$ is $K$-equivariant, i.e. $g(\phi(p)) = \phi(g(p))$ for all $g \in K$. Note that up to the sign, $B_{c,s}$ acts as a reflection on $\mathbb{R} \oplus \mathbb{F}$; therefore $K$ acts transitively on the unit sphere $S \subseteq \mathbb{R} \oplus \mathbb{F}$.

**Lemma 2**
*The point rows of $H_{n+1}\mathbb{F}$ are $d$-spheres.*

**Proof.** A point $p$ incident with the line $\ell = (u, v)$ is of the form

$$p = 1/\sqrt{2}\,(1 + B_{c,s})\,\ell$$

where $(c, s) \in S \cong \mathbb{S}^d$. ∎

**Lemma 3**
*The line pencils of $H_{n+1}\mathbb{F}$ are $d(n-1)-1$-spheres.*

**Proof.** Let $p \in \mathcal{P}$. We may assume that $\phi(p) = (1, 0)$, i.e. that $p = (w, 0)$ for some unit vector $w \in V$. Let $\ell = (u, v) \in \mathcal{L}$. Then $p$ I $\ell$ if and only if $p = 1/\sqrt{2}\,(1 + B_{1,0})\,\ell$. This implies $u = 1/\sqrt{2}w$, $(u|v) = 0$, and $|v|^2 = 1/2$. The kernel of the $\mathbb{R}$-linear map $x \longmapsto (u|x)$ is $d(n-1)$-dimensional. ∎

Therefore $H_{n+1}\mathbb{F}$ is a thick geometry, unless $n = 2$ and $\mathbb{F} = \mathbb{R}$.

**Lemma 4**
*Two lines $h, \ell \in \mathcal{L}$ are confluent if and only if $(h - \ell)/|h - \ell| \in \mathcal{P}$.*

**Proof.** The lines are confluent if and only if $(1 + B_{c,s})\ell = (1 + B_{c,s})h$ for some $(c, s) \in S$. This is equivalent with $h - \ell = (-B_{c,s})(h - \ell)$; thus $h - \ell$ has to be an invariant vector for some $B_{c,s}$. But the invariant unit vectors are precisely the points of the geometry $H_{n+1}\mathbb{F}$. ∎

**Lemma 5**
*Let $p, q \in \mathcal{P}$ be points, and put $P = B_{\phi(p)}$ and $Q = B_{\phi(q)}$. Then $p, q$ are collinear if and only if $\sqrt{2}(p - q) = (P - Q)\ell$ for some $\ell \in \mathcal{L}$. Collinearity implies that $(p - q)/|p - q| \in \mathcal{L}$.*

**Proof.** The first claim is clear. Multiplying the equation with $P - Q$ we obtain $\sqrt{2}(P - Q)(p - q) = (P - Q)^2\ell = |P - Q|^2\ell$. Since $(P - Q)/|P - Q| \in K$, the second claim follows. ∎

**Theorem 1**
*The geometry $H_{n+1}\mathbb{F}$ is a generalized quadrangle, unless $\mathbb{F} = \mathbb{R}$ and $n = 2$.*

**Proof.** Let $(p, \ell) \in \mathcal{P} \times \mathcal{L}$ be a non-incident point-line pair. We have to show that there exists a unique point $q$ which is incident with $\ell$ and collinear with $p$. Applying a suitable automorphism in $K$, we may assume that $\phi(p) = (1, 0)$, i.e. that $p = (w, 0)$. Put $\ell = (u, v)$. The non-incidence implies that $u \neq 1/\sqrt{2}w$. A typical point incident with $\ell$ is of the form $q = 1/\sqrt{2}((1 + c)u + sv, \bar{s}u + (1 - c)v)$. Collinearity of $p$ and $q$ implies by lemma 5 that

$$((1 + c)u + sv - \sqrt{2}w|\bar{s}u + (1 - c)v) = 0.$$

Note that the solution $(c, s) = (1, 0)$ is not allowed, since then $(q - p)/|q - p| \notin \mathcal{L}$. We expand the above equation as

$$s - \sqrt{2}(w|\bar{s}u - cv) = \sqrt{2}(w|v);$$

so we have to show that there is a unique solution. Consider the $\mathbb{R}$-linear map

$$f : \mathbb{R} \oplus \mathbb{F} \longrightarrow \mathbb{F}, \quad (c, s) \longmapsto s - \sqrt{2}(w|\bar{s}u - cv).$$

The kernel $N$ of $f$ has dimension at least 1. If $c = 0 \neq s$, then $f(0, s) = s - \sqrt{2}(w|\bar{s}u) \neq 0$ by Cauchy-Schwarz. Therefore, the hyperplane $0 \oplus \mathbb{F} \subseteq \mathbb{R} \oplus \mathbb{F}$ intersects $N$ trivially, and thus $\dim_{\mathbb{R}} N = 1$. It follows that $N \cap S$ consists of precisely two elements, $N \cap S = \{(1, 0), (c, s)\}$. This establishes the uniqueness of the point $q$.

To prove that $q$ is collinear with $p$, we have to check first that $(q-p)/|q-p| \in \mathcal{L}$. We can apply another automorphism such that $\phi(p) = (1, 0)$ and $\phi(q) = (c, s)$, where $s$ is a *real* number. Then the equation takes the simpler form

$$\begin{aligned} \sqrt{2}\langle w, v\rangle &= s - \sqrt{2}\langle w, \bar{s}u - cv\rangle \\ &= s - \sqrt{2}\langle w, u\rangle s + \sqrt{2}\langle w, v\rangle c. \end{aligned}$$

Using the relation $s^2 = (1 - c)(1 + c)$, it is easy to see that

$$|(1 + c)u + sv - \sqrt{2}w|^2 = |su + (1 - c)v)|^2;$$

this shows that $(q - p)/|p - q| \in \mathcal{L}$. According to lemma 5, the last condition to check is that $\sqrt{2}|p - q| = |P - Q|$. This follows from $P = B_{\phi(p)} = B_{1,0}$ and $Q = B_{\phi(q)} = B_{c,s}$. ∎

## Remarks

(1) The restriction of $\phi$ to a point row is a bijection. Therefore the fibers of $\phi$ are *ovoids*: each line meets the fiber $\phi^{-1}(c, s)$ in a unique point.

(2) The proof of theorem 1 shows that lemma 5 can be improved as follows: two points $p, q \in \mathcal{P}$ are collinear if and only if $(p - q)/|p - q| \in \mathcal{L}$.

**Proposition 1**
*If $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, then $H_n\mathbb{F}$ is the classical quadrangle belonging to the standard hermitian form of Witt index 2 on $\mathbb{F}^{n+1}$.*

**Proof.** To any point $p = (cw, sw)$ we assign the subspace $(c, s, \bar{w})\mathbb{F}$, and to any line $\ell = (u, v)$ the subspace $(1, 0, \sqrt{2}\bar{u})\mathbb{F} \oplus (0, 1, \sqrt{2}\bar{v})\mathbb{F}$. These subspaces are totally isotropic with respect to the hermitian form

$$-x_{-2}\bar{y}_{-2} - x_{-1}\bar{y}_{-1} + x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_{n-1}\bar{y}_{n-1}$$

on $\mathbb{F}^{n+1}$. It is easy to see that this correspondence is bijective and incidence preserving. ∎

**Remarks**

(1) The limits $H_\infty\mathbb{F}$ are 'stable' versions of the compact quadrangles. Note that $H_\infty\mathbb{O}$ contains all the quadrangles occuring here as subquadrangles; in particular, it contains all compact connected Moufang quadrangles, except for the three which are not standard hermitian (i.e. the complex symplectic quadrangle $W(\mathbb{C})$, the 'bigger' $\alpha$-hermitian quadrangle $H_4^\alpha\mathbb{H}$, and the real $\mathrm{E}_6$-quadrangle).

(2) The line spaces of these quadrangles are (rather obviously) the real, complex, quaternionic, or octonionic Stiefel manifolds of orthonormal 2-frames. A closer inspection of the map $\phi$ shows that the point space is the sphere bundle of the Whitney sum of $n$ copies of the Hopf bundle over the projective line $\mathbb{F}\cup\{(\infty)\}$. These bundles are closely related to Bott periodicity and $K$-theory of spheres. Thus, the underlying topological spaces of these quadrangles are quite interesting in themselves, cp. [2].

## 2 Groups of automorphisms and reconstruction

We have already seen that the group $K$ generated by the orthogonal involutions $B_{c,s}$ acts transitively on $S$. The subgroup $G$ generated by the maps

$$B_{c,s}B_{1,0} = \begin{pmatrix} c & -L_s \\ L_{\bar{s}} & c \end{pmatrix}$$

has at most index 2 in $K$. Since the involution $B_{c,s} \longmapsto B_{-c,-s}$ induces an automorphism of $K$ which fixes $G$ elementwise, $K/G \cong \mathbb{Z}/2$. Note that the group $G$ is connected. The kernel of the action on $\mathbb{R}\oplus\mathbb{F}$ has order 2, hence $G \cong \mathrm{Spin}(d+1)$, cp. Porteous [3] Ch. 15,16. The representation of $G$ on $V \oplus V = (\mathbb{F}\oplus\mathbb{F})^n$ decomposes into a direct sum of $n$ (irreducible) representations.

For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, this yields the classical quadrangles, hence we consider from now on only the case $\mathbb{F} = \mathbb{O}$, so $G \cong \mathrm{Spin}(9)$. To understand the subgroups of Spin(9), we consider the action of Spin(9) on the affine Cayley plane $\mathrm{AG}_2\mathbb{O}$. The first chapter of the book [4] by Salzmann *et al.* provides a beautiful and comprehensive introduction to the Cayley plane. Let $\ell = 1/\sqrt{2}((1,0,0,\cdots),(0,1,0,\cdots))$. The stabilizer of $\ell$ is the compact exceptional Lie group $\mathrm{G}_2$, the automorphism group of $\mathbb{O}$, cp. [4] 17.15. The orbit $G/G_\ell$ of $\ell$ has dimension $36 - 14 = 22 = 7 + 7 + 8$. The dimension of the line space is the dimension of a point row plus two times the dimension of a line pencil; in our case, we have $\dim\mathcal{L} = 8 + 2(8(n-1) - 1)$. Therefore, $G$ acts transitively on the line space $\mathcal{L}$ of $H_{n+1}\mathbb{O}$ if and only if $n = 2$.

**Proposition 2**
*$H_3\mathbb{O}$ has a line-transitive automorphism group.* ■

Suppose from now on that $n = 2$. Since $G = \mathrm{Spin}(9)$, the stabilizer of the line

$$\ell = (u, v) = 1/\sqrt{2}((1, 0, ), (0, 1))$$

is $G_\ell = \mathrm{G}_2$, cp. [4] 17.15. We compute the stabilizers of points incident with $\ell$.

A point incident with $\ell$ is of the form

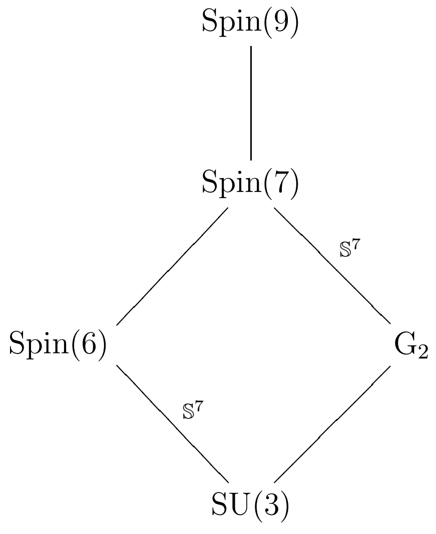
$$p = 1/\sqrt{2}((1+c)u + sv, \bar{s}u + (1-c)v) = ((1+c, s), (\bar{s}, 1-c))$$

where $(c, s) \in S$. If $s \in \mathbb{R}$, then $G_\ell$ fixes $p$, since $\mathrm{G}_2$ fixes the real subplane $\mathbb{R} \oplus \mathbb{R} \subseteq \mathbb{O} \oplus \mathbb{O}$ elementwise, $G_\ell = G_{p,\ell}$. Note that the stabilizer $G_{(a,b)}$ of a point $(a, b) \in \mathbb{O} \oplus \mathbb{O}$, fixes also the points $(ra, rb)$, for $r \in \mathbb{R}$, because the action is linear. If $(a, b) \neq (0, 0)$, then $G_{(a,b)}$ is a group isomorphic to Spin(7), cp. [4] 17.15. Thus, the stabilizer of $(1 + c, s)$ is isomorphic to Spin(7). If $s \in \mathbb{R}$, then

$$G_{(1+c,s)} = G_{(s,1-c)} = G_p,$$

since $s^2 = (1-c)(1+c)$. Therefore, the stabilizer of such a point is conjugate to the stabilizer $G_{(1,0)} \cong \mathrm{Spin}(7)$ of the point $(1, 0) \in \mathbb{O} \oplus \mathbb{O}$. The $G$-orbit of such a point is a 15-sphere.

Suppose that $s \in \mathbb{O} \setminus \mathbb{R}$. We can apply an element of $G_\ell = \mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$ such that $s \in \mathbb{C}$. Then $G_{p,\ell}$ fixes the points $(1 + c, s), (\bar{s}, 1 - c) \in \mathbb{C} \oplus \mathbb{C}$. Therefore, $G_{p,\ell} = G_{(1+c,s),(\bar{s},1-c)} = \mathrm{SU}(3)$, cp. [4] 11.34. The space $\mathbb{O} \oplus \mathbb{O}$ is in a natural way an 8-dimensional complex vector space, cp. [4] 11.34. The stabilizer of the point $p$ is the elementwise stabilizer of the complex 1-dimensional subspace spanned by $(1 + c, s)$; this is a group isomorphic to $\mathrm{SU}(4) \cong \mathrm{Spin}(6)$. We obtain the following diagram of subgroups of Spin(9).




Now $\mathrm{Spin}(7)/\mathrm{G}_2 \cong \mathbb{S}^7 \cong \mathrm{Spin}(6)/\mathrm{SU}(3)$; thus, the stabilizer $G_p$ acts in any case transitively on the line pencil through $p$. By Stroppel [5, 6], this is the crucial condition for reconstructing the geometry from the group $G$. Put

$$\mathcal{G} = \{G_p |\ p \,\mathrm{I}\, \ell\}.$$

**Proposition 3**
*The triple* $(G, \mathcal{G}, G_\ell) = (\mathrm{Spin}(9), \mathcal{G}, \mathrm{G}_2)$ *represents the geometry* $H_3\mathbb{O}$*: put*

$$\mathcal{P}' = \{gHG_\ell |\ g \in G, H \in \mathcal{G}\} \quad \text{and} \quad \mathcal{L}' = G/G_\ell,$$

*and call* $gHG_\ell$ *and* $g'G_\ell$ *incident if* $g'G_\ell \subseteq gHG_\ell$*. The resulting geometry is isomorphic to* $(\mathcal{P}, \mathcal{L}, \mathrm{I})$*, cp. Stroppel [5, 6].*

This is not completely satisfying, since the definition of the set $\mathcal{G}$ still involves the original geometry. Instead, we want a purely group-theoretic description of this collection of subgroups. In order to understand the subgroups of Spin(9), we consider again its action on the affine Cayley plane $\mathrm{AG}_2\mathbb{O}$. Identify the point space of $\mathrm{AG}_2\mathbb{O}$ with $\mathbb{O} \oplus \mathbb{O}$. There are two types of subgroups in $\mathcal{G}$: subgroups conjugate to $G_{(1,0)} \cong \mathrm{Spin}(7)$, and subgroups conjugate to $G_{(1,0),(\mathbf{i},0)} \cong \mathrm{Spin}(6)$.

Let $A = gG_{(1,0)}g^{-1}$. The fixed point set of $G_{(1,0)}$, acting on $\mathbb{O} \oplus \mathbb{O}$, is the real point row $\mathbb{R} \oplus 0$. The fixed point set of $\mathrm{G}_2$, acting on $\mathbb{O} \oplus \mathbb{O}$, is $\mathbb{R} \oplus \mathbb{R}$. If $A$ contains the group $\mathrm{G}_2$, then $g(\mathbb{R} \oplus 0) \subseteq \mathbb{R} \oplus \mathbb{R}$, and therefore $g(1,0)$ is a point with *real* coordinates. Such a group $A$ fixes a point $p$ with $\phi(p) \in \mathbb{R} \oplus \mathbb{R}$, and thus $A \in \mathcal{G}$.

Let $B = gG_{(1,0),(\mathbf{i},0)}g^{-1}$. The fixed point set of $G_{(1,0),(\mathbf{i},0)}$, acting on $\mathbb{O} \oplus \mathbb{O}$, is $\mathbb{C} \oplus 0$, and the fixed point set of SU(3), acting on $\mathbb{O} \oplus \mathbb{O}$, is $\mathbb{C} \oplus \mathbb{C}$. Thus, if $\mathrm{SU}(3) \subseteq B$, then $g(\mathbb{C} \oplus 0) \subseteq \mathbb{C} \oplus \mathbb{C}$, and therefore $A$ fixes a point $p \in \mathcal{P}$ with $\phi(p) \in \mathbb{R} \oplus \mathbb{C}$. Hence $B \in \mathcal{G}$. We have proved the following.

**Theorem 2**
*The quadrangle $H_3\mathbb{O}$ is represented by the triple*

$$(G, \mathcal{G}, G_\ell) = (\mathrm{Spin}(9)), \mathcal{G}, \mathrm{G}_2),$$

*where the collection $\mathcal{G}$ of subgroups of* Spin(9) *is given as follows. Consider the standard action of* Spin(9) *on $\mathbb{O} \oplus \mathbb{O}$. Then $\mathcal{G}$ consists of all conjugates of $G_{(1,0)} \cong \mathrm{Spin}(7)$ which contain $\mathrm{G}_2$, and of all conjugates of $G_{(1,0),(\mathbf{i},0)} \cong \mathrm{Spin}(6)$ whose intersection with $\mathrm{G}_2$ is isomorphic to* SU(3).

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