Equivariant Homotopy Epimorphisms, Homotopy Monomorphisms and Homotopy Equivalences

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1 Introduction

Recall that a morphism $f : A \longrightarrow B$ in a category $\mathcal{C}$ is an epimorphism (in short epi) if for any two morphisms $g, h : B \longrightarrow C$, $g \circ f = h \circ f$ implies $g = h$. The dual notion of a monomorphism (in short mono) is defined in the evident way.

In [4] E. Dyer and J. Roitberg proved the following result

Theorem 1.1 *If $f : X \longrightarrow Y$ is both an epimorphism and a monomorphism in $\mathcal{HCW}^*$, the homotopy category of pointed path connected CW-spaces, then f is an equivalence in $\mathcal{HCW}^*$.* ∎

The above result is interesting because a morphism in a category which is simultaneously epi and mono need not be an equivalence. Let $\mathcal{HCW}^\circ$ be the homotopy category of pointed CW-spaces. If $f : X \longrightarrow Y$ is a morphism in $\mathcal{HCW}^*$ such that when considered as a morphism in $\mathcal{HCW}^\circ$, f is a monomorphism, then f is also a monomorphism in $\mathcal{HCW}^*$. As a consequence we get a variant of Theorem 1.1

Theorem 1.2 *If $f : X \longrightarrow Y$ is a morphism in $\mathcal{HCW}^*$ such that f is an epimorphism in $\mathcal{HCW}^*$ and a monomorphism in $\mathcal{HCW}^\circ$, then f is an equivalence in $\mathcal{HCW}^*$.* ∎

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In the present paper we shall find conditions on a morphism $f : X \longrightarrow Y$ in $G\mathcal{HCW}^*$, the G-homotopy category of pointed G-path connected G-complexes, where G is a finite group, to be an equivalence.

Let $G\mathcal{HCW}^\circ$ be the G-homotopy category of pointed G-complexes. Then $G\mathcal{HCW}^*$ is a full subcategory of $G\mathcal{HCW}^\circ$. We shall prove the following result.

Theorem 1.3 *If $f : X \longrightarrow Y$ is an epimorphism in $G\mathcal{HCW}^*$ and a monomorphism in $G\mathcal{HCW}^\circ$ and X, Y are $\mathcal{A}$-admissible then f is an equivalence in $G\mathcal{HCW}^*$.*

See Definition 1.4 below for $\mathcal{A}$-admissible space.

Let O_G be the category of canonical orbits [1]. If $f : X \longrightarrow Y$ represents a morphism in $G\mathcal{HCW}^*$, then f induces a natural transformation $f_* : \underline{\pi}_* X \longrightarrow \underline{\pi}_* Y$, where $\underline{\pi}_* X$ is the O_G-group defined by setting $\underline{\pi}_*(X)(G/H) = \pi_*(X^H)$ for every object G/H in O_G, and $\underline{\pi}_*(X)(\hat{g}) = \pi_*(g)$ for every morphism $\hat{g} : G/H \longrightarrow G/K$ in O_G, f_* is given by $f_*(G/H) = \pi_*(f^H) : \pi_*(X^H) \longrightarrow \pi_*(Y^H)$.

Definition 1.4 *Let $\mathcal{A}$ denote the subcategory of $\mathcal{G}$ (the category of groups) with ob $\mathcal{A}$ = ob $\mathcal{G}$ and $Mor_{\mathcal{A}}(A, B) = Iso_{\mathcal{G}}(A, B)$, the set of isomorphisms from A to B. An O_G-group $T : O_G \longrightarrow \mathcal{G}$ is said to be $\mathcal{A}$-admissible if it is a functor from $O_G \longrightarrow \mathcal{A}$, that is, $T(\hat{g})$ is invertible for every morphism $\hat{g}$ in O_G. A G-complex X is said to be $\mathcal{A}$-admissible if $\underline{\pi}_1 X$ is $\mathcal{A}$-admissible.*

Note that when $G = \{e\}$, the $\mathcal{A}$-admissibility condition is trivially satisfied for any space X, hence Theorem 1.2 follows from Theorem 1.3 by taking $G = \{e\}$.

If X is an object in $G\mathcal{HCW}^*$ such that X is G-simply connected then X is $\mathcal{A}$-admissible. Also if X is an object in $G\mathcal{HCW}^*$ such that the equivariant 1-cells of X are of the type G/G, then X is $\mathcal{A}$-admissible.

Let $\mathcal{C}(\mathcal{G})$ denote the category whose objects are O_G-groups and morphisms are natural transformations between O_G-groups. It is easy to see that a morphism $f : S \longrightarrow T$ in $\mathcal{C}(\mathcal{G})$ is a monomorphism in $\mathcal{C}(\mathcal{G})$ if and only if $f(G/H) : S(G/H) \longrightarrow T(G/H)$ is a monomorphism in $\mathcal{G}$ for every object G/H in O_G. For suppose $f(G/H)$ is $1-1$ for each G/H in O_G. Let $Z : O_G \longrightarrow \mathcal{G}$ be an O_G-group and $\alpha, \beta : Z \longrightarrow S$ be natural transformations such that $f \circ \alpha = f \circ \beta$. Let H be a subgroup of G and $x \in Z(G/H)$. Then $f(G/H)\alpha(G/H)(x) = f(G/H)\beta(G/H)(x)$ implies $\alpha(G/H)(x) = \beta(G/H)(x)$. Thus f is a monomorphism in $\mathcal{C}(\mathcal{G})$.

Conversely, suppose that f is a monomorphism. Now $Ker\ f$ is an O_G-group defined by $(Ker\ f)(G/H) = Ker\ (f(G/H))$. We have natural transformations $i, c : Ker\ f \longrightarrow S$, where $i(G/H)$ is the inclusion and $c(G/H)$ is the zero homomorphism. Moreover, we have $f \circ i = f \circ c$. Since f is a monomorphism, $i = c$ and therefore $f(G/H)$ is a monomorphism for every object G/H in O_G.

If a morphism $f : S \longrightarrow T$ in $\mathcal{C}(\mathcal{G})$ satisfies $f(G/H)$ is onto for every object G/H in O_G, then f is an epimorphism in $\mathcal{C}(\mathcal{G})$. To see this, let $U : O_G \longrightarrow \mathcal{G}$ be an O_G-group and $\alpha, \beta : T \longrightarrow U$ be natural transformations such that $\alpha \circ f = \beta \circ f$. Let $x \in T(G/H)$, find $y \in S(G/H)$ such that $f(G/H)(y) = x$. Now

$$\alpha(G/H)(x) = \alpha(G/H)f(G/H)(y) = \beta(G/H)f(G/H)(y) = \beta(G/H)(x).$$

This implies that $\alpha = \beta$, thus f is an epimorphism in $\mathcal{C}(\mathcal{G})$.

Proposition 1.5 *If $f : S \longrightarrow T$ is an epimorphism in $\mathcal{C}(\mathcal{G})$ and S and T are $\mathcal{A}$-admissible, then $f(G/H) : S(G/H) \longrightarrow T(G/H)$ is onto for every object G/H in O_G.*

Proof Let $S_1 : O_G \longrightarrow \mathcal{G}$ be the O_G-group defined by

$$S_1(G/H) = f(G/H)(S(G/H))$$

for any object G/H in O_G and for a morphism $\hat{g} : G/H \longrightarrow G/K$ in O_G, $S_1(\hat{g}) = T(\hat{g})/S_1(G/K)$. Note that S_1 is $\mathcal{A}$-admissible. We have to show that S_1 coincides with T. Since f can be decomposed into $S \longrightarrow S_1 \longrightarrow T$, the natural transformation $i : S_1 \longrightarrow T$ is an epimorphism. Define an O_G-group P as follows.

$$P(G/H) = Perm(T(G/H)/S_1(G/H) \cup \infty),$$

the group of permutations of the union of $T(G/H)/S_1(G/H)$ with a disjoint set of one element. For any morphism $\hat{g} : G/H \longrightarrow G/K$ in O_G, $P(\hat{g}) : P(G/K) \longrightarrow P(G/H)$ is given by $T(\hat{g})^{-1}$ as follows; let

$$\alpha : T(G/K)/S_1(G/K) \cup \{\infty\} \longrightarrow T(G/K)/S_1(G/K) \cup \{\infty\}$$

be a permutation. Let $x \in T(G/H)/S_1(G/H) \cup \{\infty\}$. Suppose $x = a\ S_1(G/H)$, $a \in T(G/H)$, then define

$$P(\hat{g})(\alpha)(x) = \begin{cases} \infty & \text{if } \alpha(T(\hat{g})^{-1}\ (a)S_1(G/K)) = \infty \\ T(\hat{g})(b)S_1(G/H) & \text{if } \alpha(T(\hat{g})^{-1}\ (a)S_1(G/K)) = b\ S_1(G/K). \end{cases}$$

If $x = \infty$, define $P(\hat{g})(\alpha)(x) = \infty$ when $\alpha(\infty) = \infty$, and $P(\hat{g})(\alpha)(x) = T(\hat{g})(a)S_1(G/H)$ when $\alpha(\infty) = aS_1(G/K)$. It is easy to check that this defines an O_G group $P : O_G \longrightarrow \mathcal{G}$. Let $\sigma_H \in P(G/H)$ be the permutation which exchanges $S_1(G/H)$ and ∞, and leaves fixed all other elements. Then $\sigma_H^2 = id$. Define a natural transformation $t : T \longrightarrow P$ as follows.

$$\begin{aligned} t(G/H)(u)(v\ S_1(G/H)) &= uv\ S_1(G/H), \quad u, v \in T(G/H), \\ t(G/H)(u)(\infty) &= \infty. \end{aligned}$$

Let $s : T \longrightarrow P$ be the natural transformation defined by

$$s(G/H)(u) = \sigma_H t(G/H)(u)\sigma_H, \qquad u \in T(G/H).$$

To see that s is natural first note that for any morphism $\hat{g} : G/H \longrightarrow G/K$ in O_G, $P(\hat{g})(\sigma_K) = \sigma_H$. Thus for $\hat{g} : G/H \longrightarrow G/K$, and $u \in T(G/K)$

$$\begin{aligned} P(\hat{g})s(G/K)(u) &= P(\hat{g})(\sigma_K t(G/K)(u)\sigma_K) \\ &= P(\hat{g})(\sigma_K)P(\hat{g})(t(G/K)(u))P(\hat{g})(\sigma_K) \\ &= \sigma_H t(G/H)T(\hat{g})(u)\sigma_H \quad \text{(by naturality of } T) \\ &= s(G/H)T(\hat{g})(u) \end{aligned}$$

It is easy to see that $s \circ i = t \circ i$. Since $i : S_1 \longrightarrow T$ is an epimorphism it follows that $s = t$. Thus for each G/H, we have

$$\begin{aligned} a\ S_1(G/H) &= t(G/H)(a)(S_1(G/H)) = s(G/H)(a)(S_1(G/H)) \\ &= (\sigma_H\ t(G/H)(a)\sigma_H)(S_1(G/H)) = \sigma_H t(G/H)(a)(\infty) \\ &= \sigma_H(\infty) = S_1(G/H). \end{aligned}$$

Thus $S_1(G/H) = T(G/H)$. ■

We shall obtain Theorem 1.3 as a corollary to the following result.

Theorem 1.6 *If a morphism $f : X \longrightarrow Y$ in $G\mathcal{HCW}^*$ with X, Y $\mathcal{A}$-admissible, satisfies*

1. $f_* : \underline{\pi}_1 X \longrightarrow \underline{\pi}_1 Y$ *is an epi*
2. $f_* : \underline{\pi}_* X \longrightarrow \underline{\pi}_* Y$ *is a mono*
3. $f^* : H^*_G(Y; M) \longrightarrow H^*_G(X; f^* M)$ *is a mono for all equivariant local coefficients system M on Y,*

then f is an equivalence in $G\mathcal{HCW}^*$.

Here $H^*_G(Y; M)$ denotes the Bredon-Illman cohomology with equivariant local coefficients system M on Y. Before proceeding further let us recall ([8], [7]) the definition of $H^*_G(Y; M)$, and its properties.

2 Definition of $H^*_G(Y; M)$

For a G-space X, let ΠX denote the category whose objects are G-maps $x_H : G/H \longrightarrow X$, and a morphism from $x_H : G/H \longrightarrow X$ to $y_K : G/K \longrightarrow X$ is a pair $(\alpha, [\phi])$, where $\alpha : G/H \longrightarrow G/K$ is a G-map and $\phi : G/H \times I \longrightarrow X$ is a G-homotopy from x_H to $y_K \circ \alpha$, $[\phi]$ is the G-homotopy class rel end points of the G-homotopy ϕ. Recall that we have a homeomorphism $a : Map_G(G/H, X) \longrightarrow X^H$ given by $a(f) = f(eH)$. Therefore we may identify an object $x_H : G/H \longrightarrow X$ in ΠX with the point $x_H(eH)$ in X^H so that, if $\alpha : G/H \longrightarrow G/K$ is a morphism in O_G and $y_K : G/K \longrightarrow X$ is an object in ΠX, then the point in X^H which corresponds to $y_K \circ \alpha : G/H \longrightarrow X$ is given by $y_K \circ \alpha(eH) = \eta_X(\alpha)(y_K(eK))$, where $\eta_X : O_G \longrightarrow \mathcal{T}$ (category of spaces) is the functor such that $\eta_X(G/H) = X^H$, and for a G-map $\alpha : G/H \longrightarrow G/K$, given by a subconjugacy relation $g^{-1}Hg \subset K$, $\eta_X(\alpha) : X^K \longrightarrow X^H$ is the left translation by g. Thus a morphism $(\alpha, [\phi]) : x_H \longrightarrow y_K$ in ΠX corresponds to a homotopy class $< \alpha, [\phi] >$ rel end points of paths in X^H from $x_H(eH)$ to $\eta_X(\alpha)(y_K(eK))$.

Note that for a fixed H, the objects x_H together with morphisms $x_H \longrightarrow y_H$, which are given by the identity in the first factor and a G-homotopy class (rel $G/H \times \partial I$) of G-homotopies $\phi : G/H \times I \longrightarrow X$ from x_H to y_H in the second factor, constitute a subcategory of ΠX which is precisely the fundamental groupoid πX^H of X^H.

Definition 2.1 *An equivariant local coefficients system on a G-space X is a contravariant functor M from ΠX to the category $\mathcal{A}b$ of abelian groups.*

Note that for every $H < G$ (H is a subgroup of G) $M_H = M/\pi X^H$ is a local coefficients system on X^H and for a morphism $\hat{g} : G/H \longrightarrow G/K$ in O_G there exists a natural transformation $\underline{M}(\hat{g}) : M_K \longrightarrow g^* M_H$ defined as follows: Let $k : G/H \times I \longrightarrow X$ denote the constant G-homotopy at $x_K \circ \hat{g}$. Then $(\hat{g}, [k]) : x_K \circ \hat{g} \longrightarrow x_K$ is a morphism in ΠX. For $x \in X^K$ we define $\underline{M}(\hat{g})(x) = M(\hat{g}, [k])$. Conversely, given a local coefficients system M_H on X^H for every $H < G$, along with a natural transformation $\underline{M}(\hat{g}) : M_K \longrightarrow g^* M_H$ for every $\hat{g} : G/H \longrightarrow G/K$, we may obtain an equivariant local coefficients system M on X as follows. For $x_H : G/H \longrightarrow X$ define $M(x_H) = M_H(x_H(eH))$ and for a morphism $(\hat{g}, [\phi]) : x_H \longrightarrow y_K$ in ΠX define

$$M((\hat{g}, [\phi])) = \underline{M}(\hat{g}) \circ M_H(< \hat{g}, [\phi] >) : M(y_K) \longrightarrow M(x_H).$$

Clearly the above correspondence is a bijection.

Let X be a G-space and $x^\circ \in X^G$. Let $M : \Pi X \longrightarrow \mathcal{A}b$ be an equivariant local coefficients system on X. For every $H < G$, the point $x^\circ \in X^G$ corresponds under the homeomorphism $A : X^H \longrightarrow Map_G(G/H, X)$ (which is the inverse of 'a') to the constant map $A(x^\circ) : G/H \longrightarrow x^\circ$, which we shall denote by x°_H. Then, for every morphism $\hat{g} : G/H \longrightarrow G/K$ in O_G, there is a morphism $(\hat{g}, [k]) : x^\circ_H \longrightarrow x^\circ_K$ in ΠX, where k is the constant homotopy. Define an O_G-group $M_0 : O_G \longrightarrow \mathcal{A}b$ by $M_0(G/H) = M(x^\circ_H)$ and $M_0(\hat{g}) = M(\hat{g}, [k])$.

An element $\alpha \in \pi_1(X^H, x^\circ)$ gives rise to an equivalence $A\alpha : x^\circ_H \longrightarrow x^\circ_H$ in ΠX, and therefore an automorphism $M(A\alpha)$ of $M_0(G/H)$.

Definition 2.2 *An O_G-group T is said to act on an O_G-group S (respectively O_G -space) if there is a natural transformation $\rho : T \times S \longrightarrow S$ such that, for every $H < G$, $\rho(G/H)$ is an action of the group $T(G/H)$ on $S(G/H)$.*

The above consideration shows that if $M : \Pi X \longrightarrow \mathcal{A}b$ is an equivariant local coefficients system on X, then there exists an action $\rho : \underline{\pi}_1 X \times M_0 \longrightarrow M_0$ given by $\rho(G/H)(\alpha, m) = M(A\alpha)(m)$.

Conversely, given an O_G-group $M_0 : O_G \longrightarrow \mathcal{A}b$ along with an action of $\underline{\pi}_1 X$ we can define an equivariant local coefficients system on X and this correspondence is bijective [7].

Definition 2.3 *Let X, Y be G-spaces, and M an equivariant local coefficients system on Y. Then a G-map $f : X \longrightarrow Y$ defines a covariant functor $\Pi(f) : \Pi X \longrightarrow \Pi Y$ by $\Pi(f)(x_H) = f \circ x_H$ and $\Pi(f)(\alpha, [\phi]) = (\alpha, [f \circ \phi])$. The functor $M \circ \Pi f$ is an equivariant local coefficients system on X, which we shall denote by $f^* M$.*

We shall denote vertices of the standard n-simplex $\triangle_n$ by $e_0, e_1, \ldots, e_n$ and the j-th face operator $\triangle_{n-1} \longrightarrow \triangle_n$ by d^j_n, $0 \leq j \leq n$.

Let X be a G-space and M an equivariant local coefficients system on X. If $\sigma : \triangle_n \times G/H \longrightarrow X$ is an equivariant singular simplex in X, then σ_H will denote the G-map $G/H \longrightarrow X$ defined by $\sigma_H(gH) = \sigma(e_0, gH)$.

We define $C^n_G(X;M)$ to be the group of all functions c on equivariant singular n-simplexes $\sigma : \triangle_n \times G/H \longrightarrow X$ such that $c(\sigma) \in M(\sigma_H)$.

If $u : \triangle_q \longrightarrow \triangle_n$ is a singular q-simplex in $\triangle_n$, and $\sigma : \triangle_n \times G/H \longrightarrow X$ is an equivariant singular n-simplex in X, then $\sigma(u) : \triangle_q \times G/H \longrightarrow X$ will denote the equivariant singular q-simplex $\sigma \circ (u \times id)$, and $\sigma(u)_* : \sigma_H \longrightarrow \sigma(u)_H$ will denote the morphism $(id, [\phi])$, where $\phi : G/H \times I \longrightarrow X$ is the G-homotopy given by $\phi(gH, t) = (tu(e_0) + (1-t)e_0, gH)$.

Then $\sigma(d^i_n) : \triangle_{n-1} \times G/H \longrightarrow X$ is the j-th face $\sigma^{(j)}$ of σ. Note that $\sigma^{(j)}_H = \sigma_H$ for $j > 0$ and $\sigma(d^0_n)_* = \sigma^{(0)}_*$ is a morphism $\sigma_H \longrightarrow \sigma^{(0)}_H$.

We define coboundary $\delta : C^n_G(X;M) \longrightarrow C^{n+1}_G(X;M)$ by

$$(\delta c)(\sigma) = M(\sigma^{(0)}_*)(c(\sigma^{(0)})) + \sum_{j=1}^{n+1} (-1)^j \, c(\sigma^{(j)}),$$

where σ is an equivariant singular $(n+1)$-simplex in X. Thus we have a cochain complex $C_G(X;M) = \{C^n_G(X;M); \delta\}$.

Let $\sigma : \triangle_n \times G/H \longrightarrow X$ and $\tau : \triangle_n \times G/K \longrightarrow X$ be two equivariant singular $n-$ simplexes in X. Consider $\triangle_n \times G/H$ and $\triangle_n \times G/K$ as trivial fiber bundles over $\triangle_n$, and suppose that $h : \triangle_n \times G/H \longrightarrow \triangle_n \times G/K$ is a fiber preserving G-map such that $\sigma = \tau \circ h$. In this case we say that σ and τ are compatible under h.

The map h induces a G-map $\bar{h} : G/H \longrightarrow G/K$ given by $\bar{h}(gH) = pr_2 \circ h(e_0, gH)$, where pr_2 is the projection onto the second factor. Then $\sigma = \tau \circ h$ implies $\sigma_H = \tau_K \circ \bar{h}$. Therefore, if $k : G/H \times I \longrightarrow X$ is the constant homotopy from σ_H to $\tau_K \circ \bar{h}$, then we have a morphism $(\bar{h}, [k]) : \sigma_H \longrightarrow \tau_K$ in ΠX. We shall denote this induced morphism by h_*.

We define $S^n_G(X;M)$ to be the subgroup of $C^n_G(X;M)$ consisting of all those cochain c such that if σ and τ are equivariant singular n-simplexes in X which are compatible under h, then $c(\sigma) = M(h_*)(c(\tau))$.

It is easy to check that if $c \in S^n_G(X;M)$, then $\delta c \in S^{n+1}_G(X;M)$. Thus we have a cochain complex $S_G(X;M) = \{S^n_G(X;M); \delta\}$.

Definition 2.4 *The Bredon-Illman cohomology of X with equivariant local coefficients M is defined by $H^n_G(X;M) = H^n(S_G(X;M))$.*

It may be noted that $H^*_G(X;M)$ reduces to the Steenrod cohomology with the classical local coefficients system [10], when G is trivial.

If X is a G-map and M an equivariant local coefficients on Y, then f^*M is an equivariant local coefficients system on X and we have a cochain map

$$f^\sharp : C^n_G(Y;M) \longrightarrow C^n_G(X; f^*M),$$

defined as follows. For $c \in C^n_G(Y;M)$ and $\sigma : \triangle_n \times G/H \longrightarrow X$, $f^\sharp(c)(\sigma) = c(f \circ \sigma)$. It is straightforward to check that if $c \in S^n_G(Y;M)$, then $f^\sharp(c) \in S^n_G(X; f^*M)$. Thus f induces a homomorphism $f^* : H^n_G(Y;M) \longrightarrow H^n_G(X; f^*M)$.

An interesting feature of the Steenrod cohomology of topological space X with local coefficients M is that it can be realized as certain coholmology of its universal covering $\widetilde{X}$. If $p : \widetilde{X} \longrightarrow X$ is the covering projection, then $\pi = \pi_1(X, x^0)$ acts on $\widetilde{X}$, and $M_0 = M(x^0)$ is a π-module. Let $C^n_\pi(\widetilde{X}; M_0)$ be the group of π invariant singular n-cochains, and $H^n_\pi(\widetilde{X}; M_0)$ be the corresponding cohomology. Then a classical theorem of Eilenberg [5] [10], says that p induces an isomorphism

$$H^n(X; M) \cong H^n_\pi(\widetilde{X}; M_0).$$

We next give a similar alternative description of the Bredon-Illman cohomology. Let X be a G-space such that, for every $H < G$, the fixed point set X^H is connected, locally path connected and semilocally simply connected. For example X may be G-connected G-complex. Let $x^\circ \in X^G$, and $p_H : \widetilde{X^H} \longrightarrow X^H$ denote the universal covering of X^H. For a G-map $\hat{g} : G/H \longrightarrow G/K$ in O_G, the left translation $g : X^K \longrightarrow X^H$ induces $\tilde{g} : \widetilde{X^K} \longrightarrow \widetilde{X^H}$ such that $p_H \circ \tilde{g} = g \circ p_K$. Then we have an O_G-space $\mathcal{U}(X)$ defined by $\mathcal{U}(X)(G/H) = \widetilde{X^H}$ and $\mathcal{U}(X)(\hat{g}) = \tilde{g}$. The O_G-space $\mathcal{U}(X)$ will be called the universal O_G-covering space of X. The O_G-group $\underline{\pi}_1 X$ acts on $\mathcal{U}(X)$. This action comes from the identification of $\underline{\pi}_1 X(G/H) = \pi_1(X^H, x^\circ)$ with the deck transformation group $D(p_H)$ of $p_H : \widetilde{X^H} \longrightarrow X^H$, and the action of $D(p_H)$ on $\widetilde{X^H}$. Note that if $\alpha \in \pi_1(X^H, x^\circ)$ corresponds to $\gamma_\alpha \in D(p_H)$ and if $u : I \longrightarrow \widetilde{X^H}$ is a path from $\tilde{x}^\circ_H$ to $\gamma_\alpha(\tilde{x}^\circ_H)$, where $\tilde{x}^\circ_H \in p_H^{-1}(x^\circ)$, then the path $p_H \circ u$ represents α.

Let M be an equivariant local coefficients system on X and $M_0 : O_G \longrightarrow \mathcal{A}b$ be the associated O_G-group. Recall that $\underline{\pi}_1 X$ acts on M_0.

Let

$$\{C^n_{\underline{\pi}_1 X(G/H)}(\widetilde{X^H}; M_0(G/H)), d^n_H\}$$

be the cochain complex of Eilenberg, where $C^n_{\underline{\pi}_1 X(G/H)}(\widetilde{X^H}; M_0(G/H))$ is the subgroup of the singular cochain group $C^n(\widetilde{X^H}; M_0(G/H))$ consisting of cochains c which are equivariant with respect to the action $\underline{\pi}_1 X(G/H)$ in the sense that if $\alpha \in \pi_1(X^H, x^\circ)$, then $c(\gamma_\alpha\ \sigma) = M(A\alpha)(c(\sigma))$, for every singular n-simplex $\sigma : \triangle_n \longrightarrow \widetilde{X^H}$, where γ_α is the deck transformation corresponding to α.

Now define the cochain complex

$$C_{\underline{\pi}_1 X, G}(\mathcal{U}(X); M_0) = \{C^n_{\underline{\pi}_1 X, G}(\mathcal{U}(X); M_0), d^n\}$$

by

$$C^n_{\underline{\pi}_1 X, G}(\mathcal{U}(X); M_0) = \bigoplus_{H<G} C^n_{\underline{\pi}_1 X(G/H)}(\widetilde{X^H}; M_0(G/H)),\ d^n = \bigoplus_{H<G} d^n_H.$$

Define $S^n_{\underline{\pi}_1 X, G}(\mathcal{U}(X); M_0)$ to be the subgroup of $C^n_{\underline{\pi}_1 X, G}(\mathcal{U}(X); M_0)$ consisting of cochains $\{c_H\}_{H<G}$ such that, for every $\hat{g} : G/H \longrightarrow G/K$ and singular n-simplexes $\sigma : \triangle_n \longrightarrow \widetilde{X^H}$, $\tau : \triangle_n \longrightarrow \widetilde{X^K}$ with $\tilde{g} \circ \tau = \sigma$, the equation $M_0(\hat{g})(c_K(\tau)) = c_H(\sigma)$ holds in $M_0(G/H)$. We then have a cochain complex

$$S_{\underline{\pi}_1 X, G}(\mathcal{U}(X); M_0) = \{S^n_{\underline{\pi}_1 X, G}(\mathcal{U}(X); M_0), d^n\}.$$

Definition 2.5 *We define the $\underline{\pi}_1X$-equivariant cohomology group of $\mathcal{U}(X)$ with coefficients M_0 by*

$$H^n_{\underline{\pi}_1X,G}(\mathcal{U}(X);M_0)=H^n(S^n_{\underline{\pi}_1X,G}(\mathcal{U}(X);M_0)).$$

Theorem 2.6 *Let X is a G-space, such that for each $H<G$ the fixed point set X^H is connected, locally path connected, and semilocally simply connected. If M is an equivariant local coefficients system on X then*

$$H^n_G(X;M)\cong H^n_{\underline{\pi}_1X,G}(\mathcal{U}(X);M_0))$$

where M_0 is the O_G-group induced by M and $H^n_G(X;M)$ is the Bredon-Illman cohomology.

Sketch of the Proof. We need to introduce some notations. If $\sigma:\triangle_n\times G/H\longrightarrow X$ is an equivariant singular simplex, then $N\sigma$ will denote the corresponding non-equivariant simplex $\triangle_n\longrightarrow X^H$, $N\sigma(x)=\sigma(x,eH)$. Conversely, if $\tau:\triangle_n\longrightarrow X^H$ is a singular simplex, then $E\tau$ will denote the corresponding equivariant simplex $E\tau(x,gH)=g\ \tau(x)$. Note that we have $E(N\sigma)=\sigma$ and $N(E\tau)=\tau$. Next if $x,y\in\widetilde{X^H}$, then we shall denote by $\tilde{\xi}_H(x,y)$ a homotopy class of paths in $\widetilde{X^H}$ from x to y, and write $\xi_H(x,y)$ for $p_H\tilde{\xi}_H(x,y)$. Then $\xi_H(x,y)$ is a homotopy class of paths in X^H from $p_H(x)$ to $p_H(y)$. We shall suppose that each $\widetilde{X^H}$ comes equipped with a base point $\tilde{x}^0_H$ such that $p_H(\tilde{x}^0_H)=x^0$. When $x=\tilde{x}^0_H$ we write $\tilde{\xi}_H(y)$ and $\xi_H(y)$ instead of $\tilde{\xi}_H(\tilde{x}^0_H,y)$ and $\xi_H(\tilde{x}^0_H,y)$. Since $\widetilde{X^H}$ is simply connected, for every $y\in\widetilde{X^H}$ there is a unique class $\tilde{\xi}_H(y)$.

Define a homomorphism $\phi:C^n_G(X;M)\longrightarrow C^n_{\underline{\pi}_1X,G}(\mathcal{U}(X);M_0)$ as follows. If $c\in C^n_G(X;M)$ then $\phi(c)=\{c_H\}_{H<G}$ with $c_H\in C^n_{\pi_1X(G/H)}(\widetilde{X^H};M_0(G/H))$ is given by $c_H(\sigma)=M(A(\xi_H(\sigma(e_0)))(c(Ep_H\sigma))$, for every $\sigma:\triangle_n\longrightarrow\widetilde{X^H}$, where $A(\xi_H(\sigma(e_0))$ is the morphism $x^0_H\longrightarrow(Ep_H\sigma)_H$ corresponding to the homotopy class $\widetilde{\xi_H}(\sigma(e_0))$ of paths in X^H from x^0 to $p_H\sigma(e_0)$. Note that we have

$$A(p_H\sigma(e_0))(gH)=g\ p_H\sigma(e_0)=(Ep_H\sigma)(e_0,gH)=(Ep_H\sigma)_H(gH),$$

and therefore $A(p_H\sigma(e_0))=(Ep_H\sigma)_H$. That ϕ is well defined can be checked using the fact that c_H is $\underline{\pi}_1X(G/H)$ equivariant for every subgroup H of G. It is straightforward to check that ϕ is a cochain map and that ϕ maps $S^n_G(X;M)$ into $S^n_{\underline{\pi}_1X,G}(\mathcal{U}(X);M_0)$.

Now define a homomorphism $\psi:C^n_{\underline{\pi}_1X,G}(\mathcal{U}(X);M_0)\longrightarrow C^n_G(X;M)$ as follows. Let $c=\{c_H\}_{H<G}\in C^n_{\underline{\pi}_1X,G}(\mathcal{U}(X);M_0)$, and $\sigma:\triangle_n\times G/H\longrightarrow X$ be an equivariant simplex. Let $\tilde{\sigma}:\triangle_n\longrightarrow\widetilde{X^H}$ be the lifting of $N\sigma$ so that $p_H\tilde{\sigma}=N\sigma$. Then set

$$\psi(c)(\sigma)=M(A\xi_H(\tilde{\sigma}(d_0)))^{-1}(c_H(\tilde{\sigma})).$$

It can be verified that ψ maps $S^n_{\underline{\pi}_1X,G}(\mathcal{U}(X);M_0)$ into $S^n_G(X;M)$ and is the cochain inverse of ϕ. ■

Next we define another cohomology group of $\mathcal{U}(X)$ with coefficients in an O_G-group $\lambda : O_G \longrightarrow \mathcal{A}b$, forgetting the action of $\underline{\pi}_1 X$ on $\mathcal{U}(X)$.

Let $\mathcal{R}$ denote the category of rings, and $\Gamma : O_G \longrightarrow \mathcal{R}$ be the contravariant functor defined by $\Gamma(G/H) = \boldsymbol{Z}[\pi_1 X^H]$, the integral group ring of $\pi_1 X^H$, for object G/H in O_G and $\Gamma(\hat{g}) = g_* : \boldsymbol{Z}[\pi_1 X^K] \longrightarrow \boldsymbol{Z}[\pi_1 X^H]$ for a morphism $\hat{g} : G/H \longrightarrow G/K$ in O_G. Note that the O_G-chain complex $\underline{C}_*\mathcal{U}(X)$, defined by $\underline{C}_*\mathcal{U}(X) = C_*(\widetilde{X^H})$ is equipped with a Γ action.

Now define the cochain complex

$$C_G(\mathcal{U}(X);\lambda) = \{C^n_G(\mathcal{U}(X);\lambda); d^n\}$$

by

$$C^n_G(\mathcal{U}(X);\lambda) = \bigoplus\nolimits_{H<G} Hom_{\boldsymbol{Z}}(C_*\widetilde{X^H} \otimes_{\Gamma(G/H)} \Gamma(G/H); \lambda(G/H)), \quad d^n = \oplus_{H<G} d^n_H.$$

where

$$\begin{aligned} d^n_H \quad : \quad & Hom_{\boldsymbol{Z}}(C_n\widetilde{X^H} \otimes_{\Gamma(G/H)} \Gamma(G/H); \lambda(G/H)) \longrightarrow \\ & Hom_{\boldsymbol{Z}}(C_{n+1}\widetilde{X^H} \otimes_{\Gamma(G/H)} \Gamma(G/H); \lambda(G/H)) \end{aligned}$$

is induced from the boundary $\partial_H : C_{n+1}\widetilde{X^H} \longrightarrow C_n\widetilde{X^H}$. Define

$$S_G(\mathcal{U}(X);\lambda) = \{S^n_G(\mathcal{U}(X);\lambda); d^n\}$$

where $S^n_G(\mathcal{U}(X);\lambda)$ is the subgroup of $C^n_G(\mathcal{U}(X);\lambda)$ consisting of cochains $\{c_H\}_{H<G}$ such that for every $\hat{g} : G/H \longrightarrow G/K$ and singular n-simplexes $\sigma : \triangle_n \longrightarrow \widetilde{X^H}$ and $\tau : \triangle_n \longrightarrow \widetilde{X^K}$ with $\tilde{g} \circ \tau = \sigma$, the equation $\lambda(\hat{g})(c_K(\tau)) = c_H(\sigma)$ holds in $\lambda(G/H)$. Define

$$H^n_G(\mathcal{U}(X);\lambda) = H^n(S_G(\mathcal{U}(X);\lambda)).$$

Let $\underline{H}_*\mathcal{U}(X)$ be the O_G-group defined by

$$\underline{H}_*\mathcal{U}(X)(G/H) = H_*(\widetilde{X^H})$$

and

$$\underline{H}_*\mathcal{U}(X)(\hat{g}) = g_* : H_*(\widetilde{X^K}) \longrightarrow H_*(\widetilde{X^H}).$$

Let $\mathcal{C}_G$ denote the abelian category of abelian O_G-groups, and $Hom(S,T)$ denote the morphism set in $\mathcal{C}_G$ between objects S and T.

We define the Kronecker homomorphism

$$\kappa : H^n_G(\mathcal{U}(X);\lambda) \longrightarrow Hom(\underline{H}_n\mathcal{U}(X);\lambda)$$

as follows. For $[\{c_H\}_{H<G}] \in H^n_G(\mathcal{U}(X);\lambda), \{c_H\}_{H<G} \in S^n_G(\mathcal{U}(X);\lambda)$ define

$$\kappa[\{c_H\}_{H<G}](G/H)([\beta_H]) = c_H(\beta_H), [\beta_H] \in H_n(\widetilde{X^H}).$$

We shall show that κ is a surjection by constructing a homomorphism

$$\mu^* : Hom(\underline{H}_n\mathcal{U}(X);\lambda) \longrightarrow H^n_G(\mathcal{U}(X);\lambda)$$

such that $\kappa \circ \mu^*$ is the identity.

Let $\underline{B}_n\mathcal{U}(X)$, $\underline{Z}_n\mathcal{U}(X)$, *and* $\underline{C}_n\mathcal{U}(X)$ denote the O_G-groups: $\underline{B}_n\mathcal{U}(X)(G/H) = B_n(\widetilde{X^H})$, $\underline{Z}_n\mathcal{U}(X)(G/H) = Z_n(\widetilde{X^H})$, and $\underline{C}_n\mathcal{U}(X)(G/H) = C_n(\widetilde{X^H})$, where $C_n(\widetilde{X^H})$ is the n-th chain group of $\widetilde{X^H}$ and $B_n(\widetilde{X^H})$, and $Z_n(\widetilde{X^H})$ are the boundaries and cycles respectively. We have a natural transformation $\pi : \underline{Z}_n\mathcal{U}(X) \longrightarrow \underline{H}_n\mathcal{U}(X)$ and an exact sequence

$$0 \longrightarrow \underline{Z}_n\mathcal{U}(X) \longrightarrow \underline{C}_n\mathcal{U}(X) \longrightarrow \underline{B}_{n-1}\mathcal{U}(X) \longrightarrow 0$$

in $\mathcal{C}_G$. Thus the natural transformation π extends to a natural transformation $\mu : \underline{C}_n\mathcal{U}(X) \longrightarrow \underline{H}_n\mathcal{U}(X)$. Define a chain complex E_* in $\mathcal{C}_G$ by setting $E_n = \underline{H}_n\mathcal{U}(X)$ and the boundary homomorphism to be the zero natural transformation. Then $\mu : \underline{C}_*\mathcal{U}(X) \longrightarrow E_*$ is a chain map. This induces a cochain map

$$\mu^\sharp : Hom(E_*;\lambda) \longrightarrow Hom(\underline{C}_*\mathcal{U}(X);\lambda).$$

Next define $\nu : Hom(\underline{C}_n\mathcal{U}(X);\lambda) \longrightarrow S^n_G(\mathcal{U}(X);\lambda)$ as follows. Given $f \in Hom(\underline{C}_n\mathcal{U}(X);\lambda)$, $\nu f = \{f_H\}_{H<G} \in C^n_G(\mathcal{U}(X);\lambda)$ is given by $f_H(\sigma) = f(G/H)(\sigma)$, for $\sigma : \triangle_n \longrightarrow \widetilde{X^H}$. We claim that $\nu f = \{f_H\}_{H<G} \in S^n_G(\mathcal{U}(X);\lambda)$. For if $\sigma : \triangle_n \longrightarrow \widetilde{X^H}$, and $\tau : \triangle_n \longrightarrow \widetilde{X^K}$ and $\hat{g} : G/H \longrightarrow G/K$ are such that $\tilde{g} \circ \tau = \sigma$, then

$$\lambda(\hat{g})(f_K(\tau)) = \lambda(\hat{g})f(G/K)(\tau) = f(G/H)(\tilde{g} \circ \tau) = f(G/H)(\sigma) = f_H(\sigma)$$

by naturality of f. It is easy to see that ν is a cochain map. Thus $\nu \circ \mu^\sharp$ induces a homomorphism

$$\mu^* : Hom(\underline{H}_n\mathcal{U}(X);\lambda) \longrightarrow H^n_G(\mathcal{U}(X);\lambda)$$

in the cohomology such that $\kappa \circ \mu^*$ is identity.

Let X be an object in $G\mathcal{HCW}^*$ and $x^\circ \in X^G$ be the base point. We may assume that x° is a zero cell. Fix $\tilde{x}^\circ_G \in p_G^{-1}(x^\circ)$ and let $\tilde{x}^\circ_H = \mathcal{U}(X)(\hat{i})(\tilde{x}^\circ_G)$ where $\hat{i} : G/H \longrightarrow G/G$. Then $\mathcal{U}(X)$ is a functor from O_G to $\mathcal{HCW}^*$. Let $f : X \longrightarrow Y$ be a morphism in $G\mathcal{HCW}^*$. The map f induces a natural transformation $\tilde{f} : \mathcal{U}(X) \longrightarrow \mathcal{U}(Y)$, given by $\tilde{f}(G/H) = \tilde{f}^H$. We have a 'cofibration sequence'

$$\mathcal{U}(X) \xrightarrow{\tilde{f}} \mathcal{U}(Y) \xrightarrow{q} C_{\tilde{f}} \longrightarrow \Sigma\mathcal{U}(X) \xrightarrow{\Sigma\tilde{f}} \Sigma\mathcal{U}(Y) \longrightarrow \ldots$$

where $C_{\tilde{f}}(G/H) = C_{\tilde{f}^H}$ is the mapping cone of $\tilde{f}^H : \widetilde{X^H} \longrightarrow \widetilde{Y^H}$ and $\Sigma(\mathcal{U}X)(G/H) = \Sigma\widetilde{X^H}$, is the suspension of $\widetilde{X^H}$. Then we may deduce a long exact cohomology sequence

$$\ldots \longrightarrow H^n_G(\mathcal{U}(Y);\lambda) \xrightarrow{\tilde{f}^*} H^n_G(\mathcal{U}(X);\lambda) \longrightarrow H^{n+1}_G(C_{\tilde{f}};\lambda) \xrightarrow{q^*} H^{n+1}_G(\mathcal{U}(Y);\lambda) \xrightarrow{\tilde{f}^*} \ldots$$

We shall refer to this sequence as the long exact cohomology sequence associated to $\tilde{f}$.

3 Proof of Theorem 1.6

Let λ be an abelian O_G-group and

$$\Gamma : O_G \longrightarrow \mathcal{R}, \qquad \Gamma(G/H) = \boldsymbol{Z}[\pi_1 Y^H].$$

be the functor as defined in the previous section. We shall define an equivariant local coefficients system $Hom(\Gamma, \lambda)$ on Y as follows. For every $H < G$, we have a local coefficients system $Hom(\Gamma, \lambda)_H$ on Y^H defined by the $\pi_1 Y^H$-module $Hom_{\boldsymbol{Z}}(\Gamma(G/H), \lambda(G/H))$ as in [4]. Let $\hat{g} : G/H \longrightarrow G/K$ be a morphism in O_G. Then $g^* Hom(\Gamma, \lambda)_H$ is a local coefficients system on Y^K and is given by the $\pi_1 Y^K$-module $Hom_{\boldsymbol{Z}}(\Gamma(G/H), \lambda(G/H))$ by virtue of the isomorphism $g_* : \pi_1 Y^K \longrightarrow \pi_1 Y^H$. Note that this $\pi_1 Y^K$-module is same as the $\pi_1 Y^K$-module $Hom_{\boldsymbol{Z}}(\Gamma(G/K), \lambda(G/H))$. Thus we have a $\pi_1 Y^K$-module homomorphism

$$Hom_{\boldsymbol{Z}}(\Gamma(G/K), \lambda(G/K)) \longrightarrow Hom_{\boldsymbol{Z}}(\Gamma(G/K), \lambda(G/H))$$

given by $\alpha \mapsto \lambda(\hat{g}) \circ \alpha$. This induces a natural transformation

$$\underline{Hom}(\Gamma, \lambda)(\hat{g}) : Hom(\Gamma, \lambda)_K \longrightarrow g^* Hom(\Gamma, \lambda)_H.$$

As explained in the previous section this defines an equivariant local coefficients system on Y. Moreover note that for every $H < G$, $Hom_{\boldsymbol{Z}}(\Gamma(G/H), \lambda(G/H))$ can be regarded as a $\pi_1 X^H$-module by virtue of the isomorphism

$$f_*^H : \pi_1 X^H \longrightarrow \pi_1 Y^H$$

(by 1, 2 of Theorem 1.6, and Prop 1.5). Thus $Hom(\Gamma, \lambda)$ can be regarded as an equivariant local coefficients on X so that $f^* Hom(\Gamma, \lambda)$ is isomorphic to $Hom(\Gamma, \lambda)$. According to ([2], Prop. 5.2′) there exists a natural isomorphism

$$\bigoplus_{H<G} Hom_{\Gamma(G/H)}(C_* \widetilde{Y^H}, Hom_{\boldsymbol{Z}}(\Gamma(G/H), \lambda(G/H)) \longrightarrow$$
$$\bigoplus_{H<G} Hom_{\boldsymbol{Z}}(C_* \widetilde{Y^H} \otimes_{\Gamma(G/H)} \Gamma(G/H), \lambda(G/H)).$$

Hence we have a commutative diagram

$$\begin{array}{ccc}
C_{\underline{\pi}_1 Y, G}(\mathcal{U}(Y); Hom(\Gamma, \lambda)_0) & \longrightarrow & C_G(\mathcal{U}(Y); \lambda) \\
\Big\downarrow \tilde{f}^\sharp & & \Big\downarrow \tilde{f}^\sharp \\
C_{\underline{\pi}_1 X, G}(\mathcal{U}(X); Hom(\Gamma, \lambda)_0) & \longrightarrow & C_G(\mathcal{U}(X); \lambda)
\end{array}$$

where the horizontal maps are isomorphisms and vertical maps are induced by $\tilde{f} : \mathcal{U}(X) \longrightarrow \mathcal{U}(Y)$. It is easy to see that the horizontal maps pass into the subcomplexes S. Hence in view of Theorem 2.6 we have a commutative diagram of cohomology groups

$$\begin{array}{ccc} H^*_G(Y; Hom(\Gamma, \lambda)) & \xrightarrow{\cong} & H^*_G(\mathcal{U}(Y); \lambda) \\ \Big\downarrow \tilde{f}^* & & \Big\downarrow \tilde{f}^* \\ H^*_G(X; Hom(\Gamma, \lambda)) & \xrightarrow{\cong} & H^*_G(\mathcal{U}(X); \lambda) \end{array}$$

Now, by 3 of Theorem 1.6 $\tilde{f}^* : H^*_G(\mathcal{U}(Y); \lambda) \longrightarrow H^*_G(\mathcal{U}(X); \lambda)$ is a monomorphism for any O_G-group λ. Hence from the long exact cohomology sequence associated to $\tilde{f}$ we deduce that $q^* : H^n_G(C_{\tilde{f}}\,; \lambda) \longrightarrow H^n_G(\mathcal{U}(Y); \lambda)$ is zero for every n. By naturality of κ we have the following commutative diagram

$$\begin{array}{ccc} H^*_G(C_{\tilde{f}}\,; \lambda) & \xrightarrow{\kappa} & Hom(\underline{H}_n C_{\tilde{f}}\,; \lambda) \\ \Big\downarrow q^* = 0 & & \Big\downarrow \\ H^n_G(\mathcal{U}(Y); \lambda) & \xrightarrow[\kappa]{} & Hom(\underline{H}_n \mathcal{U}(Y); \lambda) \end{array}$$

Taking $\lambda = \underline{H}_n C_{\tilde{f}}$ we deduce that $q_* : \underline{H}_n \mathcal{U}(Y) \longrightarrow \underline{H}_n C_{\tilde{f}}$ is the zero natural transformation. From the long exact sequence

$$\ldots \longrightarrow \underline{H}_n \mathcal{U}(X) \xrightarrow{\tilde{f}_*} \underline{H}_n \mathcal{U}(Y) \xrightarrow{q_*} \underline{H}_n C_{\tilde{f}} \longrightarrow \ldots$$

in $\mathcal{C}_G$, it follows that $\tilde{f}_* : \underline{H}_* \mathcal{U}(X) \longrightarrow \underline{H}_* \mathcal{U}(Y)$ is an epi. Thus for each $H < G$, $\tilde{f}^H_* : H_* \widetilde{X^H} \longrightarrow H_* \widetilde{Y^H}$ is an epi. By 2 of Theorem 1.6 $f^H_* : \pi_* X^H \longrightarrow \pi_* Y^H$ is a mono, hence $\tilde{f}^H_* : \pi_*\widetilde{X^H} \longrightarrow \pi_*\widetilde{Y^H}$ is a mono. Now proceeding as in [4] we see that $f^H_* : \pi_* X^H \longrightarrow \pi_* Y^H$ is an isomorphism and hence f^H is a homotopy equivalence for every $H < G$. Consequently by ([9], Prop 2.7) f is a G-homotopy equivalence.

4 Proof of Theorem 1.3

Enough to prove that $f : X \longrightarrow Y$ satisfies hypothesis of Theorem 1.6. First we note that if $f : X \longrightarrow Y$ is a mono in $G\mathcal{HCW}^\circ$ then $f^H_* : \pi_* X^H \longrightarrow \pi_* Y^H$ is a monomorphism for every $H < G$. For suppose $\alpha, \beta : S^n \longrightarrow X^H$ represent any two elements of $\pi_n X^H$ such that $f^H \circ \alpha \sim f^H \circ \beta$. Let $E\alpha : S^n \times G/H \longrightarrow X$ be defined by $E\alpha(u, aH) = a\alpha(u)$ as in the proof of Theorem 2.6. Similarly define $E\beta$. Then $E(f^H \circ \alpha) = f \circ E\alpha$ and $E(f^H \circ \beta) = f \circ E\beta$. Let F be a homotopy from $f^H \circ \alpha$ to $f^H \circ \beta$. Then $EF : S^n \times G/H \times I \longrightarrow Y$, defined by $EF(u, aH, t) = aF(u, t)$ is a G-homotopy from $f \circ E\alpha$ to $f \circ E\beta$. Since f is a mono in $G\mathcal{HCW}^\circ$, it follows that $E\alpha$ is G-homotopic to $E\beta$. Thus $\alpha \sim \beta$ and 2 of Theorem 1.6 is satisfied.

Next we show that if $f : X \longrightarrow Y$ is an epi in $G\mathcal{HCW}^*$ then 1 of Theorem 1.6 is satisfied. First note that for any G-path connected G-complex X and $\lambda : O_G \longrightarrow \mathcal{G}$, there is an adjunction equivalence

$$[X, K(\lambda, 1)]_G \leftrightarrow Hom(\underline{\pi}_1 X, \lambda) \qquad (*)$$

where $K(\lambda, 1)$ is the equivariant Eilenberg-MacLane complex of the type $(\lambda, 1)$ [6]. Our assertion follows from this as in [4]. To prove (*) we proceed as follows. If $f : X \longrightarrow K(\lambda, 1)$ represents an element of $[X, K(\lambda, 1)]_G$, then the corresponding element in $Hom(\underline{\pi}_1 X, \lambda)$ is given by $f_* : \underline{\pi}_1 X \longrightarrow \lambda$. Conversely, a natural transformation $T : \underline{\pi}_1 X \longrightarrow \lambda$ induces a G-homotopy class of G-maps $T_* : K(\underline{\pi}_1 X, 1) \longrightarrow K(\lambda, 1)$ [6]. Note that X can be regarded as a G-subcomplex of $K(\underline{\pi}_1 X, 1)$. For we may obtain $K(\underline{\pi}_1 X, 1)$ from X by attaching suitable equivariant cells to X to kill the higher homotopy groups of the fixed point sets of X. The element T_*/X in $[X, K(\lambda, 1)]_G$ is then the element which corresponds to T.

In case, $\lambda : O_G \longrightarrow \mathcal{A}b$ is an abelian O_G-group we may give an alternative argument for the validity of (*) as follows. Recall from [1] that there exists a spectral sequence whose E_2 term is

$$E_2^{p,q} = Ext^p(\underline{H}_q X, \lambda) \Longrightarrow H_G^{p+q}(X; \lambda).$$

There is an edge homomorphism

$$H^n_G(X; \lambda) \longrightarrow Hom(\underline{H}_n X, \lambda)$$

which is an isomorphism if each $\underline{H}_q X$ is projective for $q < n$. We claim that $\underline{H}_0 X$ is projective. This can be seen as follows. We consider an epimorphism $\eta : S \longrightarrow T$ and an arbitrary morphism $\mu : \underline{H}_0 X \longrightarrow T$ in $\mathcal{C}_G$. Orient the cells of X in such a way that G preserves the orientation. Since X is G-path connected,

$$\underline{H}_0 X(G/H) = H_0(X^H) \cong \boldsymbol{Z}\langle x^\circ \rangle,$$

$\langle x^\circ \rangle$ being the homology class of the fixed point x°. Let $\bar{\eta}_G : \underline{H}_o X(G/G) \longrightarrow S(G/G)$ be a solution for the corresponding problem for $\underline{H}_0 X(G/G)$. Define

$$\bar{\eta} : \underline{H}_0 X \longrightarrow S$$

by

$$\bar{\eta}(G/H)(\langle x^\circ \rangle) = S(\hat{i})\bar{\eta}_G(\langle x^\circ \rangle),$$

where $\hat{i} : G/H \longrightarrow G/G$ is the morphism in O_G corresponding to the inclusion $H \subset G$. Note that since G preserves orientation, for any $\hat{g} : G/H \longrightarrow G/K$, $\underline{H}_0 X(\hat{g})$ is the identity. It is easy to check that $\bar{\eta}$ is natural and a solution for η and μ in $\mathcal{C}_G$. Thus $\underline{H}_0 X$ is projective. Hence

$$[X, K(\lambda, 1)]_G \cong H^1_G(X; \lambda) \cong Hom(\underline{H}_1 X; \lambda).$$

Let $\rho : \underline{\pi}_1 X \longrightarrow \underline{H}_1 X$ denote the natural transformation such that $\rho(G/H)$ is the Hurewicz homomorphism. Then ρ induces an isomorphism

$$Hom(\underline{H}_1 X, \lambda) \cong Hom(\underline{\pi}_1 X, \lambda)$$

and the result follows.

To prove epis in $\mathcal{GHCW}^*$ satisfy 3 of Theorem 1.6 we need a homotopy theoretical interpretation of $H^n_G(Y; \lambda)$ and $f^* : H^n_G(Y; M) \longrightarrow H^n_G(X; f^*M)$. As mentioned in section 2 an equivariant local coefficients system M on Y may be viewed as a $\underline{\pi}_1 Y$-module M_0. Then as in [7] there exists a sectioned G-fibration

$$K(M_0, n) \longrightarrow L(\underline{\pi}_1 Y, M_0, n) \xrightarrow{p} K(\underline{\pi}_1 Y, 1) \xleftarrow{s} K(\underline{\pi}_1 Y, 1).$$

This yields a sectioned G-fibration $p : E \longrightarrow Y$ with fiber $K(M_0, n)$. Then $H^n_G(Y; M)$ may be identified with the vertical G-homotopy classes of equivariant sections of $p : E \longrightarrow Y$. If now $u : Y \longrightarrow E$ is such a section and $p^* : E^* \longrightarrow X$ is the sectioned G-fibration over X induced from p via $f : X \longrightarrow Y$, then $f^*(u) \in H^n_G(X; f^*M)$ may be identified with the section $u^* : X \longrightarrow E^*$ of p^* defined by $u^*(x) = (x, u(f(x)))$. Then $f^*(u) = f^*(v)$ implies $u \circ f \sim_G v \circ f$, hence ($f$ being epi in $\mathcal{GHCW}^*$) $u \sim_G v$, that is, $f^* : H^n_G(Y; M) \longrightarrow H^n_G(X; f^*M)$ is a monomorphism. ■

Corollary 4.1 *If X, Y are G-simply connected G-complexes and $f : X \longrightarrow Y$ is an epi in $\mathcal{GHCW}^*$ and a mono in $\mathcal{GHCW}^\circ$, then f is a G-homotopy equivalence.* ■

In [3] the author has given an alternative proof of Theorem 1.1.The results in [3] can be generalized to equivariant setting in a functorial way by using the notion of universal O_G-covering space to obtain

Theorem 4.2 *If $f : X \longrightarrow Y$ is an epimorphism in $\mathcal{GHCW}^*$ where X, Y are $\mathcal{A}$-admissible, and $\underline{\pi}_k(f) : \underline{\pi}_k X \longrightarrow \underline{\pi}_k Y$ is a monomorphism for all $k \geq 0$ then f is an equivalence in $\mathcal{GHCW}^*$.* ■

Then Theorem 1.3 also follows from Theorem 4.2.

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