Harmonic morphisms, conformal foliations and shear-free ray congruences

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Abstract

A shear-free ray congruence is a foliation by null lines (light rays) of an open subset of Minkowski space satisfying a certain conformality condition. We show that (i) any real-analytic complex-valued harmonic morphism without critical points defined on an open subset of Minkowski space is conformally equivalent to the direction vector field of a shear-free ray congruence, (ii) any (real-analytic) complex-valued horizontally conformal submersion on an open subset of $\mathbb{R}^3$ is locally the boundary values at infinity of a harmonic morphism on an open subset of hyperbolic space. This provides a construction of families of minimal surfaces in hyperbolic 4-space with given boundaries at infinity.

1 Introduction

Harmonic morphisms are smooth (C^∞) maps between (semi-)Riemannian manifolds which preserve Laplace's equation in the sense that they pull back germs of harmonic functions to germs of harmonic functions. They can be characterized as harmonic maps which are *horizontally (weakly) conformal*. A smooth foliation on a Riemannian manifold is called *conformal* if Lie transport along the leaves of vectors in the normal space is conformal; locally, smooth conformal foliations are given by the fibres of horizontally conformal submersions.

A foliation by null lines (light rays) of an open subset of Minkowski space with tangent distribution V is called a *shear-free ray (SFR) congruence* if Lie transport

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along the leaves of vectors in a complement of V in $V^{\perp}$ (a *screen space*) is conformal (see below); equivalently the projection of V onto any $\mathbb{R}^3$-slice is tangent to a conformal foliation by curves. We show that the direction vector field of an SFR congruence is a harmonic morphism with values in the extended complex plane, and every real-analytic complex-valued harmonic morphism from an open subset of Minkowski space which has no critical points arises this way up to conformal equivalence (Theorem 2.3).

The boundary values at infinity of a complex-valued harmonic morphism from an open subset of hyperbolic 4-space are horizontally conformal. We show conversely that a real-analytic horizontally conformal map is locally the boundary values at infinity of a hyperbolic harmonic morphism. The construction involves only geometry and analytic continuation, no equations are solved. We interpret this construction twistorially and illustrate with an example.

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2 Harmonic morphisms and SFR congruences

Harmonic morphisms

For any (semi-)Riemannian manifolds (M^m, g), (N^n, h), a *harmonic morphism* $\phi : M^m \to N^n$ is a map which pulls back germs of harmonic functions on N^n to germs of harmonic functions on M^m. By [8, 13] for the Riemannian case and [9] for the semi-Riemannian case, these can be characterized as harmonic maps ϕ which are *horizontally weakly conformal*, i.e. at each point $p \in M$, the adjoint of $d\phi_p$ is conformal; for a complex-valued map $\phi = \phi^1 + \mathrm{i}\phi^2$ this is equivalent to the condition that the gradients of ϕ^1 and ϕ^2 be orthogonal and of the same length, i.e. that the complex gradient of ϕ be isotropic (null). In particular, let $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) : x_i \in \mathbb{R}\}$ with the standard Euclidean metric $g = \sum_{i=0}^{3} dx_i^2$ and let $\mathbb{M}^4 = \{(t, x_1, x_2, x_3) : t, x_i \in \mathbb{R}\}$ with the Minkowski metric $g^M = -dt^2 + \sum_{i=1}^{3} dx_i^2$. Then a smooth map $\phi : A^4 \to \mathbb{C}$ from an open subset of Euclidean space $\mathbb{R}^4$ is a harmonic morphism if and only if it satisfies Laplace's equation:

$$\Delta\phi \equiv \frac{\partial^2\phi}{\partial x_0^2} + \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = 0 \tag{1}$$

and the horizontal weak conformality condition:

$$\left(\frac{\partial\phi}{\partial x_0}\right)^2 + \left(\frac{\partial\phi}{\partial x_1}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2 + \left(\frac{\partial\phi}{\partial x_3}\right)^2 = 0\,, \tag{2}$$

whereas a smooth map $\phi : A^M \to \mathbb{C}$ from an open subset of Minkowski space $\mathbb{M}^4$ is a harmonic morphism if and only if it satisfies the *wave equation*:

$$\Box\phi \equiv -\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = 0 \tag{3}$$

and the horizontal weak conformality condition:

$$-\left(\frac{\partial\phi}{\partial t}\right)^2+\left(\frac{\partial\phi}{\partial x_1}\right)^2+\left(\frac{\partial\phi}{\partial x_2}\right)^2+\left(\frac{\partial\phi}{\partial x_3}\right)^2=0\,, \tag{4}$$

i.e. *a harmonic morphism from Minkowski space is a 'null' solution to the wave equation* (cf. [4]).

For a harmonic morphism $\phi : M^m \to N^n$ between (semi-)Riemannian manifolds there are three types of point $p \in M^m$:

(a) $d\phi_p = 0$; we call such points *critical points*;

(b) $d\phi_p \neq 0$ and $\ker d\phi_p$ is non-degenerate. Then $d\phi_p$ maps the *horizontal space* $(\ker d\phi_p)^\perp$ conformally onto $T_{\phi(p)}N$;

(c) $d\phi_p \neq 0$ and $\ker d\phi_p$ is degenerate. Then [9] $(\ker d\phi_p)^\perp \subset \ker d\phi_p$.

If the last case occurs we call ϕ *degenerate (at p)*; this case cannot occur if the domain manifold (M^m, g) is Riemannian. A simple example of a harmonic morphism $\phi : \mathbb{M}^4 \to \mathbb{C}$ which is degenerate everywhere is given by $\phi(t, x_1, x_2, x_3) = f(t - x_1)$ where $f : \mathbb{R} \to \mathbb{C}$ is any smooth function. Note that this has 1-dimensional image; in contrast, in the Riemannian case a harmonic morphism is always open [8]. For more theory and examples, see, for example [1, 22] for the Riemannian case, and [9] in the semi-Riemannian case. Note, in particular, in the Riemannian case, a smooth harmonic morphism is always real-analytic [7], but the above example shows that this is not so in the semi-Riemannian case.

If we include $\mathbb{R}^4$ in $\mathbb{C}^4$ by $(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_2, x_3)$ and $\mathbb{M}^4$ in $\mathbb{C}^4$ by $(t, x_1, x_2, x_3) \mapsto (-\mathrm{i}t, x_1, x_2, x_3)$ (the minus sign is unimportant and is just to avoid minus signs later on), the pairs of equations (1, 2) and (3, 4) both complexify to the pair (1, 2) where now $(x_0, x_1, x_2, x_3) \in \mathbb{C}^4$. We call a *holomorphic* map $\phi : A^C \to \mathbb{C}$ from an open subset of $\mathbb{C}^4$ which satisfies (1) a *complex-harmonic function* and one which also satisfies (2) a *complex-harmonic morphism*. We may easily adapt the argument of [13] to characterize complex-harmonic morphisms $A^C \to \mathbb{C}$ as those holomorphic maps which pull back germs of holomorphic functions to germs of complex-harmonic functions.

In all three cases, the equations for a harmonic morphism to a 2-dimensional codomain are conformally invariant in the codomain, i.e. if $\phi : A \to N^2$ is a harmonic morphism to a 2-dimensional Riemannian manifold and $\rho : N^2 \to N'^2$ is a weakly conformal map to another 2-dimensional Riemannian manifold, then the composition $\rho \circ \phi$ is a harmonic morphism. Thus, for example, the pair of equations (3,4) makes sense for a map to a Riemann surface.

Conformal foliations on $\mathbb{R}^3$

By a C^∞ *(resp.* C^ω*) non-zero vector field* on an open subset A^3 of $\mathbb{R}^3$ we mean a C^∞ (resp. C^ω) section $U : A^3 \to T\mathbb{R}^3 \setminus \{\text{zero section}\}$. Without loss of generality we may assume that U is of unit norm. To such a distribution corresponds a C^∞ (resp. C^ω) (oriented) foliation $\mathcal{C}$ of A^3 by curves given by integrating U. Note that U can be recovered from $\mathcal{C}$ as its field of (positive) unit tangents.

Let $U^\perp$ be the distribution of (oriented) subspaces of $T\mathbb{R}^3$ perpendicular to U. Then the distribution U is called *shear-free* and the corresponding foliation $\mathcal{C}$ *conformal* if Lie transport along U of vectors in $U^\perp$ is conformal. Let $J^\perp$ denote rotation through $+\pi/2$ on each oriented plane $U_p^\perp$ ($p \in A^3$); then U is shear-free if and only if $\mathcal{L}_U J^\perp = 0$ where $\mathcal{L}$ denotes Lie derivative. Now, for any $X \in C^\infty(U^\perp)$,

$$\begin{aligned}(\mathcal{L}_U J^\perp)(X) &= \{(\mathcal{L}_U(J^\perp X)\}^\perp - J^\perp\{\mathcal{L}_U(X)\}^\perp \\ &= \{\nabla_U(J^\perp X)\}^\perp - \nabla_{J^\perp X}U - J^\perp\{\nabla_U X\}^\perp + J^\perp \nabla_X U \qquad (5)\end{aligned}$$

where $\{\ \}^\perp$ denotes orthogonal projection onto $U^\perp$ (noting that, since $g(\nabla_X U, U) = \frac{1}{2}X\big(g(U,U)\big) = 0$, we have $\nabla_X U \in C^\infty(U^\perp)$). Further, since $U^\perp$ is a Hermitian connected bundle of rank 2, as for all such bundles we have

$$\{\nabla_U(J^\perp X)\}^\perp - J^\perp\{\nabla_U X\}^\perp = (\nabla_U^{\mathrm{End}\,U^\perp} J^\perp)(X) = 0\ ;$$

hence U is shear-free if and only if

$$\nabla_{J^\perp X} U = J^\perp \nabla_X U\ . \qquad (6)$$

A concrete way of obtaining conformal foliations is the following: Let $f : A^3 \to \mathbb{C}$ be a C^∞ (resp. C^ω) submersion from an open subset of $\mathbb{R}^3$. Then f is horizontally conformal if and only if

$$\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + \left(\frac{\partial f}{\partial x_3}\right)^2 = 0\ . \qquad (7)$$

Then we have the simple lemma (cf. [18]):

Lemma 2.1 *(i) If f is C^∞ (resp. C^ω) and horizontally conformal then the foliation defined by (the fibres of) f is C^∞ (resp. C^ω) and conformal.*

(ii) All C^∞ (resp. C^ω) conformal foliations are given locally in this way.

Hermitian structures

By an *almost Hermitian structure J_p at $p \in \mathbb{R}^4$* we mean an isometry $J_p : T_p\mathbb{R}^4 \to T_p\mathbb{R}^4$ with $J_p^2 = -I$. Given any orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $T_p\mathbb{R}^4_p$, setting $J_p(e_0) = e_1$, $J_p(e_2) = e_3$ defines an almost Hermitian structure J_p at p; we call J_p *positive* (resp. *negative*) according as $\{e_0, e_1, e_2, e_3\}$ is a positively (resp. negatively) oriented basis. By a *smooth almost Hermitian structure* on an open subset A^4 of $\mathbb{R}^4$ we mean a map J which assigns to each point p of A^4 an almost Hermitian structure at p in a smooth fashion, i.e. J defines a smooth section on A^4 of the bundle $E = \mathrm{End}(T\mathbb{R}^4) \to \mathbb{R}^4$. We call J *integrable* if there are local complex coordinates on A^4 with associated almost complex structure J, this is equivalent to the vanishing of the Nijenhuis tensor. A short calculation (see, e.g. [10, p. 42] or [17, p. 169]) shows that this is equivalent to

$$\nabla^E_{JX} J = J \nabla^E_X J \quad \forall X \in C^\infty_{A^4}(T\mathbb{R}^4)\,, \qquad (8)$$

i.e.

$$(\nabla^E_{JX} J)(Y) = J\Big((\nabla^E_X J)(Y)\Big) \quad \forall X, Y \in C^\infty_{A^4}(T\mathbb{R}^4)\,,$$

where ∇^E is the induced connection on the bundle $E = \mathrm{End}(T\mathbb{R}^4)$ given by the formula $(\nabla^E_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y)$. Such a J is always real-analytic.

Let $\sigma : S^2 \to \mathbb{C} \cup \{\infty\}$ denote stereographic projection from $(-1, 0, 0)$. Given a smooth almost Hermitian structure J on A^4, set

$$U = J(\partial/\partial x_0) \quad \text{and} \quad \mathrm{i}\mu = \sigma \circ U\,; \tag{9}$$

we shall say that $\mu : A^4 \to \mathbb{C} \cup \{\infty\}$ *represents* J. Then, in local coordinates, writing $z_1 = x_0 + \mathrm{i}x_1$, $z_2 = x_2 + \mathrm{i}x_3$, the equations (8) read (see, for example, [21])

$$\left(\frac{\partial}{\partial \overline{z}_1} - \mu \frac{\partial}{\partial z_2}\right)\mu = 0\,, \qquad \left(\frac{\partial}{\partial \overline{z}_2} + \mu \frac{\partial}{\partial z_1}\right)\mu = 0\,. \tag{10}$$

Let J be a Hermitian structure on an open subset A^4 of $\mathbb{R}^4$ and set $U = J(\partial/\partial x_0)$. Then, by comparing equations (6) and (8) we see that the restriction of U to any $\mathbb{R}^3$-slice: $x_0 =$ const. is shear-free and so its integral curves form a conformal foliation.

Shear-free ray congruences

Let ℓ be a smooth foliation of an open subset A^M of Minkowski space by null lines, let $v = \partial/\partial t + U$ (U a unit vector in $\mathbb{R}^3$) be its tangent vector field and write $V = \operatorname{span}\{v\}$. The distribution $V^\perp$ orthogonal to V (with respect to the Minkowski metric g^M) is three-dimensional and contains V. Choose any complement Σ of V in $V^\perp$; such a complement is called a *screen space*; then the restriction of the Minkowski metric g^M to Σ is positive definite. Let $J \in C^\infty(\operatorname{End}\Sigma)$ denote rotation through $\pi/2$. Then ℓ (or V) is said to be a *shear-free ray (SFR) congruence* [6, 12, 16] if Lie transport along V of vectors in Σ is conformal, i.e. $\mathcal{L}_v J^\perp = 0$. On calculating the Lie derivative in a similar way to (5) we see that this is equivalent to

$$(\nabla_{J^\perp X} v)^\Sigma = J^\perp (\nabla_X v)^\Sigma \text{ for all } X \in C^\infty(\Sigma) \tag{11}$$

where $(\ )^\Sigma$ indicates projection onto Σ along V; this condition is independent of the choice of screen space.

Comparing equations (11) and (6) shows that the restriction of U to any $\mathbb{R}^3$-slice: $t =$ const. is shear-free and so its integral curves form a conformal foliation. Conversely, given a shear-free vector field U on an open set A^3 of an $\mathbb{R}^3$-slice, the null lines of $\mathbb{M}^4$ tangent to $\partial/\partial t + U$ at points of A^3 define an SFR congruence ℓ on an open neighbourhood of A^3 in $\mathbb{M}^4$. We shall say that ℓ *extends* U.

Write

$$\mathrm{i}\mu = \sigma \circ U\,; \tag{12}$$

we shall say that μ *represents* ℓ. Then the shear-free condition is expressed ([14], [16, II (7.4.6)], [12, p. 50]) by

$$\left(\frac{\partial}{\partial u} - \mu \frac{\partial}{\partial z_2}\right)\mu = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial \overline{z}_2} + \mu \frac{\partial}{\partial u'}\right)\mu = 0\,, \tag{13}$$

where $u = x_1 - t$ and $u' = x_1 + t$. It quickly follows from these equations that *if ℓ is a shear-free congruence then the map $\mu : A^M \to \mathbb{C} \cup \{\infty\}$ representing it is a harmonic morphism.*

Using coordinates $z_1 = x_0 + \mathrm{i}x_1,\ \widetilde{z}_1 = x_0 - \mathrm{i}x_1,\ z_2 = x_2 + \mathrm{i}x_3,\ \widetilde{z}_2 = x_2 - \mathrm{i}x_3$ on $\mathbb{C}^4$, so that $\mathbb{R}^4$ is given by $\widetilde{z}_1 = \overline{z}_1, \widetilde{z}_2 = \overline{z}_2$, we see that both these equations and the equations (10) are restrictions of the equations in $\mathbb{C}^4$:

$$\left(\frac{\partial}{\partial \widetilde{z}_1} - \mu \frac{\partial}{\partial z_2}\right)\mu = 0\ , \quad \left(\frac{\partial}{\partial \widetilde{z}_2} + \mu \frac{\partial}{\partial z_1}\right)\mu = 0\ . \tag{14}$$

We remark that these equations express the condition that a holomorphic distribution of 2-dimensional null subspaces integrates to a foliation by null planes. Thus, given such a foliation, (i) for any $\mathbb{R}^4$-slice $\{(x_0, x_1, x_2, x_3) \in \mathbb{C}^4 : \mathrm{Im}\, x_i = \text{const.}\ (i = 0, 1, 2, 3)\}$ the almost Hermitian structure having these null subspaces as $(1, 0)$-tangent spaces is integrable, (ii) the foliation given by the intersection of the null planes with any Minkowski slice $\{(x_0, x_1, x_2, x_3) \in \mathbb{C}^4 : \mathrm{Re}\, x_0 = \text{const.}, \mathrm{Im}\, x_i = \text{const.}\ (i = 1, 2, 3)\}$ is a shear-free ray congruence.

Harmonic morphisms and shear-free ray congruences

In [21, Theorem 1.1] the following result is given:

Theorem 2.2 *Let $\phi : A^R \to N^2$ be a harmonic morphism without critical points from an open subset A^R of $\mathbb{R}^4$ to a Riemann surface. Then there exists a Hermitian structure J on A^R such that J is parallel along each connected component of the fibres of ϕ. Further, for any $p \in A^R$, there is a neighbourhood A_1^R of p in A^R and a holomorphic map $\rho : V \to \mathbb{C} \cup \{\infty\}$ from an open subset V of N^2 such that $\mu = \rho \circ \phi$ represents J on A_1^R.*

We have the following analogue in Minkowski signature:

Theorem 2.3 *Let $\phi : A^M \to N^2$ be a real-analytic harmonic morphism without critical points from an open subset A^M of Minkowski space $\mathbb{M}^4$ to a Riemann surface. Then there is a shear-free ray congruence ℓ on A^M such that each connected component of a fibre of ϕ is the union of parallel null lines of ℓ. Further, for any $p \in A^M$, there is a neighbourhood A_1^M of p in A^M, and a holomorphic map $\rho : V \to \mathbb{C} \cup \{\infty\}$ from an open subset V of N^2 such that $\mu = \rho \circ \phi$ represents ℓ on A_1^M.*

Proof: Let $p \in A^M$. By real-analyticity we may extend ϕ to an open neighbourhood A^C of p in $\mathbb{C}^4$ and then restrict it to an open subset A^4 of the $\mathbb{R}^4$-slice through p. This restriction is a harmonic morphism with respect to the Euclidean metric. Therefore, by Theorem 2.2, it is holomorphic with respect to some Hermitian structure J which is constant along each connected component of a fibre of ϕ. Replacing z_2 by $\widetilde{z}_2$ if necessary, we can assume that J is positively oriented. Representing J by $\mu : A^4 \to \mathbb{C} \cup \{\infty\}$ as in (9) then μ and ϕ satisfy (14) and

$$\left(\frac{\partial}{\partial \widetilde{z}_1} - \mu \frac{\partial}{\partial z_2}\right)\phi = 0\,, \quad \left(\frac{\partial}{\partial \widetilde{z}_2} + \mu \frac{\partial}{\partial z_1}\right)\phi = 0\,, \tag{15}$$

at points of A^4, (14) expressing integrability of J and (15) holomorphicity of ϕ with respect to J (cf. [21]). Extend μ to A^C by (15), noting that it is well-defined since not all the partial derivatives of ϕ can vanish simultaneously; μ is then a holomorphic function which we restrict to a neighbourhood A^M of p in $\mathbb{M}^4$. By analytic continuation, (14) holds at all points of A^M so that μ represents a shear-free ray congruence. By (15) ϕ is constant along any ray of the congruence so that each fibre of ϕ is the union of such rays. Further, J, and so μ, is constant along the connected components of fibres of $\phi|A^4$; by analytic continuation, $\mu : A^M \to \mathbb{C}\cup\{\infty\}$ is constant along the connected components of the fibres of $\phi : A^M \to \mathbb{C} \cup \{\infty\}$ so that the null lines of the congruence making up a connected component of a fibre of ϕ are all parallel.

Lastly, since μ is constant on the leaves of the foliation given by the fibres of ϕ, it factors through local leaf spaces as $\mu = \rho \circ \phi$. Since ϕ and μ are both holomorphic (with respect to i and J), ρ must be holomorphic.

Remarks 2.4 (i) If the fibres of ϕ are totally geodesic, there are two SFR congruences corresponding to the two Hermitian structures of [21, Theorem 1.1], otherwise there is just one.

(ii) There is a version of Theorem 2.3 for *complex*-harmonic morphisms replacing 'null line' by 'null plane' and 'shear-free ray congruence' by 'holomorphic foliation by null planes'; indeed the function $\mu : A^C \to \mathbb{C}$ in the proof defines such a foliation.

Note that, in the Euclidean case, the condition 'without critical points' is equivalent to 'ϕ is submersive'. This is not so in the Minkowski case where ϕ may be degenerate. We can actually be more precise in that case:

Corollary 2.5 *Let $\phi : A^M \to N$ be a real-analytic harmonic morphism from an open subset A^M of $\mathbb{M}^4$. Suppose that ϕ is degenerate at p with $d\phi_p \neq 0$. Then there exists a unique null direction $V_p \in T_p\mathbb{M}^4$ such that $V_p \subset \ker d\phi_p$. Furthermore, $\ker d\phi_p = V_p^{\perp}$. If, further, at each point q in the connected component of the fibre through p, ϕ is degenerate with $d\phi_q \neq 0$, then that connected component is the affine null 3-space tangent to $V_p^{\perp}$.*

Proof: By [9], $(\ker d\phi_p)^{\perp} \subset \ker d\phi_p$. This means that $\ker d\phi_p$ must be three-dimensional. But then $(\ker d\phi_p)^{\perp}$ is 1-dimensional and null. Set $V_p = (\ker d\phi_p)^{\perp}$. Then $V_p \subset \ker d\phi_p$ and $\ker d\phi_p = V_p^{\perp}$.

To prove uniqueness of V_p, suppose that $V_p' \subset \ker d\phi_p$ is another null direction. Then $V_p' \subset V_p^{\perp}$ which is easily seen to imply that $V_p' = V_p$.

This means that the distribution $p \mapsto V_p$ must be tangent to the SFR congruence of the theorem, and so each V_p is parallel for all p in a connected component of a fibre. The last assertion follows from the fact that the connected component of the fibre is 3-dimensional and has every tangent space parallel to V_p.

Examples 2.6 (i) The simplest examples of Minkowski harmonic morphisms from $\mathbb{M}^4$ to $\mathbb{C}$ are given by (a) $(t, x_1, x_2, x_3) \mapsto x_2 + \mathrm{i}x_3$ which is non-degenerate everywhere and surjective and (b) $(t, x_1, x_2, x_3) \mapsto x_1 - t$ which is degenerate everywhere and has 1-dimensional image $\mathbb{R} \subset \mathbb{C}$. Note that in both cases, the fibres are totally geodesic; in (a) the SFR congruences of Theorem 2.3 (two by Remark 2.4(i)) have leaves with (null) directions $(1, \pm 1, 0, 0)$, in (b) the SFR congruence (just one by Corollary 2.5) has leaves with (null) direction $(1, 1, 0, 0)$.

(ii) Let $f(x_1, x_2, x_3) = -\mathrm{i}x_1 \pm \sqrt{x_2^2 + x_3^2}$. This is a horizontally conformal submersion from $A^3 = \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) : x_2 = x_3 = 0\}$ to $\mathbb{C}$. Its level curves are circles in planes parallel to the (x_2, x_3)-plane and centred on points of the x_1-axis; these give a conformal foliation $\mathcal{C}$ of A^3 whose tangent vector field is the shear-free unit vector field $U : A^3 \to S^2$ given by

$$U(x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_2^2 + x_3^2}}(0, -x_3, x_2)\ . \tag{16}$$

Note that U has 1-dimensional image — the equator of S^2. We compute the tangent vector field to the SFR congruence ℓ extending U. The affine null geodesic of ℓ in $\mathbb{M}^4$ through (x_1, x_2, x_3) with direction $\partial/\partial t + U$ is given parametrically by

$$\begin{aligned} T \quad &\mapsto \quad \left(T, x_1, x_2 + T\left(\frac{-x_3}{\sqrt{x_2^2 + x_3^2}}\right), x_3 + T\left(\frac{x_2}{\sqrt{x_2^2 + x_3^2}}\right)\right) \\ &= \quad (T, X_1, X_2, X_3)\ , \text{ say.} \end{aligned} \tag{17}$$

Conversely, given $(T, X_1, X_2, X_3) \in \mathbb{M}^4$, the null geodesic of ℓ hits the $\mathbb{R}^3$-slice: $t = 0$ at (x_1, x_2, x_3) where

$$x_1 = X_1, \quad x_2 + T\left(\frac{-x_3}{\sqrt{x_2^2 + x_3^2}}\right) = X_2, \quad x_3 + T\left(\frac{x_2}{\sqrt{x_2^2 + x_3^2}}\right) = X_3\ .$$

Solving this gives

$$(x_1, x_2, x_3) = \left(X_1, \frac{R}{X_2^2 + X_3^2}(RX_2 + TX_3), \frac{R}{X_2^2 + X_3^2}(RX_3 - TX_2)\right) \tag{18}$$

where $R = \sqrt{X_2^2 + X_3^2 - T^2}$. Hence the tangent to the null geodesic of ℓ through $(t, x_1, x_2, x_3) \in \mathbb{M}^4$ is given by $v = \partial/\partial t + U$ with

$$U = U_t(x_1, x_2, x_3) = U(t, x_1, x_2, x_3) = \frac{r}{\sqrt{x_2^2 + x_3^2}}\left(0, -x_3 + \frac{t}{r}x_2, x_2 + \frac{t}{r}x_3\right) \tag{19}$$

where $r = \sqrt{x_2^2 + x_3^2 - t^2}$.

Of course, again the image of U is the equator of S^2. For each t, the integral curves of U_t give a conformal (in fact, Riemannian) foliation $\mathcal{C}_t$ of the $\mathbb{R}^3$-slice $t =$ const.; it is easy to see that the leaves of $\mathcal{C}_t$ are the involutes of circles (see [5]).

The vector field U defines a Minkowski harmonic morphism from the cone given by $A^M = \{(t, x_1, x_2, x_3) \in \mathbb{M}^4 : x_2^2 + x_3^2 > t^2\}$ to S^2 which is degenerate everywhere and has image the equator of S^2. Note that $U = \sigma(\mathrm{i}\mu)$ with

$$\mu(t, x_1, x_2, x_3) = \frac{r}{x_2^2 + x_3^2}\left\{\left(x_2 + \frac{t}{r}x_3\right) + \mathrm{i}\left(x_3 - \frac{t}{r}x_2\right)\right\}$$

giving a harmonic morphism $\mu : A^M \to \mathbb{C}$, also degenerate, with image the unit circle. The fibre of U (or μ) through any point p is the affine plane perpendicular to U_p; this is spanned by U_p, $\partial/\partial x_1$ and the vector in the (x_2, x_3)-plane perpendicular to U_p .

3 Harmonic morphisms from hyperbolic space and conformal foliations

We wish to construct harmonic morphisms from hyperbolic space with given boundary values at infinity; since such maps are always holomorphic with respect to some Hermitian structure (see below), we first examine the boundary values of a holomorphic map. For any $p = (p_0, p_1, p_2, p_3) \in \mathbb{R}^4$, denote the $\mathbb{R}^3$-slice: $x_0 = p_0$ through p by $\mathbb{R}^3_p$. (If $p = (0,0,0,0)$ we shall just write $\mathbb{R}^3$.)

Proposition 3.1 *Let $\phi : A^4 \to \mathbb{C}$ be a submersive map from an open subset of $\mathbb{R}^4$ which is holomorphic with respect to a positive Hermitian structure J on A^4. Let $p \in A^4$ and let U be the shear-free vector field on the open subset $A^3 = A^4 \cap \mathbb{R}^3_p$ of $\mathbb{R}^3_p$ defined by $U = J(\partial/\partial x_0)$. Denote the foliation of A^3 given by its integral curves by $\mathcal{C}$. If*

$$\frac{\partial \phi}{\partial x_0} = 0 \quad \text{on} \quad A^3 \tag{20}$$

then $f = \phi | A^3$ is a real-analytic horizontally conformal submersion which is constant on the leaves of $\mathcal{C}$.

Proof: Let $q \in A^3$. By holomorphicity, since $U_q = J_q(\partial/\partial x_0)$, the directional derivative $U_q(f) = 0$ so that f is constant on the leaves of $\mathcal{C}$. If $\{e_2, e_3 = Je_2\}$ is a basis for $U_q^\perp \cap T_q\mathbb{R}^3_p$, holomorphicity of ϕ implies that $e_3(f) = \mathrm{i}\, e_2(f)$ so that f is horizontally conformal. Submersivity of f easily follows from that of ϕ.

We now interpret such ϕ. Equip $\breve{\mathbb{R}}^4 \equiv \mathbb{R}^4 \setminus \mathbb{R}^3$ with the *hyperbolic metric* $g^H = \left(\sum_{i=0}^3 dx_i^2\right)/x_0^2$ so that each component $\mathbb{R}^4_+$, $\mathbb{R}^4_-$ is isometric to hyperbolic 4-space $\mathbb{H}^4$ with $\mathbb{R}^3$ as boundary at infinity. Let $\breve{A}^4$ be an open subset of $\breve{\mathbb{R}}^4$, then we call a smooth map $\phi : \breve{A}^4 \to \mathbb{C}$ a *hyperbolic harmonic function* if it is harmonic with respect to the hyperbolic metric g^H. This holds if and only if

$$x_0 \sum_{i=0}^{3} \frac{\partial^2 \phi}{\partial x_i^2} - 2\frac{\partial \phi}{\partial x_0} = 0 \tag{21}$$

at all points of $\breve{A}^4$.

Similarly, $\pi : \breve{A}^4 \to \mathbb{C}$ will be called a *hyperbolic harmonic morphism* if it is a harmonic morphism with respect to the metric g^H, by [8, 13] such maps are characterized as solutions to the pair (21, 2). Equation (21) shows that a hyperbolic harmonic morphism which is, say, C^2 up to the boundary satisfies (20). We now find a converse. Recall ([21], see also [2])

Theorem 3.2 *(i) Any submersive hyperbolic harmonic morphism $\phi : \breve{A}^4 \to \mathbb{C}$ is holomorphic with respect to some Hermitian structure J on $\breve{A}^4$ and has superminimal fibres with respect to J, i.e. $\ker d\phi \subset \ker \nabla^H J$ on $\breve{A}^4$ where ∇^H is induced by the Levi-Civita connection of the hyperbolic metric on $\breve{\mathbb{R}}^4$.*

(ii) Conversely, let J be a Hermitian structure on an open subset $\breve{A}^4$ of $\breve{\mathbb{R}}^4$, and $\phi : \breve{A}^4 \to \mathbb{C}$ a non-constant map which is holomorphic with respect to J. Then ϕ is hyperbolic harmonic if and only if, at points where $d\phi \neq 0$, its fibres are superminimal with respect to J.

To formulate this analytically, consider the twistor space $\pi : \mathbb{C}P^3 \setminus \mathbb{C}P^1_0 \to \mathbb{R}^4 = \mathbb{C}^2$ where $\mathbb{C}P^1_0 = \{[w_0, w_1, w_2, w_3] : w_0 = w_1 = 0\}$ and π is given by $\pi[w_0, w_1, w_2, w_3] = (z_1, z_2)$ with $\widetilde{z}_1 = \overline{z}_1$, $\widetilde{z}_2 = \overline{z}_2$ and

$$w_0 z_1 - w_1 \widetilde{z}_2 = w_2 \qquad w_0 z_2 + w_1 \widetilde{z}_1 = w_3 . \tag{22}$$

When $(z_1, \widetilde{z}_1, z_2, \widetilde{z}_2)$ represents a point in $\mathbb{C}^4$, the equations (22) are the well-known incidence relations for the twistor correspondence (see, e.g. [19]). Recall the Kerr Theorem [16]: Given a complex surface $\psi(w_0, w_1, w_2, w_3) = 0$ in $\mathbb{C}P^3 \setminus \mathbb{C}P^1_0$, local smooth solutions to

$$\psi(w_0, w_1, w_0 z_1 - w_1 \widetilde{z}_2, w_0 z_2 + w_1 \widetilde{z}_1) = 0 \tag{23}$$

satisfy (14) and all solutions of (14) are given this way locally. We call S the *twistor surface* of μ (or of the quantity J, ℓ etc. it represents). Now, a solution $\mu : A^4 \to \mathbb{C}$ to (10) on an open subset of $\mathbb{R}^4$ represents a Hermitian structure J on A^4, this defines a section w of the twistor bundle with image an open subset of S. Set

$$N^5 = \pi^{-1}(\mathbb{R}^3) = \{[w_0, w_1, w_2, w_3] : w_0 \overline{w}_2 + \overline{w}_0 w_2 + w_1 \overline{w}_3 + \overline{w}_1 w_3 = 0\} \subset \mathbb{C}P^3 .$$

Let Θ be the homogeneous holomorphic contact form

$$\Theta = w_1 dw_2 - w_2 dw_1 - w_0 dw_3 + w_3 dw_0 \tag{24}$$

on $\mathbb{C}P^3$. Then $\ker \Theta$ gives the horizontal spaces of the restriction of the twistor projection: $\pi : \mathbb{C}P^3 \setminus N^5 \to (\breve{\mathbb{R}}^4, g^H)$. Let $\phi : A^4 \to \mathbb{C}$ be holomorphic with respect to J, equivalently $\Phi = \phi \circ \pi$ is holomorphic on an open subset of S. Then ϕ is a hyperbolic harmonic morphism if and only if its fibres are superminimal, i.e. the fibres of Φ are horizontal, this is expressed analytically by

$$\ker d\Phi \subset \ker \Theta . \tag{25}$$

Proposition 3.3 *Let A^4 be a connected open subset of $\mathbb{R}^4$, $\mathbb{R}^4_+ \cup \mathbb{R}^3$ or $\mathbb{R}^4_- \cup \mathbb{R}^3$ with $A^3 = A^4 \cap \mathbb{R}^3$ non-empty, and let $\phi : A^4 \to \mathbb{C}$ be a non-constant C^1 map which is holomorphic with respect to a Hermitian structure J on $\breve{A}^4 = A^4 \setminus \mathbb{R}^3$ and submersive at almost all points of A^3. Then ϕ satisfies (20) on A^3 if and only if $\phi|\breve{A}^4$ is a hyperbolic harmonic morphism.*

Proof: It suffices to work at points where ϕ is submersive. At such points note that (20) holds if and only if $\ker \phi = \text{span}\{\partial/\partial x_0, J\partial/\partial x_0\}$. Now let S be the twistor surface of J and let $\Phi : S \to \mathbb{C}$ be defined by $\Phi = \phi \circ \pi$. We show that the pull-back $\theta = w^* \Theta$ to A^4 satisfies

$$\text{span}\{\partial/\partial x_0, J\partial/\partial x_0\} \subset \ker \theta \tag{26}$$

at all points of A^3. Since the (complexified) normal to $\mathbb{R}^3$ is given by the annihilator of $\text{span}\{dz_1 - d\widetilde{z}_1, dz_2, d\widetilde{z}_2\}$, it suffices to show that on $\mathbb{R}^3 = \{z_1 + \widetilde{z}_1 = 0\}$, θ is a linear combination of those three forms. To do this, taking differentials in (22) gives

$$\begin{aligned} dw_2 &= z_1 dw_0 + w_0 dz_1 - \widetilde{z}_2 dw_1 - w_1 d\widetilde{z}_2 , \\ dw_3 &= z_2 dw_0 + w_0 dz_2 + \widetilde{z}_1 dw_1 + w_1 d\widetilde{z}_1 . \end{aligned}$$

Substituting these into (24) and rearranging gives

$$\theta = (w_3 - w_0 z_2 + w_1 z_1)dw_0 - (w_2 + w_0\widetilde{z}_1 + w_1\widetilde{z}_2)dw_1 + w_0 w_1(dz_1 - d\widetilde{z}_1) - w_0^2 dz_2 - w_1^2 d\widetilde{z}_2 \,.$$

By (22), the coefficients of dw_0 and dw_1 vanish when $z_1 + \widetilde{z}_1 = 0$ so that (26) follows.

Thus condition (20) is equivalent to the superminimality of the fibres of ϕ at points of A^3, i.e. $\ker d\phi \subset \ker\theta$, or, equivalently, $\ker d\Phi \subset \ker\Theta$ on the real hypersurface $N^3 = w(A^3)$ of S. But this is a holomorphic condition, so by analytic continuation, if ϕ has superminimal fibres at points of A^3 then it has superminimal fibres on the whole of A^4 and, on applying Theorem 3.2, we are done.

Theorem 3.4 *Let $f : A^3 \to \mathbb{C}$ be a real-analytic horizontally conformal submersion on an open subset of $\mathbb{R}^3$. Then there is an open subset A^4 of $\mathbb{R}^4$ with $A^4 \cap \mathbb{R}^3 = A^3$ and a real-analytic submersion $\phi : A^4 \to \mathbb{C}$ with $\phi|A^3 = f$ such that $\phi|A^4 \setminus \mathbb{R}^3$ is a hyperbolic harmonic morphism. In fact $\phi \mapsto f = \phi|A^3$ defines a bijective correspondence between germs at A^3 of real-analytic submersions $\phi : A^4 \to \mathbb{C}$ on open neighbourhoods of A^3 in $\mathbb{R}^4$ which are hyperbolic harmonic on $A^4 \setminus \mathbb{R}^3$ and real-analytic horizontally conformal submersions $f : A^3 \to \mathbb{C}$.*

Proof: Let $\mathcal{C}$ be the conformal foliation on A^3 given by the level sets of f and let U be its unit tangent vector field given by $U = \operatorname{grad} f_1 \times \operatorname{grad} f_2 / |\operatorname{grad} f_1 \times \operatorname{grad} f_2|$ where $f = f_1 + \mathrm{i} f_2$. Let J be the unique positive almost Hermitian structure on A^3 with $U = J(\partial/\partial x_0)$ and set $\phi = f$. Then, as in Section 2, the null lines tangent to the vectors $\partial/\partial t + U$ define a shear-free ray congruence ℓ on some open neighbourhood A^M of A^3 in $\mathbb{M}^4$; we extend J and ϕ to A^M by making them constant along the leaves of ℓ; this means that they satisfy the restriction of equations (14) to A^M. We then extend them to an open neighbourhoood of A^M in $\mathbb{C}^4$ by analytic continuation, i.e. by insisting that they be complex-analytic, and finally we restrict to $\mathbb{R}^4$, thus extending J and ϕ to an open neighourhood A^4 of A^3 in $\mathbb{R}^4$. Then they satisfy (10) and the restriction of (15), hence J is a Hermitian structure on A^4 and ϕ is holomorphic with respect to J, so $\partial\phi/\partial x_0 = -\mathrm{i}\, U(\phi) = 0$. By Proposition 3.3, ϕ is hyperbolic harmonic.

Remarks 3.5 (i) The hyperbolic harmonic morphism ϕ has totally geodesic fibres if and only if the level sets of f are circles, see [3].

(ii) The extension of f can also be described in a twistorial way, as follows: U defines a 3-dimensional CR submanifold N^3 of N^5 which is the intersection of N^5 and a complex surface S of $\mathbb{C}P^3$. This defines the extension of J to a neighbourhood of A^3 in $\mathbb{R}^4$. Now f defines a CR function $F = f \circ \pi$ on N^3 which we may extend to a holomorphic function Φ on a neighbourhood of N^3 in S; on a possibly smaller neighbourhood, Φ is of the form $\phi \circ \pi$ for a unique function ϕ on an open subset of $\mathbb{R}^4$. This is the desired extension.

(iii) If f is only C^∞ but the normal distribution to the foliation $\mathcal{C}$ is nowhere integrable, then, as in [16, II, p.220 ff.], the CR manifold $N^3 \subset \mathbb{C}P^3$ has non-zero Levi form and so, by a theorem of Harvey and Lawson [11], N^3 is the boundary of a complex hypersurface S. Thus we can extend J and f to one side of $\mathbb{R}^3$: precisely, there is an open subset A^4 of $\mathbb{R}^4_+ \cup \mathbb{R}^3$ or $\mathbb{R}^4_- \cup \mathbb{R}^3$ with $A^4 \cap \mathbb{R}^3 = A^3$ and a C^∞ map $\phi : A^4 \to \mathbb{C}$ with $\phi|A^3 = f$ such that $\phi|A^4 \setminus \mathbb{R}^3$ is a hyperbolic harmonic morphism.

Corollary 3.6 *(i) Let $\mathcal{C}$ be a real-analytic conformal foliation by curves of an open subset of $\mathbb{R}^3$. Then there is a real-analytic foliation of an open subset A^4 of $\mathbb{R}^4$ by surfaces which are minimal in $A^4 \setminus \mathbb{R}^4$ with respect to the hyperbolic metric and intersect $\mathbb{R}^3$ in leaves of $\mathcal{C}$.*

(ii) Let c be a embedded real-analytic curve in $\mathbb{R}^3$. Then there is an embedded real-analytic surface s in an open subset A^4 of $\mathbb{R}^4$ which is minimal in $A^4 \setminus \mathbb{R}^3$ with respect to the hyperbolic metric and intersects $\mathbb{R}^3$ in c.

Proof: (i) Representing the leaves of $\mathcal{C}$ as the level curves of a real-analytic horizontally conformal submersion $f : A^3 \to \mathbb{C}$ on an open subset of $\mathbb{R}^3$, construct a hyperbolic harmonic morphism ϕ as in the theorem: its fibres give the desired foliation.

(ii) Embed c in a real-analytic conformal foliation by curves of an open subset of $\mathbb{R}^3$ as follows: construct the normal planes to c and integrate the vector field given by the normals to these. This gives a foliation on an open neighbourhood of c in $\mathbb{R}^3$ which has totally geodesic integrable horizontal spaces and so (see, for example, [20]) is Riemannian. (To get a conformal foliation which is not Riemannian, replace the planes by spheres, possibly of varying radii.) Now apply (i).

Remark 3.7 It is easily seen from the equations that any C^1 surface in an open subset A^4 of $\mathbb{R}^4$, $\mathbb{R}^4_+ \cup \mathbb{R}^3$ or $\mathbb{R}^4_- \cup \mathbb{R}^3$ which is minimal with respect to the hyperbolic metric on $\breve{A}^4 = A^4 \setminus \mathbb{R}^3$ hits $\mathbb{R}^3$ orthogonally.

Example 3.8 Let f be the horizontally conformal submersion of Example 2.6 (ii). Recall that its level sets are given by the leaves of the conformal foliation $\mathcal{C}$ with tangent vector field U given by (16). The extension of this to a shear-free ray congruence ℓ is described by (17). As in the proof of Theorem 3.4, extend f to a function ϕ on an open subset of $\mathbb{M}^4$ by insisting that it be constant along the leaves of ℓ. Using (18) we see that this function is

$$\begin{aligned}\phi(t, x_1, x_2, x_3) &= f\left(x_1, \frac{r}{x_2^2 + x_3^2}(rx_2 + tx_3), \frac{r}{x_2^2 + x_3^2}(rx_3 - tx_2)\right) \\ &= \mathrm{i}x_1 + \sqrt{x_2^2 + x_3^2 - t^2}\end{aligned}$$

which is smooth on the cone $x_2^2 + x_3^2 > t^2$. This extends by analytic continuation to the function

$$\phi(x_0, x_1, x_2, x_3) = \mathrm{i}x_1 + \sqrt{x_0^2 + x_2^2 + x_3^2} \qquad (27)$$

which is a complex analytic function on suitable domains of $\mathbb{C}^4$. Its restriction to $\mathbb{R}^4$ is smooth on $\mathbb{R}^4 \setminus \{x_1\text{-axis}\}$ and defines the hyperbolic harmonic morphism ϕ with boundary values at infinity given by f.

We now discuss this example from the twistorial viewpoint.

Choose as twistor surface the quadratic surface $S = \{[w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 : w_0 w_3 + w_1 w_2 = 0\}$. Then equation (23) reads

$$z_2 + \mu\widetilde{z}_1 + \mu(z_1 - \mu\widetilde{z}_2) = 0$$

which has solutions

$$\mu = \frac{z_1 + \widetilde{z}_1 \pm \sqrt{(z_1 + \widetilde{z}_1)^2 + 4z_2\widetilde{z}_2}}{2\widetilde{z}_2} = \frac{x_0 \pm s}{x_2 - \mathrm{i}x_3}$$

where $s = \sqrt{x_0^2 + x_2^2 + x_3^2}$. Note that

$$\mu|\mathbb{R}^3 = \pm\frac{\sqrt{x_2^2 + x_3^2}}{x_2 - \mathrm{i}x_3} = \pm\frac{x_2 + \mathrm{i}x_3}{\sqrt{x_2^2 + x_3^2}}$$

so that, on $\mathbb{R}^3$, $U = \sigma^{-1}(\mathrm{i}\mu)$ is given by (16) and is the tangent vector field of the conformal foliation $\mathcal{C}$ discussed above. Further, note that $\mu|\mathbb{M}^4$ is given by

$$\begin{aligned}\mu(t, x_1, x_2, x_3) &= \frac{-\mathrm{i}t \pm r}{x_2 - \mathrm{i}x_3}\\ &= \frac{r}{x_2^2 + x_3^2}\left\{\left(\pm x_2 + \frac{t}{r}x_3\right) + \mathrm{i}\left(\mp x_3 - \frac{t}{r}x_2\right)\right\}\end{aligned}$$

where $r = \sqrt{x_2^2 + x_3^2 - t^2}$ and so, on the open set $x_2^2 + x_3^2 > t^2$, U is given by (19) and $v = U + \partial/\partial t$ gives the tangent field of the shear-free ray congruence ℓ.

To find hyperbolic harmonic morphisms, we parametrize S away from $w_0 = 0$ by $(\zeta, \eta) \mapsto [1, w_1, w_2, w_3] = [1, \eta, -\zeta, \zeta\eta]$, then the incidence relations (22) read

$$\left.\begin{aligned}z_1 - \eta\widetilde{z}_2 &= -\zeta\\ z_2 + \eta\widetilde{z}_1 &= \zeta\eta\end{aligned}\right\}$$

which have solutions

$$\zeta = -\mathrm{i}x_1 \pm s \quad \text{and} \quad \eta = \frac{x_0 \pm s}{x_2 - \mathrm{i}x_3}\,;$$

these describe the section w of the twistor space with image S. In terms of (ζ, η), we have

$$\Theta = w_1 dw_2 - w_2 dw_1 - dw_3 = -\eta d\zeta + \zeta d\eta - \zeta d\eta - \eta d\zeta = -2\eta d\zeta$$

so that $\Theta(\frac{\partial}{\partial\eta}) = 0$ which means that the level surfaces $\zeta = \text{const.}$ are superminimal. This implies that ζ defines a hyperbolic harmonic morphism ϕ whose level sets on the $\mathbb{R}^3$-plane at infinity are the leaves of $\mathcal{C}$, namely the map given by (27) above. Note that this harmonic morphism has fibres given by the Euclidean spheres having these circles as diameters, these spheres are totally geodesic in $(\breve{R}^4, g^H)$.

If we introduce a parameter $a = (a_0, a_1, a_2, a_3) \in \mathbb{C}^4$ we can find the hyperbolic harmonic morphism whose level sets at infinity are the leaves of the foliation $\mathcal{C}_t$ of Example 2.6 (ii). For setting

$$\begin{aligned}\Theta_a &= 2a_0 dw_1 + w_1 dw_2 - w_2 dw_1 - dw_3\\ &= 2a_0 d\eta - \eta d\zeta + \zeta d\eta - \zeta d\eta + \eta d\zeta\,,\end{aligned}$$

it is easily seen that $\ker \Theta_a$ gives the horizontality condition for $(\mathbb{R}^4_a \setminus \mathbb{R}^3_a, g^H_a)$ where g^H_a denotes the hyperbolic metric $g^H_a = \left(\sum_{i=0}^3 dx_i^2\right) \Big/ (x_0 - \operatorname{Re} a_0)^2$. Then ϕ is a hyperbolic harmonic morphism if and only if

$$\eta \frac{\partial \phi}{\partial \eta} + a_0 \frac{\partial \phi}{\partial \zeta} = 0$$

which has a solution

$$\phi_a = \zeta - a_0 \ln \eta \tag{28}$$

giving the complex-valued map on a dense subset of $\mathbb{C}^4$:

$$\phi_a(x_0, x_1, x_2, x_3) = -\mathrm{i}x_1 \pm s - a_0 \ln \frac{x_0 \pm s}{x_2 - \mathrm{i}x_3}\,. \tag{29}$$

For any $a \in \mathbb{C}^4$ this restricts to a complex-valued hyperbolic harmonic morphism on a dense subset of $\mathbb{R}^4_a \setminus \mathbb{R}^3_a$. Note that when $a = 0$ this simplifies to (27).

Putting $a_0 = -\mathrm{i}t$ in (29) and restricting to the open set $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 > t^2\}$ gives the horizontally conformal map on the $\mathbb{R}^3$-slice: $t = \text{const.}$ given by

$$\begin{aligned} \phi_t = \phi_a &= -\mathrm{i}x_1 + r + \mathrm{i}t \ln \frac{r - \mathrm{i}t}{x_2 - \mathrm{i}x_3} \\ &= -\mathrm{i}x_1 + r - t \arg \frac{r - \mathrm{i}t}{x_2 - \mathrm{i}x_3}\,, \end{aligned}$$

the level curves of this being the leaves of the conformal foliation $\mathcal{C}_t$.

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