Strongly near-standard functions in Lebesgue's spaces

P. Borde

Abstract

The framework of this paper is Internal Set Theory (IST in [6]). Let P be an interval of $\mathbb{R}^N$. We give a characterization of functions $f \in L^p(P)$ $(1 \leq p < +\infty)$ which are near-standard with respect to the norm of $L^p(P)$ (i.e. $\exists^{st} f_0 \in L^p(P)$ such that $\int_P |f - f_0|^p \approx 0$). We shall find some applications of this result in reaserch of compact sets in Lebesgue's spaces, but also in operator theory because an operator of $L^p(P)$ is compact if, and only if, it transforms any limited function into a near-standard one.

1 Introduction.

Some mathematicians have already established necessary and sufficient conditions to prove the integrability of the shadow, according to a different definition, of a given function. Peter Loeb is the reference on this subject (see [4], [5]). He defines a specific notion, the "Loeb integral" and a notion of S-integrability, which makes sure we obtain an integrable function by a sort of projection on the standard. More exactly, Loeb works in an $\aleph_1$ saturated enlargement $V(^*S)$ of a superstructure $V(S)$. Fix an internal probability space $(\Lambda, \mathcal{A}, \mu)$ of $V(^*S)$. The Loeb space associated to $(\Lambda, \mathcal{A}, \mu)$ is denoted by $(\Lambda, L_\mu(\mathcal{A}), \hat{\mu})$. An arbitrary subset $N \subset \Lambda$ (N may be external) is called a $\hat{\mu}$-*nullset*, if the outer measure of N equals 0, i.e. $\inf\{ {}^o\mu(T), N \subset T \in \mathcal{A}\} = \prime$.

If $T \in \mathcal{A}$ and $U \subset \Lambda$, then T is called a $\hat{\mu}$-approximation of U, if $T \triangle U$ is a $\hat{\mu}$-nullset. The Loeb σ-algebra $L_\mu(\mathcal{A})$ is the set of all subsets $U \in \Lambda$ with $\hat{\mu}$-approximations in $\mathcal{A}$ and the Loeb measure $\hat{\mu}$ on $L_\mu(\mathcal{A})$ is defined by setting $\hat{\mu}(U) = {}^o\mu(T)$, if T is a $\hat{\mu}$-approximation of U.

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We say that a function $f : \Lambda \to {}^*\mathbb{R}$ is *S_μ-integrable* if and only if for all $A \in \mathcal{A}$,

- $\int_A |f|$ is limited,
- $\mu(A) \approx 0 \Longrightarrow \int_A |f| \approx 0$.

We can find many equivalent definitions of the S_μ-integrability. This notion is one of the most important ideas in non standard measure theory. We could use this notion to define the usual integrability of functions $f : \Lambda \to \mathbb{R} \cup \{-\infty, +\infty\}$. These functions are called $\hat{\mu}$-integrable, if there exists an S_μ-integrable lifting $F : \Lambda \to {}^*\mathbb{R}$ of f. One theorem of the theory ensures that f is $L_\mu(\mathcal{A})$-measurable if f is $\hat{\mu}$-integrable. The integrale of f could be defined by setting

$$\int_\Lambda f d\hat{\mu} = {}^o\int_\Lambda F d\mu.$$

It can be seen that this definition of integrability coincides with the usual definition. For arbitrary S_μ-integrable function $F : \Lambda \to {}^*\mathbb{R}$, ${}^oF$, the standard part of F is $\hat{\mu}$-integrable. Moreover,

$$\int_\Lambda {}^oF d\hat{\mu} = {}^o\int_\Lambda F d\mu.$$

But there is nothing to prove that ${}^oF$ is close to F with respect to "the topology defining by the norm of L^1"; in fact, it is not always right.

In this article, we want to characterize the strongly near standard functions of $L^p(P)$. The solution requires some results about signed measures which will prove in the first part. Finding characterizations of existence of a strong shadow can help us in research of relatively compact sets in the Lebesgue spaces more directly than by the classical theorem of Frechet-Kolmogorov. We shall also find other applications in the study of compact operators of $L^p(P)$. An operator of $L^p(P)$ is compact if and only if it transforms any limited function (in $L^p(P)$) into a near-standard one (with respect to the norm of $L^p(P)$).

Let us give some notations and definitions.

Consider N a standard natural. If x and y are two points of $\mathbb{R}^N$, we say that x is *infinitely close* to y, and we denote by $x \approx y$, if and only if, for any standard $\varepsilon > 0$, $\|x - y\|_{\mathbb{R}^N} < \varepsilon$. We call *shadow* of x and we denote by ${}^ox$, the unique standard point of $\mathbb{R}^N$ (if it exists) such that $x \approx {}^ox$. We say that x is *near-standard* if it admits a shadow.

Let P be a standard interval of $\mathbb{R}^N$ ($\overline{P} = [a, b] = \prod_{n=1}^{N} [a_n, b_n]$, where a_n and b_n are possibly ∞). A function $f : P \to \mathbb{R}$ is said to be *S-continuous* if and only if for any standard $x \in P$, for any $y \in P$, $[x \approx y \Longrightarrow f(x) \approx f(y)]$. We can easily extend this definition to the $\overline{\mathbb{R}}$ valued functions if we admit that ${}^ox = +\infty$ (resp $-\infty$) if x is illimited and $x > 0$ (resp $x < 0$).

If f is a S-continuous function, we can define a standard $\overline{\mathbb{R}}$ valued function ${}^of$ such that for any standard $x \in P$, $f(x) \approx ({}^of)(x)$. We say that ${}^of$ is the *shadow* of f.

A function $f : P \to \mathbb{R}$ is said to be *of the class* S^0, if and only if, it is S-continuous and takes near-standard values at standard points. We have the important following theorem.

Continuous shadow theorem. *Any function f of the class S^0 admits a continuous shadow on P.* (see [2] for the proof).

The definition of ${}^of$ implies that for any standard $x \in P$, $f(x) \approx ({}^of)(x)$; but if x is not standard, it is quite possible that $f(x)$ is not infinitely close to $({}^of)(x)$. We say that f admits a *uniform shadow* on P if and only if there exists a standard function (which is usually denoted by ${}^{\underline{o}}f$) such that $\forall x \in P$, $f(x) \approx ({}^{\underline{o}}f)(x)$. If f admits a uniform shadow, it clearly admits a shadow and ${}^{\underline{o}}f = {}^of$.

In the following, we denote by $\mathcal{O}(P)$ the set of open sets of P, by $\mathcal{O}_{fin}(P)$ the set of finite union of open intervals of P and by μ the Lebesgue measure on P. We put $\Omega \approx \infty$ if, for all $x \in \Omega$, $\|x\|_{\mathbb{R}^N} \approx +\infty$ and we denote by $\mathcal{A}(P)$ the class of μ-measurable sets of P.

A *signed measure* is an extended real valued, countably additive set function F on $\mathcal{A}(P)$ such that $\mu(\phi) = 0$ and F assumes at most one of the values $+\infty$ or $-\infty$ (see [3]).

If F is a signed measure, there exists a decomposition $F = F^+ - F^-$, where F^+ and F^- are measures and are called respectively, the *upper variation* and the *lower variation* of F. We define the *total variation* of F as the function defined on $\mathcal{A}(P)$ by $|F|(A) = F^+(A) + F^-(A)$. We say that a signed measure is of *S-bounded* total variation if and only if $|F|$ is bounded by a standard real.

We say that a signed measure F is *absolutely continuous* if and only if $F(A) = 0$ for any nullset of $\mathcal{A}(P)$. It is easy to show that F is absolutely continuous if and only if for any $\varepsilon > 0$, it exists $\delta > 0$ such that for any $\Omega \in \mathcal{O}_{fin}(P)$ (resp $\Omega \in \mathcal{A}(P)$), $\mu(\Omega) < \delta \Longrightarrow |F|(\Omega) < \varepsilon$.

A signed measure is said to be *S-absolutely continuous* if and only if

$$\forall \Omega \in \mathcal{A}(P),\ \mu(A) \approx 0 \Longrightarrow |F|(A) \approx 0.$$

Remark. We easily show that this notion is equivalent to the following,
$\forall \Omega \in \mathcal{O}_{fin}(P),\ \mu(A) \approx 0 \Longrightarrow |F|(A) \approx 0.$

Moreover, this notion is equivalent, for standard signed measure, to the absolute continuity.

2 Some results about signed measures.

Now we shall give some sufficient conditions so that the shadow of a signed measure be an absolutely continuous signed measure.

Proposition 1 *Let F be a S-absolutely continuous signed measure, of S-bounded total variation such that, for any $\Omega \approx \infty$, $|F|(\Omega) \approx 0$, then ${}^oF$ is a signed measure on A(P).*

Proof. It suffices to consider the case of a finite measure (and so positive). If F is a finite measure, it is easy to show that ${}^oF$ is an additive set function on $\mathcal{A}(P)$. We must show the complete additivity of ${}^oF$.

Let $(A_i)_{i\in\mathbb{N}}$ be a standard sequence of nonoverlapping measurable sets of P. Put $B = \bigcup_{i\in\mathbb{N}} A_i$ and, for any $k \in \mathbb{N}$, $B_k = \bigcup_{i=1}^{k} A_i$. The definition and the additivity of ${}^oF$ imply that for any standard k, $({}^oF)(B_k) \approx F(B_k)$, $({}^oF)(B - B_k) \approx F(B - B_k)$ and $({}^oF)(B_k) = \sum_{i=1}^{k}({}^oF)(A_i)$. Then, for any standard k, ${}^oF(B) = ({}^oF)(B_k) + ({}^oF)(B - B_k) \approx ({}^oF)(B_k) + F(B - B_k)$. We deduce that, for any standard finite subset $\underline{k}$ of $\mathbb{N}$, there exists $k_0 \in \mathbb{N}$ ($k_0 = 1 + \max(k, k \in \underline{k})$) such that

$$\forall k \in \underline{k},\ [k < k_0 \text{ and } \forall m \leq k_0,\ {}^oF(B) \approx ({}^oF)(B_m) + F(B - B_m)]$$

We find with the principle of idealisation of I.S.T. (see [2] for example), an illimited ω such that $({}^oF)(B_\omega) \approx F(B_\omega)$ and ${}^oF(B) \approx ({}^oF)(B_\omega) + F(B - B_\omega) \approx \sum_{i=1}^{\omega}({}^oF)(A_i) + F(B - B_\omega)$.

The increasing standard sequence $\left(\sum_{i=1}^{n}({}^oF)(A_i)\right)_{n\in\mathbb{N}}$ is bounded by $|F|(P) + 1$, so it converges. If we prove that $F(B - B_\omega) \approx 0$, the limit of this sequence will be $({}^oF)(B)$ and we will conclude that $({}^oF)(B) = \sum_{i\in\mathbb{N}}({}^oF)(A_i)$ which correspond to the complete additivity of ${}^oF$ for standard sequence. We will conclude with a transfer.

So, let us show that $F(B - B_\omega) \approx 0$. For any $p \in \mathbb{N}$, put $K_p = \prod_{i=1}^{N}[-p, p]$. For any standard p, $\mu(B \cap K_p)$ is limited, so $\mu((B - B_\omega) \cap K_p) \approx 0$; the Fehrele principle (see [2]) gives us an illimited p_0 which satisfies $\mu((B - B_\omega) \cap K_{p_0}) \approx 0$. But $P - K_{p_0} \in \mathcal{O}_{fin}(P)$ and $P - K_{p_0} \approx \infty$ imply that $F(P - K_{p_0}) \approx 0$; as F is increasing, $F((B - B_\omega) \cap (P - K_{p_0})) \approx 0$. So we conclude that $F(B - B_\omega) \approx 0$. ∎

Proposition 2 *Let F be a S-absolutely continuous signed measure, of S-bounded total variation such that, for any $\Omega \approx \infty$, $|F|(\Omega) \approx 0$, then ${}^oF$ is an absolutely continuous signed measure.*

Proof. We know with proposition 1 that ${}^oF$ is a signed measure. The definition of ${}^oF$ and the S-absolute continuity of F imply that for any standard nullset A, $({}^oF)(A) \approx F(A) = 0$. As ${}^oF(A)$ is standard we deduce $({}^oF)(A) = 0$. By transfer, this property is true for any nullset A. ∎

Now, let us recall the classical Radon-Nicodym theorem (see [3]).

Theorem. *Let F be a signed measure; F is absolutely continuous if and only if it exists a measurable function $f : P \to \mathbb{R}$ such that for any mesurable set A of P, $F(A) = \int_A f$. The function f, called density of F, is unique in the sense that if also $F(A) = \int_A g$, $A \in \mathcal{A}(P)$, then $f = g\mu$ almost everywhere.*

3 Necessary and sufficiency of strongly near-standardness.

In the following, if f and g are two functions of $L^p(P)$, we say that $f \approx_{L^p} g$ if and only if $\int_P |f-g|^p \approx 0$.

a) Functions which are strongly infinitely close to 0.

Theorem 1. *Let P be a standard compact interval of $\mathbb{R}^N$ and $f : P \to \overline{\mathbb{R}}$ be a function of $L^p(P)$. For any $\varepsilon > 0$, we denote by $E_\varepsilon = \{x \in P,\ |f(x)| > \varepsilon\}$. We have*

$$f \approx_{L^p} 0 \Longleftrightarrow \exists \varepsilon \approx 0;\ \mu(E_\varepsilon) \approx 0 \ \ \textit{and} \ \int_{E_\varepsilon} |f|^p \approx 0.$$

Proof. Suppose $\int_P |f|^p \approx 0$; it is obvious that $\int_{E_\varepsilon} |f|^p \approx 0$ for all $\varepsilon > 0$. Moreover, for any standard $\varepsilon > 0$, we have $0 \approx \int_{E_\varepsilon} |f|^p > (\varepsilon)^p \mu(E_\varepsilon) \geq 0$. As ε and p are standard, we deduce $\mu(E_\varepsilon) \approx 0$. Consider the internal set $\{n \in \mathbb{N}; \mu(E_{\frac{1}{n}}) < \frac{1}{n}\}$. It contains all standard naturals. The permanence principle (see [2]), ensures us that it contains an illimited.

The converse is obvious if we write $P = E_\varepsilon \cup (P - E_\varepsilon)$ and if we use Minkowsky's inequality. ∎

In what follows, theorem 3 gives a characterization of strongly near standardness when P is a bounded interval and $1 \leq p < +\infty$. We begin to expose the case of $p = 1$ because we need this result to prove theorem 3, but also because it is not necessary, in this case, to restrict our considerations to P bounded.

b) Strongly near standardness in $L^1(P)$.

Theorem 2. *Let f be a Lebesgue-integrable function on P, then f admits a shadow with respect to the norm of $L^1(P)$ if and only if*

$$\begin{cases} 1)\ \forall \Omega \in \mathcal{O}_{fin}(P),\ \mu(\Omega) \approx 0 \ \textit{or} \ \Omega \approx \infty \Longrightarrow \int_\Omega |f| \approx 0, \\ 2)\ \int_P |f| \ \textit{is limited}, \\ 3)\ \forall \Omega \in \mathcal{O}_{fin}(P), \int_\Omega f = F(\Omega) \approx (\,{}^oF)(\Omega). \end{cases}$$

Proof. Necessary; if f_0 is the strong shadow of f on P, then, for any measurable set A of P, $\int_A f \approx \int_A f_0$ and $\int_A |f| \approx \int_A |f_0|$. These properties imply obviously conditions 1), 2) and 3); moreover we obtain that f_0 is the density of ${}^oF$.

For the converse, we have two problems; first, the existence of a density function f_0 of ${}^oF$. Second, if such a f_0 exists, have we got $\int_P |f - f_0| \approx 0$?

Conditions 1) and 2) of the theorem and proposition 2 imply that ${}^oF$ is absolutely continuous. By using the Radon Nicodym theorem, we can conclude to the existence of a standard integrable function on P, f_0, such that, for all measurable sets of P, E, $(\,{}^oF)(E) = \int_E f_0$. Now, let us prove that $\int_P |f - f_0| \approx 0$.

We put $E_1 = \{x \in P; f(x) \geq f_0(x)\}$ and $E_2 = \{x \in P; f(x) < f_0(x)\}$. These sets are measurable and $P = E_1 \cup E_2$. As the Lebesgue measure is regular, there exists an open set W in P such that $E_1 \subset W$ and $\mu(W - E_1) \approx 0$.

Consider $W = \bigcup_{i\in\mathbb{N}} U_i$ a decomposition of W where each U_i is an open interval of P, and put $W_k = \bigcup_{i=1}^{k} U_i$. We deduce from property 3) that for any k, $\int_{W_k} f - f_0 \approx 0$ and consequently $\int_W f - f_0 \approx 0$.

Absolute continuity of the functions F and ${}^oF$, and the property $\mu(W - E_1) \approx 0$ imply that $\int_{W-E_1} f - f_0 \approx 0$.

As $\int_W f - f_0 = \int_{W-E_1} f - f_0 + \int_{E_1} f - f_0$, we have $\int_{E_1} f - f_0 \approx 0$.

Similarly, we find $\int_{E_2} f \approx \int_{E_2} f_0$ and finally

$$\int_P |f - f_0| = \int_{E_1} f - f_0 + \int_{E_2} f_0 - f \approx 0.$$

■

c) Case of P is bounded.

Theorem 3. *Let f be a Lebesgue-integrable function on P, then f admits a shadow with respect to the norm of $L^p(P)$ $(1 \leq p < +\infty)$ if and only if*

$$\begin{cases} 1_p)\ \forall \Omega \in \mathcal{O}_{fin}(P),\ \mu(\Omega) \approx 0 \Longrightarrow \int_\Omega |f|^p \approx 0, \\ 2_p)\ \int_P |f|^p \text{ is limited}, \\ 3)\ \forall \Omega \in \mathcal{O}_{fin}(P), \int_\Omega f = F(\Omega) \approx (\ {}^oF)(\Omega). \end{cases}$$

Proof. Necessary; suppose that the strong shadow of f in $L^p(P)$ exists and denote by f_0 this function. We have $\left(\int_A |f|^p\right)^{\frac{1}{p}} \approx \left(\int_A |f_0|^p\right)^{1/p}$ for any $A \in \mathcal{A}(P)$. Moreover, the standard signed measure with a density $|f_0|^p$ is absolutely continuous, and of bounded total variation. These facts imply conditions 1_p) and 2_p).

Let q be the real such that $\frac{1}{p} + \frac{1}{q} = 1$, the Hölder inequality gives us $\int_P |f - f_0| \leq \left(\int_P |f - f_0|^p\right)^{1/p} (\mu(P))^{1/q}$. As P is bounded and $f \approx_{L^p} f_0$, we deduce condition 3).

Conversely; the conditions 1_p) and 2_p) are true if $p = 1$ (Hölder) on P. As F is absolutely continuous and has a S-bounded total variation, the signed measure ${}^oF$ admits a density function g, which is the strong shadow of f in $L^1(P)$ (see theorem 2). Now, it suffices to prove that $f \approx_{L^p} g$.

First, suppose that $\int_P |g|^p$ is limited. This hypothesis implies the absolute continuity of the standard signed measure of density $|g|^p$. We deduce from theorem 1 the existence of a positive infinitesimal ε such that $\mu(E_\varepsilon) \approx 0$ and $\int_{E_\varepsilon} |f-g| \approx 0$ where $E_\varepsilon = \{x \in P; |f(x)-g(x)| > \varepsilon\}$. Now, we have

$$\left(\int_P |f-g|^p\right)^{1/p} \leq \left(\int_{E_\varepsilon} |f-g|^p\right)^{1/p} + \left(\int_{P-E_\varepsilon} |f-g|^p\right)^{1/p}.$$

But, $\int_P |f-f_0|^p \leq \varepsilon^p \mu(P) \approx 0$, and

$$\left(\int_{E_\varepsilon} |f-g|^p\right)^{1/p} \leq \left(\int_{E_\varepsilon} |f|^p\right)^{1/p} + \left(\int_{E_\varepsilon} |g|^p\right)^{1/p}.$$

In this last sum, all terms are infinitesimals because of the S-absolute continuity of the signed measures of density $|f|^p$ and $|g|^p$. So, in the case of $\int_P |g|^p$ is limited, we have $f \approx_{L^p} g$.

Now, it suffices proving that $g \in L^p(P)$ to finish the proof. Let n be a natural and consider the functions $f_n = \inf(n, \sup(f, -n))$ and $g_n = \inf(n, \sup(g, -n))$. For any standard n, f_n and g_n are in $L^p(P)$. The first part of the present proof implies that $f_n \approx_{L^p} g_n$. Moreover, the property

$$\left|\left(\int_P |f_n|^p\right)^{1/p} - \left(\int_P |g_n|^p\right)^{1/p}\right| \leq \left(\int_P |f_n - g_n|^p\right)^{1/p} \approx 0,$$

implies that for any standard $n \in \mathbb{N}$, $\left(\int_P |g_n|^p\right)^{1/p} \leq \left(\int_P |f|^p\right)^{1/p} + 1$, which is limited. By the transfer principle and the monotone convergence theorem, we find that $g \in L^p(P)$. ■

The easy proof of the following proposition is left to the reader.

Proposition 3. *Let P be a standard bounded interval of $\mathbb{R}^N$, and q a standard real number which is strictly greater than 1. If $f : P \to \mathbb{R}$ satisfies*

a) $\int_P |f|^q$ *is limited,*

b) $\forall \Omega \in \mathcal{O}_{fin}(P), \int_\Omega f = F(\Omega) \approx ({}^oF)(\Omega)$,

then, for all $p \in [1, q[$, f is strongly near standard in $L^p(P)$.

Remarks.

1) Practically, condition 3) of the previous theorems is not easy to check, but is essential. Let us give an example. Put $P = [0,1] \subset \mathbb{R}$; we denote by N an even illimited natural and we construct the subdivision $(I_n = [x_n, x_{n+1}[)_{n=1..N-1}$ of P such that $x_0 = 0 < x_1 = \frac{1}{N} < ... < x_i = \frac{i}{N} < ... < x_N = 1$. Now, let us consider the non-negative real function f on P defined by $f = 1 + \sum_{n=0}^{N-1} (-1)^n \mathbb{1}_{I_n}$; as a step function, f is integrable on P. Let us prove that f satisfies conditions 1) and 2) of the theorem 2.

$$\int_P |f| = \int_P f = 1 + \sum_{n=1}^{N} \frac{(-1)^n}{N} = 1 < \infty;$$

then 2) is right.

Let $\Omega = \bigcup_{k\in\mathbb{N}} I_k$ be an open set of P such that $\mu(\Omega) \approx 0$. We have

$$\int_\Omega f = \int_\Omega 1 \ + \ \sum_{n=1}^{N}(-1)^n \int_{\Omega\cap I_n} 1 \leq 2\mu(\Omega) \approx 0.$$

Then 1) is right.

We have, for any x in P, $F(]0,x[) = x + \int_{[0,x]} \sum_{n=0}^{N-1} \frac{(-1)^n}{N} \mathbb{I}_{I_n} \approx x$, so, $({}^oF)(]0,x[) = x$; this implies $f_0 = 1$ almost everywhere. It is clear that if we suppose f strongly near standard in L^1, its strong shadow is f_0 (almost everywhere). But,

$$\int_P |f-1| = \sum_{n=1}^{N} \frac{1}{N} = 1,$$

then f is not infinitely closed to 1 in $L^1(P)$. So f is not strongly near standard in $L^1(P)$.

2) It is interesting to see that condition 3) does not depend on p.

3) We can solve our problem without using the measure theory. We only need of $\mathcal{O}_{fin}(P)$.

d) When P is unbounded.

Consider $p > 2$. The real function defined by $f(x) = \varepsilon x^{1/p}\mathbb{I}_{[\frac{1}{\varepsilon},\frac{2}{\varepsilon}]}$, ($\varepsilon$ a nonnegative infinitesimal) is strongly infinitely close to 0 in $L^p(\mathbb{R})$ but f does not satisfy the condition 3) of the theorem 2 (or theorem 3). So this condition is not adapted to the general case if P is unbounded.

4 Special cases.

We shall now study special, but useful cases.

Proposition 4. *If P is a standard compact interval of $\mathbb{R}^N$, p is a standard natural greater than 1 and $f \in L^p(P)$ is a function of the class S^0 on P, then, ${}^of$ which is continuous on P, is the strong shadow of f in L^p.*

Proof. Hypothesis on f imply that for any $x \in P$, $|f(x) - ({}^of)(x)| \approx 0$, so, for any standard $\varepsilon > 0$, for any $x \in P$, $|f(x) - ({}^of)(x)|^p < \varepsilon$.

We deduce $\int_P |f(x) - ({}^of)(x)|^p < \varepsilon\mu(P)$ for all standard $\varepsilon > 0$. ∎

Proposition 5. *Let P be a standard compact interval and p be a standard natural greater than 1. Let $f \in L^p(P)$ be a S-continuous function which satisfies conditions 1_p) and 2_p) of theorem 3, then ${}^of$ (as a $\overline{\mathbb{R}}$ valued function), is the strong shadow of f in L^p.*

Proof. Suppose that f is S-continuous on a standard compact interval of $\mathbb{R}^N$, P. It is clear that, for any $n \in \mathbb{N}$, the functions $f_n = \inf(n, \sup(f, -n))$ are S-continuous.

Then, for any standard n, f_n is of the class S^0, so we can apply proposition 4 and obtain a standard continuous function g_n such that for all standard $\varepsilon > 0$, $\int_P |g_n - f_n|^p < \varepsilon$.

The construction principle infers the existence of a standard sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$ such that, for any standard n, $g_n = {}^o(f_n)$ and for any standard $\varepsilon > 0$, $\int_P |g_n - f_n|^p < \varepsilon$.

It is easy to show that this sequence converges to ${}^o f$, which is in $L^1(P)$.

The monadic collection $\{n \in \mathbb{N};\ \forall^{st}\varepsilon > 0 \int_P |g_n - f_n|^p < \varepsilon\}$ contains all standard points of $\mathbb{N}$ and by the Fehrele principle, we deduce the existence of an $\omega \approx +\infty$ such that $\int_P |g_\omega - f_\omega|^p \approx 0$. But,

$$\left(\int_P |g_0 - f|^p\right)^{1/p} \leq \left(\int_P |g_0 - g_\omega|^p\right)^{1/p} + \left(\int_P |g_\omega - f_\omega|^p\right)^{1/p} + \left(\int_P |f_\omega - f|^p\right)^{1/p}.$$

In this last sum, all terms are infinitesimals. The second according to the definition of ω, the third as a consequence of hypothesis 1_p) et 2_p), since 2_p) implies that the set $E^\omega = \{x; |f(x)| \geq \omega\}$ has an infinitesimal measure and, moreover, we have $\left(\int_P |f_\omega - f|^p\right)^{1/p} \leq \left(\int_{E^\omega} |f|^p\right)^{1/p}$. And the first because of the absolute continuity of the signed measure of density $|g|^p$ and the assertion $\int_P |g|^p \geq \omega^p \mu(E_g^\omega)$ where $E_g^\omega = \{x; |g(x)| \geq \omega\}$ (ω is any illimited), which implies that $\mu(E_g^\omega) \approx 0$.

So $f \approx_{L^p} g$. ■

Remark. We can easily generalize this proposition to the case of any standard interval but also for quasi-S-continuous functions on P. One function f is said to be *quasi S-continuous on* P if and only if there exists a standard subdivision of P, $S = (P_0, P_1, ..., P_n)$, such that f be S-continuous on each P_i.

We can apply these results for example to show that any function which is M-Lipschitz on P (M is a standard real) is strongly near standard in $L^p(P)$.

Proposition 6 *Let Ω be a standard bounded interval of $\mathbb{R}$, M be a standard integer and f be a bounded function in $L^1(\Omega)$, which is derivable on Ω; let us suppose its derivative is finite, integrable on Ω and bounded by M in $L^r(\Omega)$ ($r > 1$). Then f is strongly near-standard in $L^p(\Omega)$ for any p and near-standard in $C^0(\Omega)$ to the uniform topology.*

Proof. As before, Hölder's inequality implies, for any x and y in Ω,

$$|f(y) - f(x)| = |\int_{[x,y]} f'| \leq \int_{[x,y]} |f'| \leq M(\mu([x,y]))^{1-\frac{1}{r}};$$

so we deduce the S-continuity of f on Ω. As $f \in L^1(\Omega)$, it exists $a \in \Omega$ such that $f(a)$ is limited. Then, for any standard natural p,

$$|f(x) - f(a)|^p = |\int_{[a,x]} f'|^p \leq \left(\int_{[a,x]} |f'|\right)^p \leq M^p \mu(\Omega)^{p(1-\frac{1}{r})}.$$

This implies that f is limited on Ω; so f is of the class S^0 on the compact interval Ω; we deduce that f is near-standard in $C^0(\Omega)$ with respect to the norm of the uniform convergence, and strongly near-standard in $L^1(\Omega)$.

We can also write

$$\begin{aligned}\left(\int_A |f|^p\right)^{1/p} &\leq \left(\int_A |f-f(a)|^p\right)^{1/p} + \left(\int_A |f(a)|^p\right)^{1/p} \\ &\leq M\mu(\Omega)^{(1-1/r)}\mu(A)^{1/p} + |f(a)|\mu(A)^{1/p}.\end{aligned}$$

This last sum is infinitesimal if $\mu(A) \approx 0$ and limited if $A = \Omega$. Then $|f|^p$ satisfies conditions 1) and 2) of theorem 2; this implies the strongly near-standardness of f in $L^p(\Omega)$. ∎

Remark. We can easily generalize these results when P is any measurable set of $\mathbb{R}^N$: we extend f by setting it equal to zero outside P and we integrate the extension over $\mathbb{R}^N$.

N.B. If we admit the framework of T.R.E.(see [9]), all our results can be generalized to non standard functions and non-standard spaces. If f is a non standard function, the propositions of T.R.E. allow us to attribute a level α of standardicity and we will then solve the problem with a change of indexes in the formulation of the definitions so that we can "adjust to level α". In particular, we can choose N infinitely large.

References

[1] H. Brezis. *Analyse fonctionnelle, théorie et applications.* Masson.

[2] F. Diener et G. Reeb. *Analyse non standard.* Hermann, 1989.

[3] P.R. Halmos. *Measure Theory.* Van Nostrand, 1950.

[4] A.E. Hurd, P. Loeb. *An introduction to nonstandard analysis.* Academic Press, 1985.

[5] P. Loeb. *Conversion from nonstandard to standard measure spaces and applications in probability theory.* Trans . Amer. Math. Soc., 211, 1975, p113.

[6] E. Nelson. *Internal Set Theory.* Bull. Amer. Math. Soc. 83,1165-1198, 1977.

[7] E. Nelson. *Radically elementary probability theory.* Ann of Math Studies, 117, Princeton, 1987.

[8] H. Osswald. *An introduction to Analysis on the Wiener space using Infinitesimals.* Preprint, 1996.

[9] Y. Péraire. *Introduction à l'analyse non standard relative.* Proceeding of the 5^{th} Workshop in Analysis and its Applications, Missouri-Rolla, 1995.

[10] I. P. Van Den Berg. *Extended use of I.S.T.* Annals of Pure and Applied Logic 58, 1992. p 73-92

Université Blaise Pascal (Clermont II),
Département de mathématiques,
63177 Aubière Cedex,
France.
Tel: 04.73.40.70.88,
Fax: 04.73.40.70.64,
E-mail: borde@ucfma.univ-bpclermont.fr.