

# Differential Inclusions at Resonance

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## Abstract

The existence of periodic solutions are studied for certain differential inclusions at resonance. Landesman–Lazer type conditions are derived. Applications are given to discontinuous differential equations.

## 1 Introduction

Many problems of nature are modeled by differential equations with discontinuous nonlinearities [4], [9], [15]. In this paper, we deal with discontinuous differential equations which are at resonance in infinity. This problem for continuous nonlinear operator equations is handled by Landesman–Lazer type results [1], [3], [18]. The purpose of this paper is to extend this method to discontinuous nonlinear operator equations, i.e. to operator inclusions at resonance in infinity. A similar question is investigated in [2]. To study such inclusions, we are motivated by recent papers [6–8], [11], [13], [16]. Consequently, in Section 2 we present existence results of Landesman–Lazer type, and then in Section 3, we apply them to discontinuous differential equations. We also remark that our method can be used to certain implicit scalar differential inclusions, see Section 4, and to certain implicit differential equations as well [10], [12], see Section 5. Multi-valued boundary value problems are studied also in [17]. Finally, we have to point out that we apply our method for the existence of periodic solutions of nonlinear systems, but our method works also for discontinuous partial differential equations similar to [4].

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## 2 Abstract Setting

Let  $H$  be a Banach space with a continuous inner product  $(\cdot, \cdot)$  and with the corresponding norm  $|\cdot|$ . Let  $Z$  be a Banach space continuously embedded into  $H$  and let  $Y$  be a reflexive Banach space compactly embedded into  $Z$ .

**Definition 2.1.** A mapping  $F : Z \rightarrow 2^H \setminus \{\emptyset\}$  is said

- to be weakly upper semi-continuous (denote w.u.s.c.), if for any  $Z \ni u_n \rightharpoonup u \in Z$ ,  $F(u_n) \ni z_n \rightharpoonup$  (weakly)  $z \in H$ , it holds  $z \in F(u)$ .
- to be uniformly bounded, if its range is a bounded set.

In this section, we consider the operator inclusion

$$h \in Lu + F(u), \quad (2.1)$$

where  $L : Y \rightarrow H$  is bounded linear, Fredholm with index 0, symmetric with respect to  $(\cdot, \cdot)$ ,  $h \in H$  and  $F$  is uniformly bounded, w.u.s.c. with convex set values. Let us put

$$S = \{u \in \ker L \mid |u| = 1\}. \quad (2.2)$$

Let  $Q : H \rightarrow \text{im } L$  be the orthogonal projection with respect to  $(\cdot, \cdot)$  and consider a bounded linear operator  $B : Y \rightarrow H$  such that

$$(C) \quad B(\text{im } L \cap Y) \subset \text{im } L, \quad B(\ker L) \subset \ker L, \quad \ker(B/\ker L) = 0.$$

Of course,  $Bu = u$ ,  $\forall u \in Y$  satisfies (C). The following result is an extension of Landesman–Lazer type ones [1] to operator inclusions.

**Theorem 2.2** Suppose the existence of a mapping  $\phi : S \rightarrow \mathbb{R}$  such that

$$\varliminf_{\rho \rightarrow \infty} \left( \inf_{z \in F(\rho w + v)} (z, Bw) \right) \geq \phi(w) \quad (H1)$$

uniformly with respect to  $w \in S$ , and bounded  $v \in \text{im } L \cap Y$  as well. If there is a  $\delta > 0$  such that

$$(h, Bw) < \phi(w) - \delta \quad \forall w \in S, \quad (H2)$$

then (2.1) is solvable.

*Proof.* In spite of the fact that the proof is standard, we present it here for the reader convenience. First of all, we consider

$$h \in Lu + \varepsilon Bu + F(u), \quad (2.3)$$

where  $\varepsilon > 0$  is sufficiently small. Since (C) holds, (2.3) has the form

$$u \in (L + \varepsilon B)^{-1}(h - F(u)), \quad u \in Z. \quad (2.4)$$

The right-hand side of (2.4) is upper semi-continuous with compact convex set values and uniformly bounded as well. Indeed, if  $u_i \rightarrow u$  and  $v_i \rightarrow v$  in  $Z$  are such that

$$v_i \in (L + \varepsilon B)^{-1}(h - F(u_i)),$$

then  $v_i$  are uniformly bounded in  $Y$ . Since  $Y$  is reflexive and compactly embedded into  $Z$ , we can suppose that  $v_i \rightharpoonup v$  in  $Y$ . Hence  $(L + \varepsilon B)v_i \rightharpoonup (L + \varepsilon B)v$  in  $H$ . On the other hand,  $F$  is w.u.s.c. and it holds

$$(L + \varepsilon B)v_i \in h - F(u_i).$$

Consequently,

$$(L + \varepsilon B)v \in h - F(u),$$

and the right-hand side of (2.4) has the abovementioned properties.

By [5], (2.4) has a solution  $u_\varepsilon$  for any sufficiently small  $\varepsilon > 0$ . We take

$$u_\varepsilon = w_\varepsilon + v_\varepsilon, \quad w_\varepsilon \in \ker L, \quad v_\varepsilon \in \operatorname{im} L \cap Y.$$

Then (2.3) gives

$$h = Lu_\varepsilon + \varepsilon Bu_\varepsilon + f_\varepsilon, \quad f_\varepsilon \in F(u_\varepsilon),$$

and by (C), we have

$$(h, Bw_\varepsilon) = \varepsilon |Bw_\varepsilon|^2 + (f_\varepsilon, Bw_\varepsilon), \quad f_\varepsilon \in F(w_\varepsilon + v_\varepsilon). \tag{2.5}$$

Moreover, if  $h = h_1 + h_2$ ,  $h_1 \in \ker L$ ,  $h_2 \in \operatorname{im} L$ , then

$$\begin{aligned} h_1 &= (L + \varepsilon B)v_\varepsilon + Qf_\varepsilon \\ v_\varepsilon &= (L + \varepsilon B)^{-1}(h_1 - Qf_\varepsilon). \end{aligned}$$

So  $v_\varepsilon$  is uniformly bounded in  $Y$ . Now we show that  $w_\varepsilon$  is also uniformly bounded. Let us suppose that  $w_{\varepsilon_i} \rightarrow \infty$  as  $\varepsilon_i \rightarrow 0_+$ . Then (H1-2) and (2.5) give for  $i$  sufficiently large

$$\phi(w_{\varepsilon_i}) - \delta > (h, Bw_{\varepsilon_i}) \geq (f_{\varepsilon_i}, Bw_{\varepsilon_i}) \geq \phi(w_{\varepsilon_i}) - \delta.$$

This contradiction gives the uniform boundedness of  $u_\varepsilon$  in  $Y$ . Since  $Y$  is reflexive and compactly embedded into  $Z$  as well, there is a subsequence  $u_{\varepsilon_i} \rightarrow u \in Z$  and  $Lu_{\varepsilon_i} \rightharpoonup Lu$ . On the other hand,  $F$  is w.u.s.c., hence by passing to the limit in (2.3) as  $\varepsilon_i \rightarrow 0_+$ , we obtain

$$h \in Lu + F(u).$$

The proof is finished. ■

Similarly we have the following result.

**Theorem 2.3** *Suppose the existence of a mapping  $\psi : S \rightarrow \mathbb{R}$  such that*

$$\overline{\lim}_{\rho \rightarrow \infty} \left( \sup_{z \in F(\rho w + v)} (z, Bw) \right) \leq \psi(w) \tag{H3}$$

*uniformly with respect to  $w \in S$ , and bounded  $v \in \operatorname{im} L \cap Y$  as well.*

*If there is a  $\delta > 0$  such that*

$$(h, Bw) > \psi(w) + \delta \quad \forall w \in S, \tag{H4}$$

*then (2.1) is solvable.*

The next result is trivial.

**Theorem 2.4** *Let us put*

$$\sup_{f \in \cup_{u \in Z} F(u)} (f, Bw) = M(w), \quad \inf_{f \in \cup_{u \in Z} F(u)} (f, Bw) = m(w).$$

We note that  $M(\cdot) < \infty$  and  $M(-w) = -m(w)$ . If (2.1) has a solution then  $h$  satisfies

$$m(w) \leq (h, Bw) \leq M(w) \quad \forall w \in S.$$

### 3 Discontinuous Differential Equations at Resonance

Let us consider the equation

$$x'' + x + g(x) = h(t), \quad (3.1)$$

where  $h \in L^2_{2\pi}(\mathbb{R}) = \{x \in L^2_{\text{loc}}(\mathbb{R}) \mid x \text{ is } 2\pi\text{-periodic}\}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is uniformly bounded. We put

$$g_+(x) = \overline{\lim}_{s \rightarrow x} g(s) \\ g_-(x) = \underline{\lim}_{s \rightarrow x} g(s).$$

Then  $g_{\pm}$  are Borel measurable. We take

$$H = Z = L^2_{2\pi}(\mathbb{R}), \quad Y = \{x \in H \mid x', x'' \in H\} \\ F(u) = \{y \in H \mid g_-(u(s)) \leq y(s) \leq g_+(u(s)) \text{ a.e. on } \mathbb{R}\} \\ Lu = u'' + u, \quad Bu = u, \quad (u, v) = \int_0^{2\pi} u(s)v(s) ds.$$

Then  $F$  is w.u.s.c. [2], [4], and uniformly bounded as well. We have

$$\ker L = \{c \sin(t + \tau) \mid c, \tau \in \mathbb{R}\}.$$

Hence for (2.2), we can in this case consider

$$S = \{\sin(t + \tau) \mid \tau \in \mathbb{R}\}.$$

We suppose

(i) There are constants  $A_+ \geq A_-$  such that

$$A_- \leq g_-(x) \leq g_+(x) \leq A_+ \quad \forall x \in \mathbb{R}.$$

(ii) There are constants  $A_+ \geq B_- \geq B_+ \geq A_-$  such that

$$\underline{\lim}_{x \rightarrow \infty} g_-(x) = B_-, \quad \overline{\lim}_{x \rightarrow -\infty} g_+(x) = B_+.$$

If  $f \in F(u)$  and  $w = \sin(t + \tau)$ , then

$$(f, w) = \int_0^{2\pi} f(s) \sin(s + \tau) ds.$$

Since  $A_- \leq f(s) \leq A_+$ , we have

$$\begin{aligned} A_- \int_0^{2\pi} \sin^+(s + \tau) ds + A_+ \int_0^{2\pi} \sin^-(s + \tau) ds &\leq \int_0^{2\pi} f(s) \sin(s + \tau) ds \\ &\leq A_+ \int_0^{2\pi} \sin^+(s + \tau) ds + A_- \int_0^{2\pi} \sin^-(s + \tau) ds, \end{aligned}$$

where  $\sin^+ t = \max\{\sin t, 0\}$ ,  $\sin^- t = \min\{\sin t, 0\}$ . Hence

$$\left| \int_0^{2\pi} f(s) \sin(s + \tau) ds \right| \leq 2(A_+ - A_-) \quad \forall \tau \in \mathbb{R}. \tag{3.2}$$

Consequently we have

$$M(w) \leq 2(A_+ - A_-), \quad -2(A_+ - A_-) \leq m(w)$$

in Theorem 2.4 for this case.

Now let  $f \in F(\rho w + v)$ , where  $\sin(s + \tau) = w \in S$  and  $v \in Y$  is uniformly bounded. We note that  $v$  is also uniformly bounded in the space  $C[0, 2\pi]$ . We show the validity of (H1) for this case. The proof is standard [1]. We take small  $\varepsilon > 0$ . Then there is  $r > 0$  such that  $|\sin(s + \tau)| \geq r$  for any  $s \in [0, 2\pi] \setminus \Omega_r = \Omega'_r$ , where  $\text{mes} \Omega_r \leq \varepsilon$ . Uniformly with respect to  $\tau$ , we compute for large  $\rho > 0$

$$\begin{aligned} \int_0^{2\pi} f(s) \sin(s + \tau) ds &= \int_{\Omega_r} f(s) \sin(s + \tau) ds + \int_{\Omega'_r} f(s) \sin(s + \tau) ds \\ &\geq -(|A_+| + |A_-|)\varepsilon + \int_{\Omega'_r} f(s) \sin^+(s + \tau) ds + \int_{\Omega'_r} f(s) \sin^-(s + \tau) ds \\ &\geq -(|A_+| + |A_-|)\varepsilon + (B_- - \varepsilon) \int_{\Omega'_r} \sin^+(s + \tau) ds + (B_+ + \varepsilon) \int_{\Omega'_r} \sin^-(s + \tau) ds \\ &= 2((B_- - \varepsilon) - (B_+ + \varepsilon)) - (|A_+| + |A_-|)\varepsilon \\ &\quad - (B_- - \varepsilon) \int_{\Omega_r} \sin^+(s + \tau) ds - (B_+ + \varepsilon) \int_{\Omega_r} \sin^-(s + \tau) ds \\ &\geq 2(B_- - B_+) - 4\varepsilon - (|A_+| + |A_-| + |B_+ - \varepsilon| + |B_+ + \varepsilon|)\varepsilon. \end{aligned}$$

Hence (H1) holds with

$$\phi(w) = 2(B_- - B_+),$$

and (H2) has now the form

$$\int_0^{2\pi} h(s) \sin(s + \tau) ds < 2(B_- - B_+) - \delta \quad \forall \tau \in \mathbb{R}. \tag{3.3}$$

We note that (3.2), respectively (3.3), are equivalent to

$$\left| \int_0^{2\pi} h(s) e^{is} ds \right| \leq 2(A_+ - A_-), \quad (3.4)$$

respectively

$$\left| \int_0^{2\pi} h(s) e^{is} ds \right| < 2(B_- - B_+) - \delta. \quad (3.5)$$

By applying Theorems 2.2 and 2.4, we obtain the following result.

**Theorem 3.1** Consider (3.1) satisfying (i) and (ii). Then (3.1) may have a  $2\pi$ -periodic solution only for  $h$  satisfying (3.4). On the other hand, if  $h$  is such that

$$\left| \int_0^{2\pi} h(s) e^{is} ds \right| < 2(B_- - B_+), \quad (3.6)$$

then (3.1) has a  $2\pi$ -periodic solution.

*Remark 3.2.* (3.1) is considered in [16] with  $g(x) = a \operatorname{sgn} x$ ,  $a > 0$ . Then  $A_+ = B_- = a$ ,  $A_- = B_+ = -a$ , and (3.4), (3.6) become

$$\left| \int_0^{2\pi} h(s) e^{is} ds \right| \leq 4a, \quad \left| \int_0^{2\pi} h(s) e^{is} ds \right| < 4a,$$

respectively. We have recovered a result of [16].

**Theorem 3.3** Consider (3.1) satisfying (i) and the following condition holds as well:

(iii) There are constants  $A_+ \geq D_- \geq D_+ \geq A_-$  such that

$$\overline{\lim}_{x \rightarrow \infty} g_+(x) = D_+, \quad \underline{\lim}_{x \rightarrow -\infty} g_-(x) = D_-.$$

Then (3.1) has a  $2\pi$ -periodic solution for any  $h$  satisfying

$$\left| \int_0^{2\pi} h(s) e^{is} ds \right| < 2(D_- - D_+). \quad (3.7)$$

*Proof.* We apply Theorem 2.3. Like in the above proof, we derive

$$\psi(w) = 2(D_+ - D_-).$$

The proof is finished. ■

Now we consider the equation

$$x'' + x + g(x') = h(t), \tag{3.8}$$

where  $h \in L^2_{2\pi}(\mathbb{R})$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is uniformly bounded. The set up of (3.8) is similar as for (3.1), only  $B$  now has the form  $Bu = u'$  and  $Z = H^1_{2\pi}(\mathbb{R})$ . We note that this strategy was used in [6]. We verify (C) of Section 2. If  $v \in \text{im } L \cap Y$ ,  $0 \neq c \sin(s + \tau) = w \in \ker L$ , then

$$\begin{aligned} (Bv, w) &= c \int_0^{2\pi} v'(s) \sin(s + \tau) ds \\ &= c \left[ v(s) \sin(s + \tau) \right]_0^{2\pi} - c \int_0^{2\pi} v(s) \cos(s + \tau) ds = 0, \\ Bw &= c \cos(s + \tau) \neq 0. \end{aligned}$$

Consequently, the condition (C) holds.

**Theorem 3.4** Consider (3.8) satisfying (i) and (ii), respectively (i) and (iii). Then (3.8) may have a  $2\pi$ -periodic solution only for  $h$  satisfying (3.4). On the other hand, if  $h$  satisfies (3.6), respectively (3.7), then (3.8) has a  $2\pi$ -periodic solution.

*Proof.* The proof is the same like above, so we omit it. ■

*Remark 3.5.* (3.8) is considered in [6] with  $g(x) = a \text{sgn } x$ ,  $a > 0$ . Hence we arrive at the same situation like in Remark 3.2.

We combine the above examples in the next one

$$x'' + x + e(x') + d(x) = h(t), \tag{3.9}$$

where  $h \in L^2_{2\pi}(\mathbb{R})$  and  $e, d : \mathbb{R} \rightarrow \mathbb{R}$  are uniformly bounded satisfying the following assumption:

(H5) There are constants  $E^j_i, G^j_i, i, j \in \{+, -\}$  such that

$$\begin{aligned} E^j_- &= \underline{\lim}_{x \rightarrow j\infty} e_-(x), & G^j_- &= \underline{\lim}_{x \rightarrow j\infty} d_-(x) \\ E^j_+ &= \overline{\lim}_{x \rightarrow j\infty} e_+(x), & G^j_+ &= \overline{\lim}_{x \rightarrow j\infty} d_+(x). \end{aligned}$$

We use the framework of (3.8) with  $Bu = u' + u$ . Hence

$$B \sin(t + \tau) = \cos(t + \tau) + \sin(t + \tau) = \sqrt{2} \sin\left(\frac{\pi}{4} + t + \tau\right).$$

It is again clear that the condition (C) holds for our case. Let  $u = \rho \sin(t + \tau) + v$ ,  $v \in Y$  be uniformly bounded and  $\rho > 0$  be sufficiently large. Let  $f \in H$  be such that

$$f(s) \in [e_-(u'(s)) + d_-(u(s)), e_+(u'(s)) + d_+(u(s))]$$

a.e. on  $\mathbb{R}$ . Then we have

$$(f, B \sin(t + \tau)) = \sqrt{2} \int_0^{2\pi} f(s) \sin\left(\frac{\pi}{4} + s + \tau\right) ds.$$

We compute

$$\int_0^{2\pi} f(s) \sin\left(\frac{\pi}{4} + s + \tau\right) ds = \sum_{j=0}^7 \int_{\frac{\pi}{4}j - \tau}^{\frac{\pi}{4}(j+1) - \tau} f(s) \sin\left(\frac{\pi}{4} + s + \tau\right) ds. \tag{3.10}$$

We note that all functions  $\sin(\frac{\pi}{4} + t + \tau)$ ,  $\sin(t + \tau)$ ,  $\cos(t + \tau)$  do not change their signs on the intervals  $(\frac{\pi}{4}j - \tau, \frac{\pi}{4}(j + 1) - \tau)$ ,  $0 \leq j \leq 7$ . By estimating (3.10) like above (3.3) (we omit this tedious computation), we arrive at the following result.

**Theorem 3.6** Consider (3.9) under the condition (H5). Let us put

$$K_1 = \left(1 + \frac{\sqrt{2}}{2}\right)(E_-^+ + G_-^+ - E_+^- - G_+^-) + \left(1 - \frac{\sqrt{2}}{2}\right)(E_-^- + G_-^- - E_+^+ - G_+^+)$$

$$K_2 = \left(1 + \frac{\sqrt{2}}{2}\right)(E_-^- + G_-^- - E_+^+ - G_+^+) + \left(1 - \frac{\sqrt{2}}{2}\right)(E_-^+ + G_-^+ - E_+^- - G_+^-).$$

If  $K_1 > 0$ , respectively  $K_2 > 0$ , and  $h$  satisfies

$$\left| \int_0^{2\pi} h(s) e^{ts} ds \right| < K,$$

where  $K = K_1$ , respectively  $K = K_2$ , then (3.9) has a  $2\pi$ -periodic solution.

*Remark 3.7.* Similar problems for the continuous case are studied in [14].

### 4 Scalar Implicit Differential Inclusions

In this section, we consider the differential inclusion

$$f(u'', u', u, t) \in S(u', u, t), \tag{4.1}$$

where  $f : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}$  is continuous,  $S : \mathbb{R}^2 \times [0, 1] \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$  is upper semi-continuous with compact and convex values. Moreover, we suppose

- (I)  $f = f(z, w, u, t)$  is monotone in  $z \in \mathbb{R}$ .
- (II)  $\lim_{|z| \rightarrow \infty} |f(z, w, u, t)| = \infty$  for any  $w, u, t$ .

We put

$$T(w, u, t) = \left\{ v \in \mathbb{R} \mid f(v, w, u, t) = h, \quad h \in S(w, u, t) \right\}.$$

Now we show some properties of the mapping  $T$ .

**Lemma 4.1**  $T : \mathbb{R}^2 \times [0, 1] \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$  is upper semi-continuous with compact and convex set values.



*Proof.* If  $v_i \rightarrow v, w_i \rightarrow w, u_i \rightarrow u, t_i \rightarrow t$  are such that

$$f(v_i, w_i, u_i, t_i) = h_i, \quad h_i \in S(w_i, u_i, t_i),$$

then we can assume that  $h_i \rightarrow h$ . The upper semi-continuity of  $S$  and continuity of  $f$  imply

$$f(v, w, u, t) = h \in S(w, u, t).$$

So  $T$  is upper semi-continuous with compact values. The convexity of  $T(w, u, t)$  follows from (I) and from the convexity of  $S(w, u, t)$ . ■

Lemma 4.1 gives that (4.1) is equivalent to

$$u'' \in T(u', u, t). \tag{4.2}$$

By considering a suitable boundary value condition for (4.1) and growth conditions for  $f$  and  $S$  as well, Theorems 2.2, 2.3 and 2.4 can be applied to (4.2) with  $h = 0$ , like in Section 3. To be more concrete, and for the simplicity as well, we consider (4.1) of the form

$$f(u'') + u + g(u) = h(t), \tag{4.3}$$

where  $f$  is nondecreasing satisfying

$$(iv) \quad \sup_{u \in \mathbb{R}} |u - f(u)| < \infty.$$

Moreover,  $g : \mathbb{R} \rightarrow \mathbb{R}$  satisfies (i) and (ii) of Section 3, and  $h$  is continuous  $2\pi$ -periodic. The mapping  $T$  of (4.2) has now the form

$$T(u, t) = \left\{ v \in \mathbb{R} \mid f(v) + u = p, \quad p \in [-g_+(u) + h(t), -g_-(u) + h(t)] \right\}.$$

Hence (4.3) has the form

$$0 \in u'' + u + W(u, t),$$

where

$$W(u, t) = \left\{ v \in \mathbb{R} \mid f(-v - u) + u = p, \quad p \in [-g_+(u) + h(t), -g_-(u) + h(t)] \right\}.$$

By (iv), we put

$$\inf_{v \in \mathbb{R}} (f(v) - v) = C_-, \quad \sup_{v \in \mathbb{R}} (f(v) - v) = C_+.$$

Then  $v \in W(u, t)$  implies

$$g_-(u) - h(t) + C_- \leq v \leq g_+(u) - h(t) + C_+.$$

By repeating the same arguments like in Section 3 above (3.3), we obtain the following result.

**Theorem 4.2** *Consider (4.3) under the above conditions. Then (4.3) may have a  $2\pi$ -periodic solution only for  $h$  satisfying*

$$\left| \int_0^{2\pi} h(s) e^{2s} ds \right| \leq 2(A_+ - A_- + C_+ - C_-).$$

On the other hand, if  $B_- - B_+ + C_- - C_+ > 0$  and  $h$  satisfies

$$\left| \int_0^{2\pi} h(s) e^{is} ds \right| < 2(B_- - B_+ + C_- - C_+),$$

then (4.3) has a  $2\pi$ -periodic solution.

For instance, the equation

$$x'' - \sin(x'' + t) + x + a \operatorname{sgn} x = h(t), \quad a > 0$$

may have a  $2\pi$ -periodic solution only for  $h$  satisfying

$$\left| \int_0^{2\pi} h(s) e^{is} ds \right| \leq 4(a + 1),$$

and if  $a > 1$  then it has a  $2\pi$ -periodic solution provided

$$\left| \int_0^{2\pi} h(s) e^{is} ds \right| < 4(a - 1).$$

## 5 Implicit Operator Equations

Let  $L, H, Y, Z$  be given as in Section 2. Moreover,  $H$  is a Hilbert space with respect to  $(\cdot, \cdot)$ .

Ideas of previous section can be extended to more general implicitly given operator equations of the form

$$G(Lu, u) = 0, \quad (5.1)$$

where  $G : H \times Z \rightarrow H$  is continuous and monotone in the first variable. Moreover, we suppose

$$\lim_{|v| \rightarrow \infty} |G(v, u)| = \infty \quad \text{for any } u \in Z.$$

We put

$$V(u) = \{v \in H \mid G(v, u) = 0\}. \quad (5.2)$$

We show some properties of the mapping  $V$ .

**Lemma 5.1**  $V : Z \rightarrow 2^H \setminus \{\emptyset\}$  is w.u.s.c. with bounded convex set values.

*Proof.* We know from [5]

$$G(v, u) = 0 \iff (G(z, u), z - v) \geq 0 \quad \forall z \in H. \quad (5.3)$$

If  $u_i \rightarrow u$ ,  $V(u_i) \ni v_i \rightarrow v$ , then (5.3) implies

$$(G(z, u_i), z - v_i) \geq 0 \quad \forall z \in H. \quad (5.4)$$

By using the continuity of  $G$ , (5.4) implies

$$(G(z, u), z - v) \geq 0 \quad \forall z \in H.$$

Hence (5.3) gives  $G(v, u) = 0$ . So  $V$  is w.u.s.c. The convexity of  $V(u)$  follows also from (5.3). ■

Consequently, (5.1) is equivalent to the operator inclusion

$$Lu \in V(u),$$

which is of the form of (2.1).

Now we consider the implicit system

$$x'' + Ax + \Gamma(x'', x) = h(t), \tag{5.5}$$

where  $A$  is a symmetric matrix,  $\Gamma \in C(\mathbb{R}^{2n+1}, \mathbb{R}^n)$  is uniformly bounded and  $h \in L^2_{2\pi}(\mathbb{R}, \mathbb{R}^n)$ . We suppose that the equation  $x'' + Ax = 0$  has only one linearly independent  $2\pi$ -periodic solution  $w_0 \sin(t + \tau)$ ,  $\tau \in \mathbb{R}$ ,  $|w_0| = 1$ . We take

$$\begin{aligned} H = Z &= L^2_{2\pi}(\mathbb{R}, \mathbb{R}^n), \quad Y = \{x \in H \mid x', x'' \in H\} \\ G(v, u) &= v + \Gamma(v - Au, u) - h(t). \end{aligned}$$

Moreover, we assume that  $z + \Gamma(z, x)$  is monotone in  $z$ . Consequently, Lemma 5.1 holds for (5.5), and (5.5) has the form

$$\begin{aligned} 0 &\in Lu + F(u) \tag{5.6} \\ Lu &= -u'' - Au \\ F(u) &= \left\{ v \in H \mid v(t) + \Gamma(v(t) - Au(t), u(t)) - h(t) = 0 \right. \\ &\quad \left. \text{a.e. on } \mathbb{R} \right\}. \end{aligned}$$

$F$  is clearly uniformly bounded. For (2.2), we take  $S = \{w_0 \sin(t + \tau) \mid \tau \in \mathbb{R}\}$ . Let  $\langle \cdot, \cdot \rangle$  be the scalar product on  $\mathbb{R}^n$ .

**Theorem 5.2** *Consider (5.5) under the above conditions. In addition, we suppose the existence of a continuous mapping  $\omega : \{x \in \mathbb{R}^n \mid |x| = 1\} = S^{n-1} \rightarrow \mathbb{R}$  such that*

$$\liminf_{\rho \rightarrow \infty} \langle \Gamma(z, \rho w), w \rangle \geq \omega(w)$$

*uniformly with respect to  $w \in S^{n-1}$  and  $z$  as well. If  $\omega(w_0) > \omega(-w_0)$  and*

$$\left| \int_0^{2\pi} \langle h(s), w_0 \rangle e^{is} ds \right| < 2(\omega(w_0) - \omega(-w_0)),$$

*then (5.5) has a  $2\pi$ -periodic solution.*

*Proof.* We apply Theorem 2.3 with  $h = 0$  to (5.6). For any  $u \in Y$  uniformly bounded

and  $v \in F(\tilde{u})$ ,  $\tilde{u} = \rho w_0 \sin(t + \tau) + u$ , we have

$$\begin{aligned} & \overline{\lim}_{\rho \rightarrow \infty} \int_0^{2\pi} \langle v(s), \rho w_0 \sin(s + \tau) \rangle ds \\ &= - \underline{\lim}_{\rho \rightarrow \infty} \int_0^{2\pi} \langle \Gamma(v(s) - A\tilde{u}(s), \rho w_0 \sin(s + \tau) + u(s)) - h(s), w_0 \sin(s + \tau) \rangle ds \\ &\leq \int_0^{2\pi} \langle h(s), w_0 \sin(s + \tau) \rangle ds - \int_0^{2\pi} \omega(w_0 \operatorname{sgn} \{ \sin(s + \tau) \}) |\sin(s + \tau)| ds \\ &= \int_0^{2\pi} \langle h(s), w_0 \sin(s + \tau) \rangle ds - 2\omega(w_0) + 2\omega(-w_0). \end{aligned}$$

Hence (H3) holds with

$$\psi(w_0 \sin(t + \tau)) = \int_0^{2\pi} \langle h(s), w_0 \sin(s + \tau) \rangle ds - 2\omega(w_0) + 2\omega(-w_0).$$

(H4), with  $h = 0$ , has the form

$$0 > \int_0^{2\pi} \langle h(s), w_0 \sin(s + \tau) \rangle ds - 2\omega(w_0) + 2\omega(-w_0),$$

i.e.

$$\left| \int_0^{2\pi} \langle h(s), w_0 \rangle e^{is} ds \right| < 2(\omega(w_0) - \omega(-w_0)).$$

The proof is finished. ■

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