Contribution to the modelling of the hump effect by the study of an equation of Hamilton-Jacobi type

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Abstract

This paper is a contribution to the mathematical modelling of the hump effect. We present a mathematical study (existence, homogenization) of an Hamilton-Jacobi problem which represents the propagation of a front flame in a striated media.

1 Introduction

The physical problem consists in an anomaly of overvelocity observed in the combustion room of propellers during the combustion of some solid propellants blocks. This anomaly, called 'Hump effect', attains its maximum in the middle of the burning block. The reduced mathematical model of this phenomenon (hump effect) is the following Hamilton-Jacobi problem:

$$P_\xi \quad \begin{cases} \dfrac{\partial \xi}{\partial t} + R_0(\xi, s_2)\sqrt{1 + \left(\dfrac{\partial \xi}{\partial s_2}\right)^2} = 0 & \forall t > 0\ ,\ s_2 \in \mathbb{R} \\ \xi(s_2, 0) = \xi_0(s_2) & s_2 \in \mathbb{R} \end{cases}$$

where the unknown $s_1 = \xi(s_2, t)$ is the position of the flame front. We show in this paper that the anomaly results from the heterogeneity of the propellant blocks.

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Effectively, the blocks are striated (with the linner) and we prove by our study that the combustion velocity of the flame front is an increasing function of the angle between the striations (which are supposed here to be straight lines) and the flame front. Thus, we consider 3 cases: vertical striations ($\alpha = 0$), horizontal striations ($\alpha = \pi/2$) and oblique striations ($0 < \alpha < \pi/2$). We define some parameters: $L_0 > 0$, $L_1 = L_0/cos(\alpha)$ and $L_2 = L_0/sin(\alpha)$ like in FIG.1. $R_0(s_1, s_2)$ is a positive, périodic function in s_1 with period L_1 and in s_2 with period L_2. When $\alpha = 0$ (resp $\alpha = \pi/2$), R_0 depends periodically only in s_1 (resp s_2) with period L_0. The couch formed by the striations are called 'linner' and the second one is 'charge'. L_0 is the sum of the thickness of the 'linner' and the 'charge'.

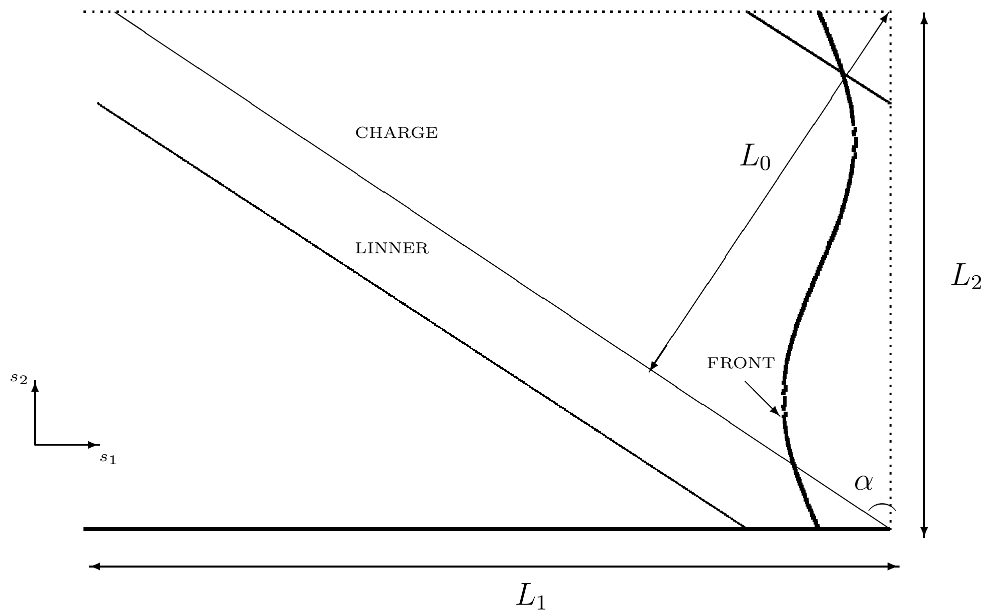

Figure 1: Domain of study (One period)

2 Existence and uniqueness

2.1 Vertical case

In this case, we have $R_0 = R_0(\xi)$ and the flame front can be reduced to a point and the problem becomes an ordinary differential equation of the form:

$$P_\xi^V \quad \begin{cases} \dfrac{d\xi}{dt} = -R_0(\xi) \qquad t > 0 \\ \xi(0) = \xi_0 \end{cases}$$

One knows that P_ξ^V has a unique solution $\xi \in W^{k+1,\infty}(0,T)$ $\forall k > 0$ et $T > 0$ provided $R_0 \in W^{k,\infty}(\mathbb{R})$. From the uniqueness of ξ, we have the following proposition

Proposition 1. *Let T be the real defined by: $\xi(T) - \xi(0) = -L_0$ where L_0 is the period of R_0. Then the speed $\dfrac{d\xi}{dt}$ is a periodic function of t with period T which is the time necessary to the front to cover the spacial period L_0.*

2.2 Horizontal case

In this section, one looks for periodic or quasi-periodic solutions. R_0 is a regular periodic and positive function of s_2 with period L_0. So we have the following problem:

$$P_\xi^H \quad \begin{cases} \dfrac{\partial \xi}{\partial t} + R_0(s_2)\sqrt{1+\left(\dfrac{\partial \xi}{\partial s_2}\right)^2} = 0 & \forall t > 0\ ,\ s_2 \in \mathbb{R} \\ \xi(s_2, 0) = \xi_0(s_2) & s_2 \in \mathbb{R} \end{cases}$$

Let $\Omega = \Omega_0$ be a subset of $\mathbb{R}$. We note $\Omega_0 = \Omega$, $\Omega_T = \Omega \times]0, T[$ for $T > 0$ and $E_T = C(\Omega_T)$ or $C(\Omega_T) \cap L^\infty(\Omega_T)$ or $W^{1,\infty}(\Omega_T)$. The function R_0 is supposed to verify:

$$R_0 \in C^2(\mathbb{R}), \quad \min_{\mathrm{x}\in\mathbb{R}} \mathrm{R_0(x) = R_{0l} \leq R_0(x) \leq R_{0c}} = \max_{\mathrm{x}\in\mathbb{R}} \mathrm{R_0(x)} \quad \forall \mathrm{x} \in \mathbb{R}.$$

Let $H(s_2, v) = R_0(s_2)\sqrt{1+v^2}$. Then we have the following theorem due to Crandall-Lions (see CL83):

Theorem 1. *If $\xi_0 \in E_0$, then the problem P_ξ^H has a unique viscosity solution $\xi \in E_T$ i.e satisfying: if (x_0, t_0) is a local maximum (resp minimum) point of $\xi - u$, then $\dfrac{\partial u}{\partial t}(x_0, t_0) + H\left[x_0, \nabla u\,(x_0, t_0)\right] \leq 0$ (resp ≥ 0). In addition, we have the following inequalities: if $\xi_0 \in W^{1,\infty}(\mathbb{R})$, then the viscosity solution $\xi \in W^{1,\infty}(\mathbb{R}\times]0,\mathrm{T}[)$ verifies:*

$$\|\frac{\partial \xi}{\partial t}\|_{L^\infty(\mathbb{R}\times]0,\infty[)} \leq c_1 \quad \text{and} \quad \|\frac{\partial \xi}{\partial s_2}\|_{L^\infty(\mathbb{R}\times]0,\infty[)} \leq c_2$$

where c_1 and c_2 are constants depending only on $\nabla\ \xi_0$.

The uniqueness of the viscosity solution of P_ξ^H yields the periodicity of ξ. Let formally define $\psi = \dfrac{\partial \xi}{\partial s_2}$. One remarks that if ξ is a viscosity solution of P_ξ^H, then ψ is an entropic solution (in the Kruzkov sense) of the problem Q_ψ^H below:

$$Q_\psi^H \quad \begin{cases} \dfrac{\partial \psi}{\partial t} + \dfrac{\partial}{\partial s_2}\left[R_0(s_2)\sqrt{1+\psi^2}\right] = 0 & \forall t > 0\ ,\ s_2 \in \mathbb{R} \\ \psi(s_2, 0) = \psi_0(s_2) & s_2 \in \mathbb{R} \end{cases}$$

The stationary solutions of Q_ψ^H verify $\psi(s_2) = \pm\sqrt{\left[\dfrac{c}{R_0(s_2)}\right]^2 - 1}$ where c is a positive constant $\geq R_{0c}$. We denote ψ_c the corresponding solution of ψ. This yields a sequence of solutions $(\psi_c)_{c \geq R_{0c}}$.

Lemma 1. *The stationary solutions $(\psi_c)_{c \geq R_{0c}}$ are discontinuous.*

Proof

We have: $[P1]:\ \exists y^* \in \mathbb{R};\ \psi_c(\mathrm{y}^*) = 0$, $\quad [\mathrm{P2}]:\ \displaystyle\int_0^{\mathrm{L_0}} \psi_c(\mathrm{s_2})\mathrm{ds_2} = 0$. Applying $[P1]$ we find $c = R_{0c} \equiv c^*$. $[P2]$ implies that ψ_c is negative and positive as well. As $c = c^*$, from the definition of R_0 (see FIG.2), we have $\psi_c(s_2) = 0 \Longleftrightarrow R_0(s_2) = R_{oc}$

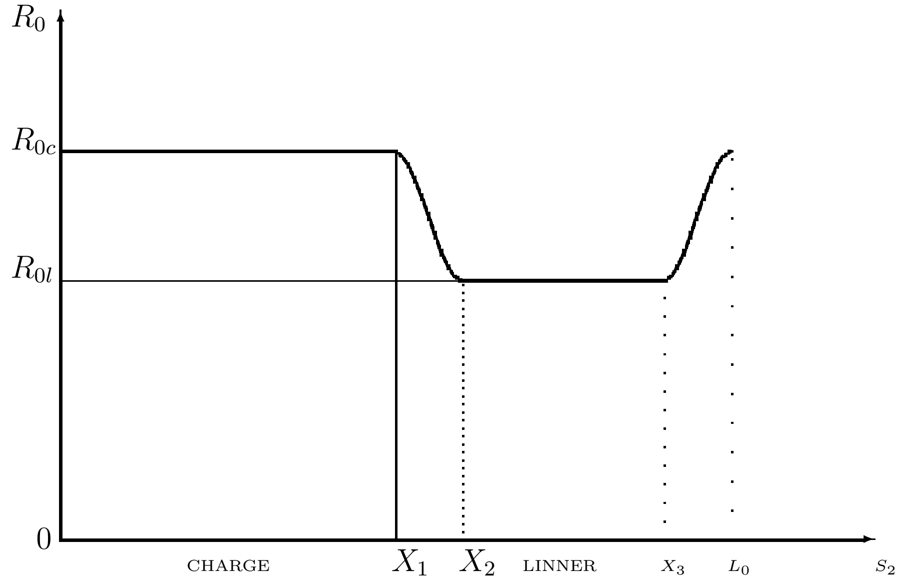

Figure 2: Function R_0

i.e $s_2 \in$ 'charge'. If ψ_c was not discontinuous, one can find y $\notin$ 'charge' with $\psi_c(y) = 0$. Then since $R_0(y) < R_{oc}$ we have $c^* = R_0(y) < R_{0c} = c^*$ which is absurd. We conclude that ψ_c is not continuous. The physical solution ξ verifies $R_{0l} \leq |\frac{\partial \xi}{\partial t}| \leq R_{0c}$. In these conditions, c^* is the unique value of c which satisfies this inequality. From the curve of R_0 and the value of $c^*, \psi_{c^*}(s_2) = 0 \quad \forall s_2 \in [0, X_1]$ (see FIG.2) i.e ψ_{c^*} is continuous on this interval. Since ψ_{c^*} is discontinuous, it exists $x^* \in [X_1, L_0]$ so that $\forall s_2 \in [X_1, L_0]$, $\psi_{c^*}(s_2) = \sqrt{\left[\frac{c^*}{R_0(s_2)}\right]^2 - 1}$ if $X_1 \leq s_2 < x^*$ and $-\sqrt{\left[\frac{c^*}{R_0(s_2)}\right]^2 - 1}$ if $x^* < s_2 \leq L_0$. The inverse is not possible. In fact, in these conditions, the discontinuty in x^* will be increasing thus inadmissible i.e the solution ψ_{c^*} will not be entropic because H is convex in $\nabla \xi \quad \forall s_2 \in \mathbb{R}$. One easily verifies that ψ_{c^*} has a unique point of discontinuty on $[0, L_0]$ equal to $x^* = \frac{X_2 + X_3}{2}$. Then the function ψ_{c^*} is defined as follow:

$$\psi_{c^*}(s_2) = \begin{cases} 0 & \text{if } 0 \leq s_2 \leq X_1 \\ \sqrt{\left(\frac{c^*}{R_0(s_2)}\right)^2 - 1} & \text{if } X_1 \leq s_2 < x^* \\ -\sqrt{\left(\frac{c^*}{R_0(s_2)}\right)^2 - 1} & \text{if } x^* < s_2 \leq L_0 \end{cases}$$

We then prove the following theorem:

Theorem 2. *ψ_{c^*} is the unique periodic stationary solution of $\psi_t + \left[R_0(s_2)\sqrt{1+\psi^2}\right]_{s_2} = 0$ and P_ξ^H has a unique wave and explicit solution ξ_{c^*} of the form: $\xi_{c^*}(s_2, t) = -c^*.t + \int_0^{s_2} \psi_{c^*}(x)dx$*

Remark 1. *By considering the quasi-periodic solutions we prove that P_{ξ}^{H} has a unique wave and explicit solution verifying $\xi_{c^*}(s_2) = \xi_{c^*}(s_2 + L_0) \pm D$ where D is the gap (to the right or left: see section 3.3). The corresponding solution ψ_{c^*} is periodic and of the form: $\psi_{c^*}^{D}(s_2) = 0$ if $0 \le s_2 \le X_1$ and $\pm\sqrt{\left[\frac{c^*}{R_0(s_2)}\right]^2 - 1}$ if $X_1 \le s_2 < L_0$*

3 Homogenization

3.1 Vertical case

Let ε be a positive parameter tied up to the dimension of the period and destinated to tightened to 0. We define R_0^ε by: $R_0^\varepsilon(s_1) = R_0\left(\frac{s_1}{\varepsilon}\right)$ and look for $\xi^\varepsilon(t)$ verifying the problem:

$$P_{\xi^\varepsilon}^{V} \quad \begin{cases} \dfrac{d\xi^\varepsilon}{dt} + R_0^\varepsilon(\xi^\varepsilon) = 0 \quad \forall t > 0 \\ \xi^\varepsilon(0) = \xi_0 \end{cases}$$

We know that it exists an unique $\xi^\varepsilon \in W^{k+1,\infty}(0,T)$ since $R_0 \in W^{k,\infty}(\mathbb{R})$ for fixed ε. R_0^ε periodic in s_1 with period εL_0. For $\varepsilon \longrightarrow 0$, we have $R_0^\varepsilon \longrightarrow \frac{1}{L_0}\int_0^{L_0} R_0(s_1)ds_1 \overset{\text{déf}}{\equiv} \mathcal{M}_{L_0}(R_0)$ which is the average of R_0. Let ϕ a test function on [0,T]. We have $\int_0^T \frac{1}{R_0^\varepsilon(\xi^\varepsilon)}\phi(t)d\xi^\varepsilon = -\int_0^T \phi(t)dt$. Let $\tau = \xi^\varepsilon(t)$ and $\xi^\varepsilon(0) = 0$ to simplify then we find $\int_0^{\xi^\varepsilon(T)} \frac{1}{R_0^\varepsilon(\tau)}\phi\left[(\xi^\varepsilon)^{-1}(\tau)\right]d\tau = -\int_0^T \phi(t)dt$. We also have $\frac{1}{R_0^\varepsilon(\tau)} \overset{\varepsilon \to 0}{\longrightarrow} \mathcal{M}_{L_0}\left(\frac{1}{R_0}\right)$ $L^\infty(\mathbb{R})$ weak star and $\xi^\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \xi^0$ uniformly on [0,T].
So for $\varepsilon \longrightarrow 0$, we obtain $\int_0^{\xi^0(T)} \mathcal{M}_{L_0}\left(\frac{1}{R_0}\right)\phi\left[\left(\xi^0\right)^{-1}(\tau)\right]d\tau = -\int_0^T \phi(t)dt$.
By $t = \left(\xi^0\right)^{-1}(\tau)$, we find $\int_0^T \mathcal{M}_{L_0}\left(\frac{1}{R_0}\right)\frac{d\xi^0}{dt}\phi(t)dt = -\int_0^T \phi(t)dt$ i.e $\frac{d\xi^0}{dt} = -R_0^h$ where R_0^h is the harmonic average of R_0. The following theorem is then proved.

Theorem 3. *The solution ξ^ε of the problem $P_{\xi^\varepsilon}^{V}$ converges when $\varepsilon \longrightarrow 0$ to ξ^0 verifying: $\xi^0(t) = -R_0^h t + \xi_0$.*

Remark 2. *ξ^0 is a progressive wave with velocity $-R_0^h$ where $-R_0^h$ is exactly the average velocity of the front.*

3.2 Horizontal case

As in the vertical case, let's have $R_0^\varepsilon(s_2) = R_0\left(\frac{s_2}{\varepsilon}\right)$ and the following Cauchy problem which is to find ξ^ε verifying :

$$P_{\xi^\varepsilon}^{H} \quad \begin{cases} \dfrac{\partial\xi^\varepsilon}{\partial t} + R_0^\varepsilon(s_2)\sqrt{1 + \left(\dfrac{\partial\xi^\varepsilon}{\partial s_2}\right)^2} = 0 & (s_2, t) \in \mathbb{R}\times]0,\mathrm{T}[\\ \xi^\varepsilon(s_2, 0) = \xi_0(s_2) & s_2 \in \mathbb{R} \end{cases}$$

We look for periodic solutions in s_2 with period L_0. For fixed ε, $P^H_{\xi^\varepsilon}$ has a unique viscosity solution $\xi^\varepsilon \in W^{1,\infty}(\mathbb{R}\times]0,T[)$ provided $\xi_0 \in W^{1,\infty}(\mathbb{R})$. The asymptotic development of ξ^ε is in the form $\xi^\varepsilon(s_2,t) = \xi^0(s_2,t,y) + \sum_{i\geq 1} \varepsilon^i \xi^i(s_2,t,y)$ where we have $y = s_2/\varepsilon$. Let $Y =]0, L_0[$; then R_0 is Y-periodic in y. For $i \geq 1$, the functions ξ^i are Y-periodic in y. The differenciations with regards to t and s_2 become

$$\frac{\partial \xi^\varepsilon}{\partial t} = \frac{\partial \xi^0}{\partial t} + \sum_{i\geq 1} \varepsilon^i \frac{\partial \xi^i}{\partial t} \qquad \text{and} \qquad \frac{\partial \xi^\varepsilon}{\partial s_2} = \frac{1}{\varepsilon}\frac{\partial \xi^0}{\partial y} + \sum_{i\geq 0} \epsilon^i \left(\frac{\partial \xi^i}{\partial s_2} + \frac{\partial \xi^{i+1}}{\partial y}\right)$$

We take the square of the equality $\frac{\partial \xi^\varepsilon}{\partial t} = -R_0^\varepsilon(s_2)\sqrt{1 + \left(\frac{\partial \xi^\varepsilon}{\partial s_2}\right)^2}$ after replacing $\frac{\partial \xi^\varepsilon}{\partial t}$ and $\frac{\partial \xi^\varepsilon}{\partial s_2}$ by their development. After calculations and identifying the terms in front of ε, we find

$$[R_0(y)]^2 \left(\frac{\partial \xi^0}{\partial y}\right)^2 = 0 \qquad (1)$$

$$[R_0(y)]^2 \frac{\partial \xi^0}{\partial y}\left(\frac{\partial \xi^0}{\partial s_2} + \frac{\partial \xi^1}{\partial y}\right) = 0 \qquad (2)$$

$$\left(\frac{\partial \xi^0}{\partial t}\right)^2 - [R_0(y)]^2 \left[1 + \left(\frac{\partial \xi^0}{\partial s_2} + \frac{\partial \xi^1}{\partial y}\right)^2 + 2\left(\frac{\partial \xi^0}{\partial y}\right)\left(\frac{\partial \xi^1}{\partial s_2} + \frac{\partial \xi^2}{\partial y}\right)\right] = 0 \qquad (3)$$

The equation (3) gives $\frac{\partial \xi^0}{\partial t} + R_0(y)\sqrt{1 + \left(\frac{\partial \xi^0}{\partial s_2} + \frac{\partial \xi^1}{\partial y}\right)^2} = 0$ from $P^H_{\xi^\varepsilon}$ and (1). We denote $\bar{H}(p) = R_0(y)\sqrt{1 + \left(\frac{\partial \xi^0}{\partial s_2} + \frac{\partial \xi^1}{\partial y}\right)^2}$ where $p = \frac{\partial \xi^0}{\partial s_2}$; it doesn't depend on y. Let $v = \xi^1$, the problem to solve is

$$P_v \quad \begin{cases} \text{Find } v \text{ viscosity solution of} \\ R_0(y)\sqrt{1 + \left(p + \frac{\partial v}{\partial y}\right)^2} = \bar{H}(p) \\ v \text{ Y-periodic in y; p is a "parameter"} \end{cases}$$

We have $\frac{\partial v}{\partial y} = \pm\sqrt{\left[\frac{\bar{H}(p)}{R_0(y)}\right]^2 - 1} - p$ with $\bar{H}(p) \geq R_0(y) \ \ \forall y \in \mathbb{R}$. Let $y_0 \in \mathbb{R}$ with $R_0(y_0) = R_{0c}$. We consider the function f defined by:

$$f(y) = \frac{1}{L_0}\int_{y_0}^{y} \sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\, d\tau - \frac{1}{L_0}\int_{y}^{y_0+L_0} \sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\, d\tau.$$

So $f(y_0) = -\frac{1}{L_0}\int_{y_0}^{y_0+L_0} \sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\, d\tau$ and $f(y_0+L_0) = -f(y_0)$. As f is continuous, for all p as $|p| \leq \frac{1}{L_0}\int_{y_0}^{y_0+L_0} \sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}\, d\tau$, $\exists \bar{y} \in [y_0, y_0 + L_0]\,;\ f(\bar{y}) = p$

i.e

$$\int_{y_0}^{\bar{y}} \left[\sqrt{\left(\frac{R_{0c}}{R_0(\tau)}\right)^2 - 1} - p \right] d\tau = \int_{\bar{y}}^{y_0+L_0} \left[\sqrt{\left(\frac{R_{0c}}{R_0(\tau)}\right)^2 - 1} + p \right] d\tau$$

We define then a function v by: $v(y) = \displaystyle\int_{y_0}^{y} \left[\sqrt{\left(\frac{R_{0c}}{R_0(\tau)}\right)^2 - 1} - p \right] d\tau$ if $y_0 \leq y \leq \bar{y}$ and $\displaystyle\int_{y}^{y_0+L_0} \left[\sqrt{\left(\frac{R_{0c}}{R_0(\tau)}\right)^2 - 1} + p \right] d\tau$ if $\bar{y} \leq y \leq y_0 + L_0$ and extend v to all $\mathbb{R}$ periodically. One easily verifies that $\forall p$ with $|p| \leq \displaystyle\frac{1}{L_0}\int_{y_0}^{y_0+L_0} \sqrt{\left[\frac{R_{0c}}{R_0(\tau)}\right]^2 - 1}$, the function v definied above is a viscosity solution of P_v.

Lemma 2. $\bar{H}(p) = \max\limits_{y\in\mathbb{R}} R_0(y) \equiv R_{0c}$.

Proof:

We have $\displaystyle\frac{\partial v}{\partial y}(y_0^+) = \sqrt{\left[\frac{\bar{H}(p)}{R_0(y_0)}\right]^2 - 1} - p$ and $\displaystyle\frac{\partial v}{\partial y}(y_0^-) = -\sqrt{\left[\frac{\bar{H}(p)}{R_0(y_0)}\right]^2 - 1} - p$. We so obtain $\displaystyle\frac{\partial v}{\partial y}(y_0^+) \geq \frac{\partial v}{\partial y}(y_0^-)$. In the same way, we have $\displaystyle\frac{\partial v}{\partial y}(\bar{y}^+) \leq \frac{\partial v}{\partial y}(\bar{y}^-)$. As v is a viscosity solution, the following inequalities hold:

$$\begin{aligned} R_0(y_0)\sqrt{1 + (p+\eta)^2} - \bar{H}(p) \ &\geq 0 \qquad \forall \eta;\ \frac{\partial v}{\partial y}(y_0^+) \geq \eta \geq \frac{\partial v}{\partial y}(y_0^-) \\ R_0(\bar{y})\sqrt{1 + (p+\zeta)^2} - \bar{H}(p) \ &\leq 0 \qquad \forall \zeta;\ \frac{\partial v}{\partial y}(\bar{y}^+) \leq \zeta \leq \frac{\partial v}{\partial y}(\bar{y}^-) \end{aligned}$$

We deduce that $\bar{H}(p) = R_{0c}$. So the formal homogenized problem is then:

$$P_{\xi^0}^{\bar{H}} \quad \begin{cases} \dfrac{d\xi^0}{dt} + R_{0c} = 0 & t > 0 \\ \\ \xi^0(0) = \mathcal{M}_{L_0}(\xi_0) \end{cases}$$

and the solution ξ^0 is: $\xi^0(t) = \xi_0 - R_{0c}t \quad \forall t \geq 0$. It doesn't depend on s_2; the "homogenized" front is a vertical line which velocity doesn't depend on the **presence** of the striations (linner). The absolute value of the velocity of the wave solution is R_{0c} and it is greater than the one in the vertical case (R_0^h).

Theorem 4. *For all* $\xi_0^\varepsilon \in W^{1,\infty}(\mathbb{R})$*, the solution* ξ^ε *of* $P_{\xi^\varepsilon}^H$ *converges uniformly on* $\mathbb{R} \times [0, \mathrm{T}]$ $\forall \mathrm{T}$ $(\mathrm{T} < +\infty)$ *to the viscosity solution* ξ^0 *of the problem* $P_{\xi^0}^{\bar{H}}$ *in* $C(\mathbb{R} \times [0, \mathrm{T}])$.

Proof

The uniqueness of ξ^ε of $P_{\xi^\varepsilon}^H$ yields a contraction (in sup norm) semi-group $S^\varepsilon(t)$ on $W^{1,\infty}(\mathbb{R})$ which converges on compact set of $\mathbb{R} \times [0, +\infty[$ to $S(t)$. By the inverse theorem of P.L. Lions and M. Nisio (see Lio85) and the unicity of $\bar{H}(p)$, one can conclude that ξ^ε converge uniformly to ξ^0 which satisfies $P_{\xi^0}^H$.

3.3 Oblique case

Here, we look for solutions ξ verifying the conditions below (see FIG.3):

i) θ is the angle between the front and the vertical where $R_0(s_2) = R_{0l}$,

ii) $0 \leq \theta \leq \alpha$,

iii) $\dfrac{\partial \xi}{\partial s_2} = 0$ where $R_0(s_2) = R_{0c}$,

iv) The front spreads with constant velocity in the direction of the striations.

Let R_0 be discontinuous with two constant states i.e R_{0c} and R_{0l}. Then we obtain the following relation: $R_{0c}\,(1 - cotg\alpha tg\theta) = R_{0l}\sqrt{1+tg^2\theta}$. We deduce the equation for $tg\theta$ of the form:

$$\left(R_{0l}^2 - R_{0c}^2 cotg^2\alpha\right) tg^2\theta + (2R_{0c}^2 cotg\alpha) tg\theta + \left(R_{0l}^2 - R_{0c}^2\right) = 0$$

where $\Delta' = -R_{0l}^4 + R_{0l}^2 R_{0c}^2\,(1 + cotg^2\alpha) > 0$ for all R_0 and $\alpha \neq 0$. The relation ii) gives:

$$\theta = arctg\left[\left(-R_{0c}^2 cotg\alpha + \sqrt{\Delta'}\right) / \left(R_{0l}^2 - R_{0c}^2 cotg^2\alpha\right)\right]$$

If the initial condition is a front with gradient null in the 'charge' and presenting an angle θ in the 'linner', one verifies that these solutions don't distort i.e the angle θ is preserved and the velocity in the direction of the striations is constant. Those solutions are not periodic but staggered from one period to another with (see FIG.3):

$$D = e\frac{sin\theta}{sin(\alpha - \theta)}$$

where e is the thickness of the striations. In these conditions, one can resolve the problem in the bounded domain $]0, \bar{Y}[$ with the following boundary conditions $\xi(0) = \xi(\bar{Y}) - D$ for the staggering to the left. In the general case, the staggering to the right doesn't produce fronts with constant velocity in the direction of the striations. Concretly, it is to solve the Hamilton-Jacobi problem with the staggered condition. So we have:

$$P_\xi^D \quad \begin{cases} \dfrac{\partial \xi}{\partial t} + R_0(\xi, s_2)\sqrt{1 + \left(\dfrac{\partial \xi}{\partial s_2}\right)^2} = 0 & \forall (s_2, t) \in]0, \bar{Y}[\times]0, T[, \\ \xi(s_2, 0) = \xi_0(s_2) & s_2 \in]0, \bar{Y}[\\ \xi(0, t) = \xi(\bar{Y}, t) - D & t \geq 0 \end{cases}$$

with $\bar{Y}$ defined by: $\bar{Y} = L_0 + (L_0 - e/sin\alpha)\,\dfrac{tg\theta}{tg\alpha - tg\theta}$.

Remark 3. *In the horizontal case, $\theta_1 = -\theta_2$. Then one can have the two staggerings i.e $\xi(0) = \xi(\bar{Y}) \pm D$ if we wish to stagger to left or right.*

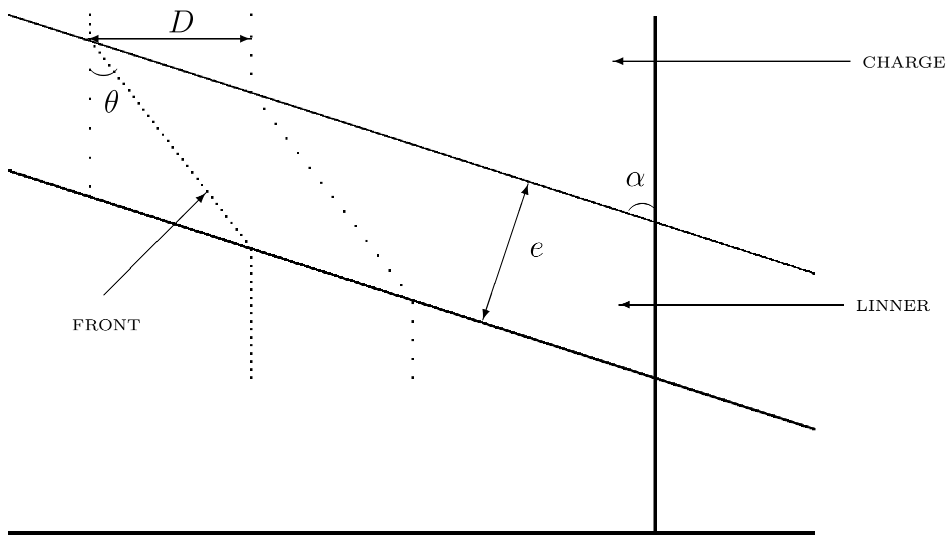

Figure 3: Staggered front

3.3.1 The average velocity

We recall that $R_0(s_1, s_2)$ is periodic in s_1 and s_2 with period $L_1 = L_0/cos\alpha$ and $L_2 = L_0/sin\alpha$ respectively, for $0 < \alpha < \pi/2$. The average velocity is the quotient of L_1 by the time necessary to the front (or a point of the front) to cover the distance L_1. Let L_c and L_l be the lengths of the 'charge' and the 'linner' respectively on a period; T_c and T_l the corresponding times. Let r be the quotient of the thickness of the 'charge' by the one of the 'linner'. Then we have:

$$e = \frac{L_0}{1+r} \qquad L_l = \frac{L_0}{(1+r)cos\alpha} \qquad L_c = L_1 - L_l$$

$$T_l = \frac{L_l}{R_{0l}\sqrt{1+tg^2\theta}} \qquad T_c = \frac{L_1 - L_l}{R_{0c}}$$

The velocity of the front is equal to $V_c = -R_{0c}$ in the 'charge and $V_l = -R_{0l}\sqrt{1+tg^2\theta}$ in the 'linner'. Let V_m the absolute value of the average velocity. It is a function of r and α with $\theta = \theta(\alpha)$, let note it $V_m(r, \alpha)$. Then it verifies: $V_m(r, \alpha) = \frac{L_1}{T_c + T_l}$. By replacing L_1 , L_l , T_c , T_l... by their values, one finds after simplification:

$$V_m(r, \alpha) = \frac{1+r}{\left(\frac{r}{R_{0c}} + \frac{1}{R_{0l}\sqrt{1+tg^2\theta}}\right)}$$

- In the vertical case, we have: $\alpha = \theta = 0$ and $V_m(r, 0) = R_0^h$.
- In the horizontal case, $\alpha = \pi/2$, $R_{0c} = R_{0l}\sqrt{1+tg^2\theta}$ and $V_m(r, \pi/2) = R_{0c}$.

These values are the same we found previously. One easily verifies that $V_m(r, \alpha)$ is an increasing function of r and α for fixed R_0.

3.3.2 The overvelocity coefficient

For fixed r, it is the rate of the growth of $V_m(r,\alpha)$ between 0 and $\pi/2$. We note it $G(r)$ and have:

$$G(r) = 1 - \frac{V_m(r,0)}{V_m(r,\pi/2)} = 1 - \frac{R_0^h}{R_{0c}}$$

It is an decreasing function of r. For reasonnable values of r which determines the lenght of the striations, we observe an overvelocity coefficient analogous to the one found experimentally.

References

[Bar92] G. Barles, *Solutions de viscosité des équations d'Hamilton-Jacobi du premier ordre et applications*, Faculté des Sciences et Techniques; Parc de Grandmont Tours-France

[BENS90] C.M. Brauner, T.K. Edarh-Bossou, G. Namah and C. Schmidt-Lainé, *Pré-étude sur l'homogénéisation de l'effet "Hump",* Rapport, Univ Bordeaux I, Ens-Lyon, Juin 1990

[CEL84] M.G. Crandall, L.C. Evans and P.L Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations,* AMS 282(1984), no 2

[CIL92] M.G. Crandall, H. Ishii and P.L Lions, *User's guide to viscosity solutions of second order partial differential equation,* AMS 27(1992) no 1 1-63

[CL83] M.G. Crandall and P.L Lions, *Viscosity solutions of Hamilton-Jacobi equations,* AMS 277(1983), no 1

[Eda89] T.K. Edarh-Bossou, *Modélisation numérique de la combustion d'un bloc de propergol solide - effet hump,* DEA, Univ Claude-Bernard Lyon I-ENS Lyon, 1989

[Eda93] T.K. Edarh-Bossou, *Etude de la propagation d'un front de flamme dans un milieu strié,* thèse de Doctorat, Univ Claude-Bernard Lyon I-ENS Lyon, 1993

[Lio82] P.L. Lions *Generalized solutions of Hamilton-Jacobi equations,* Pitman, London 1982

[Lio85] P.L Lions *Some properties of the viscosity semigroups for Hamilton-Jacobi equations,* Nonlinear Differential Equations ed. J.K. Hale and P. Martinez-Amores, Pitman London, 1985

[LPV87] P.L. Lions, G. Papanicolau and S.R.S Varadan, *Homogenization of Hamilton-Jacobi equation,* Preprint, Univ Paris dauphine, 1987

[Nam90] G. Namah, *Etude de deux modèles de combustion en phase gazeuse et en milieu strié,* thèse de Doctorat, Univ Bordeaux I, 1990

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