Transitive and Co–Transitive caps

A. Cossidente O.H. King

1 Introduction

Let $PG(r, q)$ be the projective space of dimension r over $GF(q)$. A *k–cap* $\bar{K}$ in $PG(r, q)$ is a set of k points, no three of which are collinear [10], and a k–cap is said to be *complete* if it is maximal with respect to set–theoretic inclusion. The maximum value of k for which there is known to exist a k–cap in $PG(r, q)$ is denoted by $m_2(r, q)$. Some known bounds for $m_2(r, q)$ are given below.

Suppose that $\bar{K}$ is a cap in $PG(r, q)$ with automorphism group $\bar{G}_0 \leq P\Gamma L(r+1, q)$. Then $\bar{K}$ is said to be *transitive* if $\bar{G}_0$ acts transitively on $\bar{K}$, and *co-transitive* if $\bar{G}_0$ acts transitively on $PG(r, q) - \bar{K}$.

Our main result is the following theorem.

Theorem 1. *Suppose $\bar{K}$ is a transitive, co–transitive cap in $PG(r, q)$. Then one of the following occurs:*

1. *$\bar{K}$ is an elliptic quadric in $PG(3, q)$ and q is a square when q is odd;*
2. *$\bar{K}$ is the Suzuki–Tits ovoid in $PG(3, q)$ and $q = 2^h$, with h odd and ≥ 3;*
3. *$\bar{K}$ is a hyperoval in PG(2,4);*
4. *$\bar{K}$ is an 11–cap in $PG(4, 3)$ and $\bar{G}_0 \simeq M_{11}$;*
5. *$\bar{K}$ is the complement of a hyperplane in $PG(r, 2)$;*
6. *$\bar{K}$ is a union of Singer orbits in $PG(r, q)$ and $G_0 \leq \Gamma L(1, p^d) \leq GL(d, p)$.*

Received by the editors March 1999.
Communicated by J. Thas.
1991 *Mathematics Subject Classification :* Primary 51E22; Secondary 20B15, 20B25.
Key words and phrases : Caps, Rank 3 permutation groups.

In each of 1–5 $\bar{K}$ *is indeed a transitive co–transitive cap.*

Our conclusion is that transitive, co–transitive caps are rare with the possible exception of unions of Singer cyclic orbits.

The origin of this problem are papers by Hill [8], [7], in which he studies such caps whose automorphism group acts 2–transitively on the cap. [As he notes [8, Theorem 1], it is trivial to show that if $\bar{K}$ is a subset of $PG(r,q)$ lying in no proper subspace and admitting a 3–transitive group then $\bar{K}$ must be a cap.] Hill gives a short list of possibilities (omitting Suzuki–Tits ovoids) but excludes caps in $PG(r,q)$ for $q>2$ and $r\geq 13$. We find no new caps but show that any other transitive, co–transitive cap is a union of Singer cyclic orbits.

The known upper bounds on cap sizes are summarised in the following Result.

Result 2. *[10, Theorem 27.3.1]*
$m_2(2,q)=q+1$ *(for q odd);*
$m_2(2,q)=q+2$ *(for q even);*
$m_2(3,q)=q^2+1$ *for* $q>2$*;*
$m_2(r,2)=2^r$*; and*
$m_2(r,q)\leq q^{r-1}$ *for* $q>2$ *and* $r\geq 4$.

The bounds for $q>2$ and $r\geq 4$ are not the best known, but they are sufficient here.

We begin by showing that as a consequence of Result 2, a cap must be smaller than its complement (with one exception). It then follows that in considering subgroups of $P\Gamma L(r+1,q)$ having two orbits, we need only consider the smaller orbit when looking for transitive, co-transitive caps.

Lemma 3. *Suppose that $\bar{K}$ is a cap in $PG(r,q)$. Then either $|\bar{K}|<(q^{r+1}-1)/2(q-1)$, or $q=2$ and $\bar{K}$ is the complement of a hyperplane.*

Proof. It is easy to deduce from Result 2, that the result holds when $q\neq 2$. Thus suppose now that $q=2$ and that $|\bar{K}|\geq(2^{r+1}-1)/2$. The only possiblity is that $|\bar{K}|=2^2$. Let $x\in\bar{K}$. For each $y\in\bar{K}-\{x\}$ there is a line through x and y and the 2^r-1 such lines must be distinct since $\bar{K}$ is a cap. However x lies on exactly 2^r-1 lines in $PG(r,2)$ and so every line in $PG(r,2)$ through x meets $\bar{K}$ in two points and $PG(r,q)-\bar{K}$ in one point. Therefore any line meeting $PG(r,q)-\bar{K}$ in at least two points is contained in $PG(r,q)-\bar{K}$. This shows that $PG(r,q)-\bar{K}$ is a subspace of $PG(r,2)$ and its size shows that it is a hyperplane. ∎

Using Result 6 we shall know orbit lengths when looking at candidates for transitive, co-transitive caps. Lemma 5 below helps in eliminating a number of possibilities.

Definition 4. *Suppose that $\bar{K}$ is a cap in $PG(r,q)$. For any $x\in PG(r,q)$, the* chord-number *of x is the number of chords (2-secants) of $\bar{K}$ passing through x.*

Lemma 5. *Suppose that $\bar{K}$ is a tranistive, co-transitive cap in $PG(r,q)$ and suppose that $x\in PG(r,q)-\bar{K}$. Let $k=|\bar{K}|$ and $m=|PG(r,q)-\bar{K}|$. Then the chord-number, c, of x is given by*

$$c = \frac{k(k-1)(q-1)}{2m}.$$

In particular the expression for c always yields an integer.

Proof. We count combinations of chords and points of $PG(r,q) - \bar{K}$ in two ways. Firstly there are $k(k-1)/2$ chords of $\bar{K}$ and each has $q-1$ points not in $\bar{K}$. There is a subgroup $\bar{G_0}$ of $\Gamma L(r+1,q)$ acting transitively on $PG(r,q)-\bar{K}$, so each of these m points has the same chord-number c and a second count gives mc chord-point combinations. Thus $mc = k(k-1)(q-1)/2$ leading to the required expression for c.

■

The main tool in our investigation is the substantial result by M.W. Liebeck [12], where the affine permutation groups of rank three are classified.

Result 6. *[12] Let G be a finite primitive affine permutation group of rank three and of degree $n = p^d$, with socle V, where $V \simeq (Z_p^d)$ for some prime p, and let G_0 be the stabilizer of the zero vector in V. Then G_0 belongs to one of the following families:*
(A) *11 Infinite classes;*
(B) *Extraspecial classes with $G_0 \le N_{\Gamma L(d,p)}(R)$, where R is a 2–group or 3–group irreducible on V;*
(C) *Exceptional classes. Here the socle L of $G_0/Z(G_0)$ is simple (where $Z(G_0)$ denotes the centre of G_0).*

We shall recall the details of the groups belonging to the classes in **(A)**, **(B)** and **(C)** as we need them.

Suppose $\bar{K}$ is a cap in $PG(r,q)$ such that a subgroup $\bar{G_0}$ of $P\Gamma L(r+1,q)$ acts transitively on each of $\bar{K}$ and its complement. Then $\bar{G_0}$ corresponds to a subgroup G_0 of $GL(d,p)$ having three orbits on the vectors of $V(d,p)$, where p is prime and $p^d = q^{r+1}$. Moreover G_0 will contain matrices corresponding to scalar multiplication by elements of $GF(q)^*$. As we demonstrate shortly, with one exception, $V(d,p)\cdot G_0$ is primitive as a permutation group, so Liebeck's theorem may be applied. Notice that since we are interested in groups G_0 containing $GF(q)^*$ we avoid the possibility of two orbits of vectors in $V(d,p)$ giving rise to a single orbit of points in $PG(r,q)$.

Clearly G_0 may be embedded in $\Gamma L(r+1,q)$. At the beginning of Section 1 of [12], Liebeck notes that in his result $G_0 \le GL(d,p)$ is embedded in $\Gamma L(a, p^{d/a})$ with a minimal. Thus $r+1 \ge a$ i.e. $q \le p^{d/a}$. Moreover in almost all cases it is clear that the groups he identifies have orbits that are unions of 1–dimensional subspaces of $V(a, p^{d/a})$ (excluding the zero vector). If a 1-dimensional subspace over $GF(p^{d/a})$ does contains vectors u, v that are linearly independent over $GF(q)$, then u, v and $u+v$ correspond to three collinear points in $PG(r,q)$ and the orbit in $PG(r,q)$ cannot be a cap. Thus in our setting we usually have $q = p^{d/a}$: there is just one exception, the class A1, although we have to justify $q = p^{d/a}$ for the class A2.

Lemma 7. *Suppose $\bar{K}$ is a transitive, co-transitive cap in $PG(r,q)$ with $\bar{G_0} \le P\Gamma L(r+1,q)$ acting transitively on each of $\bar{K}$ and $PG(r,q) - \bar{K}$ and suppose that G_0 is the pre-image of $\bar{G_0}$ in $GL(d,p)$. Let $H = V(d,p)\cdot G_0$. Then H is imprimitive on $V = V(d,p)$ if and only if $q = 2$ and $\bar{K}$ is the complement of a hyperplane.*

Proof. Suppose that H is imprimitive on V. Let Ω be a block containing 0. Then the two orbits of non–zero vectors of G_0 are $\Omega \setminus 0$ and $V \setminus \Omega$. Let u and v be any two vectors in Ω, then $\Omega + v$ is a block containing $0 + v$ and $u + v$ so $\Omega + v = \Omega$. In other words $u + v$ is in Ω and so Ω is a $GF(p)$–subspace of V. More than this G_0 contains the scalars in $GF(q)^*$ and so Ω is actually a $GF(q)$–subspace. Thus Ω cannot correspond to a cap. In $PG(r, q)$ our two orbits consist of points in a subspace and the complement. A line not in the subspace meets the subspace in at most one point so the complement cannot form a cap except perhaps when $p = q = 2$ and the subspace has projective dimension $r - 1$. Conversely, as is well known, the complement of a hyperplane is indeed a cap in $PG(r, 2)$ and it is the only way in which the complement of a subspace is a cap. It is easy to see that this cap is transitive and co–transitive. ■

We recall for the reader that the *socle* of a finite group is the product of its minimal normal subgroups. In our setting $V(d, p) \cdot G_0$ has V as its unique minimal normal subgroup.

Liebeck's theorem tells us the possibilities for G_0 and gives two orbits of G_0 on the non–zero vectors of $V(d, p)$. We denote these by K_1 and K_2, and the corresponding sets of points in $PG(r, q)$ by $\bar{K_1}$ and $\bar{K_2}$. We assume that neither K_1 nor K_2 lies in a subspace of $V(r + 1, q)$; given $GF(q)^* \leq G_0$ this means that neither K_1 nor K_2 lies in a subspace of $V(d, p)$. We may henceforth assume that $V(d, p) \cdot G_0$ is a finite primitive affine permutation group of rank 3 and degree p^d, so we may apply Result 6.

We begin with the case by case analysis. In many cases we use data from Result 6 and apply Lemmas 3, 5, but there are occasions when we need to look at the structure of orbits in detail; there are also occasions when using the structure of the orbits is more illuminating and yet no less efficient than the bound and chord-number arguments.

2 The infinite classes A

2.1 The class A1

In this case G_0 is a subgroup of $\Gamma L(1, p^d)$ containing $GF(q)^*$. Such a subgroup is generated by ω^N and $\omega^e \alpha^s$, for some N, e, s where ω is a primitive element of $GF(p^d)$ andα is the generating automorphism $x \mapsto x^p$ of $GF(p^d)$; if we write $p^d = q^a$, then N divides $(q^a - 1)/(q - 1)$. Let H_0 be the subgroup of $\Gamma L(1, p^d)$ generated by ω^N. Then H_0 is a Singer subgroup of $GL(1, p^d)$ and the orbits of H_0 in $PG(r, q)$ are called Singer orbits. Clearly if G_0 has two orbits in $PG(r, q)$, then each orbit is the union of Singer orbits. If the smaller orbit is to be a cap, then each Singer orbit must itself be a cap. A precise criterion for deciding when Singer orbits are caps in $PG(r, q)$ is given by Szőnyi [14, Proposition 1].

Precise criteria for there to be two orbits for G_0 on non–zero vectors of $V(d, p)$ are given by Foulser and Kallaher [5]. These involve numbers m_1 and v such that the primes of m_1 divide $p^s - 1$, v is a prime $\neq 2$ and $\mathrm{ord}_v p^{sm_1} = v - 1$ (meaning $p^{sm_1} \equiv v - 1 \bmod v$), $(e, m_1) = 1$, $m_1 s(v - 1) | d$, $N = v m_1$. The orbit lengths are $m_1(p^d - 1)/N$ and $(v - 1)m_1(p^d - 1)/N$. Notice that when $p = 2$ the smaller orbit has odd size. Hill [8] suggests the possibility of transitive, co–transitive caps of size 78 in $PG(5,4)$ and 430 in $PG(6,4)$. It is now clear that these cannot be caps from class $A1$ and our main theorem then shows that they cannot be caps at all.

2.2 The class A2

G_0 preserves a direct sum $V_1 \oplus V_2$, where V_1, V_2 are subspaces of $V(d,p)$. One orbit must be $K_1 = (V_1 \cup V_2) - \{0\}$ and the other $K_2 = \{v_1 + v_2 : 0 \neq v_1 \in V_1, 0 \neq v_2 \in V_2\}$. We first show that V_1, V_2 are subspaces over $GF(q)$. Observe that for any $\lambda \in GF(q)^* \leq G_0$, $\lambda V_1 = V_1$ or V_2 and let $F = \{\lambda \in GF(q)^* : \lambda V_1 = V_1\} \cup \{0\}$. Then F is a subfield of $GF(q)$ having order greater than $q/2$ so must be $GF(q)$. It is now clear that V_1, V_2 are subspaces of $V(r+1,q)$ of dimension $t = (r+1)/2$. Given that $r \geq 2$, we must have $t \geq 2$, so $\bar{K_1}$ contains lines of $PG(r,q)$ and cannot be a cap. Moreover $|\bar{K_1}| = 2(q^t - 1)/(q-1) < (q^{r+1} - 1)/2$ so $\bar{K_1}$ is the smaller orbit and therefore $\bar{K_2}$ cannot be a cap.

2.3 The class A3

G_0 preserves a tensor product $V_1 \otimes V_2$ over $GF(q)$, with V_1 having dimension 2 over $GF(q)$. This means that if V_1 and V_2 have basis $\{x_1, x_2\}$ and $\{y_j\}$, respectively, then $V_1 \otimes V_2$ has basis $x_i \otimes y_j$. A group stabilizing this tensor product fixes the sets of subspaces $\{x \otimes V_2 : 0 \neq x \in V_1\}$ and $\{V_1 \otimes 0 \neq y \in V_2\}$. Hence, from a projective point of view, a group stabilizing such a tensor product preserves a Segre variety $\mathcal{S}_{1,t}$ with indices 1 and t [10], where $t+1$ is the dimension of V_2. Here one orbit must be $K_1 = \{v_1 \otimes v_2 : 0 \neq v_1 \in V_1, 0 \neq v_2 \in V_2\}$ and the other $K_2 = V - (K_1 \cup \{0\})$.

Consider the $GF(q)$–subspace $V_1 \otimes v_2$ of V for some $0 \neq v_2 \in V_2$. It has dimension 2 in $V(r+1,q)$ so corresponds to a line in $PG(r,q)$ inside $\bar{K_1}$. Hence $\bar{K_1}$ is not a cap.

Let b be the dimension of V_2 over $GF(q)$. Then $r+1 = 2b$ and $|\bar{K_1}| = (q+1)(q^b-1)/(q-1)$ ([12, Table12]) so $|\bar{K_2}| = q(q^b-1)(q^{b-1}-1)/(q-1) > |\bar{K_1}|$ except when $q=2, b=2$ (i.e., $r+1 = d = 4$). Thus there is only one case in which $\bar{K_2}$ can possibly be a cap.

Suppose that $q = p = 2$ and $d = 4$, i.e. we are reduced to considering caps in $PG(3,2)$. In $PG(3,2)$, we see that $|\bar{K_1}| = 9$ and $|\bar{K_2}| = 6$. Thus here $\bar{K_1}$ is too big and for $\bar{K_2}$ it is simplest to note that $(6.5.1)/(2.9) \notin \mathbb{Z}$, so neither is a cap (by Lemmas 2 and 5).

2.4 The class A4

$G_0 \trianglerighteq SL(a,s)$ and $p^d = s^{2a}$. Here $q = s^2$, $a = r+1$ and $p^d = q^a$ with $SL(a,s)$ embedded in $GL(d,p)$ as a subgroup of $SL(a,q)$: let $e_1, e_2, ..., e_a$ be a basis for V over $GF(q)$ then with respect to this basis $SL(a,s)$ consists of the matrices in $SL(a,q)$ having every entry in $GF(s)$. If G_0 has two orbits on non-zero vectors of V then those orbits must be $K_1 = \{\gamma \sum \lambda_i e_i \ (\lambda_i \in GF(s)$, not all 0),$0 \neq \gamma \in GF(q)\}$ and K_2 the set of all remaining non-zero vectors. In other words $\bar{G_0}$ preserves a subgeometry of $PG(r,q)$, and this is the subgeometry $PG(a-1,s)$ of $PG(r,q)$. We have $r > 1$ so that $a \geq 3$. Thus three collinear points of $PG(r,s)$ are still three collinear points in $PG(r,q)$ and so $\bar{K}_1$ is not a cap.

Let us turn to $\bar{K}_2$. As noted above, $r > 1$ so $a \geq 3$. Let $u = e_1 + \sigma e_2$, $v = e_2 + \sigma e_3$, where $\sigma \in GF(q) \backslash GF(s)$. Then u, v and $u+v = e_1 + (\sigma+1)e_2 + \sigma e_3 \in K_2$ correspond to collinear points of $PG(r,q)$, all in $\bar{K}_2$. Hence $\bar{K}_2$ is not a cap.

2.5 The class A5

$G_0 \trianglerighteq SL(2,s)$ and $p^d = s^6$. Here $q = s^3$ and $p^d = q^2$ with $SL(2,s)$ embedded in $GL(d,p)$ as a subgroup of $SL(2,q)$. However $r = 1$ in this case so it does not concern us.

2.6 The class A6

$G_0 \trianglerighteq SU(a,q')$ and $p^d = ((q')^2)^a$. In this case $q = (q')^2$ and $a = r+1$. Here one orbit K_1 consists of the non-zero isotropic vectors and the other orbit K_2 consists of the non-isotropic vectors with respect to an appropriate non-degenerate hermitian form. Each orbit is a union of 1–dimensional subspaces of $V(a,q)$ (excluding the zero vector). To begin with, a non–isotropic line of $PG(r,q)$ contains at least three isotropic points, i.e., three points of $\bar{K}_1$. Therefore $\bar{K}_1$ cannot be a cap.

Now consider $\bar{K}_2$. Given $a \geq 3$, consider a line of $PG(r,q)$ that is isotropic but not totally isotropic, then it contains one point of $\bar{K}_1$ and $q \geq 4$ points of $\bar{K}_2$. Hence $\bar{K}_2$ is not a cap.

2.7 The class A7

$G_0 \trianglerighteq \Omega^{\pm}(a,q)$ and $p^d = (q)^a$ with a even (and if q is odd , G_0 contains an automorphism interchanging the two orbits of $\Omega^{\pm}(a,q)$ on non-singular 1-spaces). The arguments here are similar to the Unitary case. K_1 consists of the non-zero singular vectors and K_2 consists of the non-singular vectors. Let b be the Witt index of the appropriate quadratic form on $V(a,q)$ i.e., the dimension of a maximal totally singular subspace. Then a is one of $2b, 2b+2$. Any totally singular line would be a line of $PG(r,q)$ lying inside $\bar{K}_1$. Given that $a \geq 3$, it follows that the only possibility for $\bar{K}_1$ being a cap is when $\bar{K}_1$ is an elliptic quadric in $PG(3,q)$. In passing we note that for odd q, the necessary automorphism is contained in G_0 only when q is square; in this case and in the case q even, the elliptic quadric gives a well known cap.

Let us turn to $\bar{K}_2$. Any line skew to the quadric of $PG(r,q)$ lies inside $\bar{K}_2$ so $\bar{K}_2$ can never be a cap.

2.8 The class A8

$G_0 \trianglerighteq SL(5,q)$ and $p^d = (q)^{10}$ (from the action of $SL(5,q)$ on the skew square $\bigwedge^2(V(5,q))$. From a projective point of view, a group stabilizing $\bigwedge^2(V(5,q))$ preserves the Grassmannian of lines of $PG(4,q)$ in $PG(9,q)$ [10]. Here one orbit of non-zero vectors must be $K_1 = \{0 \neq u \bigwedge v : u, v \in V(5,q)\}$ with the other non-zero vectors belonging to K_2. One can argue in a similar manner to the case of the tensor product. However it is quicker here to note that the orbits of $\bar{G}_0$ on $PG(r,q)$ have sizes $k = (q^5-1)(q^2+1)/(q-1)$ and $m = q^2(q^5-1)(q^3-1)/(q-1)$ ([12, Table12]) with $k < m$ for all values of q. The chord-number is then given by $c = k(k-1)(q-1)/2m$ by Lemma 5 i.e., $c = (q^2+1)(q^3+q+1)/2q \notin \mathbb{Z}$. Hence neither $\bar{K_1}$ nor $\bar{K_2}$ is a cap.

2.9 The class A9

$G_0/Z(G_0) \trianglerighteq \Omega(7,q) \cdot Z_{(2,q-1)}$ and $p^d = q^8$ (from the action of $B_3(q)$ on a spin module) [3], [11]. The study of Clifford algebras leads to the construction of "spin modules" for $P\Omega(m,q)$. When $m = 8$ this leads to the triality automorphism of $P\Omega^+(8,q)$. One finds that it is possible (via this automorphism) to embed $\Omega(7,q) \cong P\Omega(7,q)$ inside $P\Omega^+(8,q)$ as an irrdeucible subgroup. The important thing from our point of view is that two non–trivial orbits of G_0 must be the set of all non–zero singular vectors and the set of all non–singular vectors with respect to a non–degenerate quadratic form on $V(8,q)$. In this setting the arguments employed for class $A7$ apply: neither orbit can be a cap.

2.10 The class A10

$G_0/Z(G_0) \trianglerighteq P\Omega^+(10,q)$ and $p^d = q^{16}$ (from the action of $D_5(q)$ on a spin module) [3], [11]. Once again we have a spin representation, this time of $P\Omega^+(10,q)$ on $PG(15,q)$. On this occasion it is quickest to work from the orbit lengths.

The orbits of $\bar{G}_0$ on $PG(r,q)$ have sizes $k = (q^8-1)(q^3+1)/(q-1)$ and $m = q^3(q^8-1)(q^5-1)/(q-1)$ ([12, Table12]) with $k < m$ for all values of q. The chord-number is then given by $c = k(k-1)(q-1)/2m$ by Lemma 5 i.e., $c = (q^3+1)(q^5+q^2+1)/2q^2 \notin \mathbb{Z}$. Hence neither $\bar{K_1}$ nor $\bar{K_2}$ is a cap.

2.11 The class A11

$G_0 \trianglerighteq Sz(q)$ and $p^d = (q)^4$, with $q \geq 8$ an odd power of 2 (from the embedding $Sz(q) \leq Sp(4,q)$). Here the smaller orbit $\bar{K_1}$ on $PG(3,q)$ is a Suzuki–Tits ovoid containing q^2+1 points and this is indeed a cap [15], [9, 16.4.5].

3 The Extraspecial classes

In most cases here $G_0 \leq M$ where M is the normalizer in $\Gamma L(a,q)$ of a 2–group R, where $p^d = (q)^a$ and $a = 2^m$ for some $m \geq 1$; either R is an extraspecial group 2^{1+2m} or R is isomorphic to $Z_4 \circ 2^{1+2m}$. In all cases here p is odd. There are two types of extraspecial group 2^{1+2m}, denoted R_1^m and R_2^m; the first of these has the structure $D_8 \circ D_8 \circ \ldots D_8$ (m copies) and the second $D_8 \circ D_8 \circ \cdots \circ D_8 \circ Q_8$ ($m-1$ copies of D_8), where D_8 and Q_8 are respectively the dihedral and quaternion groups of order 8, and '$\circ$' indicates a central product. The group $Z_4 \circ 2^{1+2m}$ is again a central product, this time $Z_4 \circ D_8 \circ D_8 \circ \cdots \circ D_8$ (m copies of D_8) and is denoted by R_3^m. Notice that R modulo its centre is an elementary abelian 2–group, i.e. a $2m$–dimensional vector space over $GF(2)$ and in fact M/RZ (Z being the centre of $\Gamma L(a,q)$) may be embedded in $GSp(2m,2)$. In just one case $G_0 \le M$ with M the normalizer in $\Gamma L(3,4)$ of a 3–group of order 27. We record from [12, Table 13] that in this case the non–trivial orbit sizes of G_0 on $V(3,4)$ are 27 and 36, i.e. the point orbit sizes in $PG(2,4)$ are 9 and 12, but the largest possible size of a cap (here better termed an arc) in $PG(2,4)$ is 6. Hence there are no caps here and we may henceforth assume that R is a 2–group, with p odd.

There are sixteen instances where G_0 has two non–trivial orbits on $V(d,p) \simeq V(a,q)$, but ten of these have $a = 2$ (i.e. $m = 1$) and so refer to action on a projective line, i.e. $r < 2$; note that two of these cases have $q > p$. Thus we concentrate on the remaining six cases. In each of these cases $q = p$ and in all but the last case the vector space is $V(4,p)$. In the last case the vector space is $V(8,3)$. Four cases follw immediately from known bounds - they are listed in the table below.

p=q	r	R	smaller orbit size	max. cap size
3	3	R_1^2	16	10
5	3	R_2^2	60	26
5	3	R_3^2	60	26
7	3	R_2^2	80	50

The case $\mathbf{p = q = 3,\ r = 7,\ R = R_2^3}$.

In this case smaller orbit of $\bar{G}_0$ on $PG(7,3)$ has size 720, while the maximum size for a cap in $PG(7,3)$ is only known to be ≤ 729. Instead we use Lemma 5: the larger orbit has size 2560 and $(720.719.2)/(2.2560) \notin \mathbb{Z}$.

The case $\mathbf{p = q = 3,\ r = 3,\ R = R_2^2}$.

Here Liebeck notes that R has five orbits of size 16 on $V(4,3)$ and M permutes these orbits acting as S_5, the symmetric group of degree 5. Thus there are a number of possibilities for G_0 having two non–trivial orbits on $V(4,3)$. However it is straightforward to construct generating matrices for R and we see immediately that one orbit of size 16 on $V(4,3)$ cannot correspond to a cap in $PG(3,3)$. Therefore none of the orbits of size 16 can correspond to a cap and hence no possible choices of G_0 can give rise to a cap.

4 The Exceptional classes

Finally we turn to the exceptional classes where the socle L of $G_0/Z(G_0)$ is simple. There are just thirteen different possibilities for L, although on occasion more than one possibility for G_0 corresponds to a given L. For example for $L = A_5$ there are seven different possibilities for G_0 (one of which leads to a single orbit in $PG(d-1,p)$); however all of these lead to $r < 2$ and so do not concern us.

We employ a variety of techniques to tackle these cases. Liebeck [12, Table 14] gives the orbit sizes in $V(d,p)$ and sometimes we can use these to rule out the possibility of caps. On other occasions we can use the fact that the chord-number is an integer. On two occasions, neither of these appraoches works and we have to investigate the known structure of the smaller orbit. There remain two cases where a cap does occur.

The cases where caps occur.

When $L = A_6$ and $(d, p) = (6, 2)$, L admits an embedding in $PSL(3, 4)$ (so here $q = 4$) and G_0 has an orbit of size 6. In fact this in a hyperoval in $PG(2, 4)$ [2],[6] so we do have a cap.

When $L = M_{11}$ and $(d, p) = (5, 3)$ there is a representation of L in which one orbit has size 11 and in fact this is a cap. In passing we note that this cap arises as an orbit of a Singer cyclic subgroup of $PG(4, 3)$ [4]; moreover $PG(4, 3)$ is partitioned into eleven 11–caps (the eleven orbits of the Singer cyclic subgroup). Note also that there is a second representation of $L = M_{11}$ on $PG(4, 3)$ (see below). In fact both representations appear in the context of the ternary Golay code [1, Ch. 6].

Cases where known bounds rule out caps.

In each of the following cases the smaller orbit is larger than the known upper bound for a cap size, so cannot be a cap. In the table k is the smaller orbit size.

L	(d, p)	r	q	k	max. cap size
A_6	$(4, 5)$	3	5	36	26
A_7	$(4, 7)$	3	7	120	50
M_{11}	$(5, 3)$	4	3	55	≤ 27
J_2	$(6, 5)$	5	5	1890	≤ 625
J_2	$(12, 2)$	5	4	525	≤ 256

Cases where c an integer rules out caps.

In each of the following cases a calculation $c = k(k-1)(q-1)/2m$ yields a non-integer and so by Lemma 5, the smaller orbit does not correspond to a cap. In the table k is the smaller orbit size and m the larger orbit size.

L	(d, p)	r	q	k	m
A_9	$(8, 2)$	7	2	120	135
A_{10}	$(8, 2)$	7	2	45	210
$L_2(17)$	$(8, 2)$	7	2	102	153
M_{24}	$(11, 2)$	10	2	276	1771
M_{24}	$(11, 2)$	10	2	759	1288
Suz or J_4	$(12, 3)$	11	2	65520	465920

The case $\mathbf{L = A_7}$ and $\mathbf{(d, p) = (8, 2)}$.

Here L is embedded in $PSL(4, 4)$ (so $q = 4$). In fact L may actually embedded in $A_8 \simeq PSL(4, 2) \leq PSL(4, 4)$. The group A_8 and therefore A_7 preserve a subgeometry whose 15 points form the smaller orbit. There are numerous examples of three points on a line in the subgeometry. Thus we have no caps.

The case $\mathbf{L} = \mathbf{PSU(4,2)}$ and $\mathbf{(d,p) = (4,7)}$.
The vectors in the smaller orbit are given by Liebeck [12, Lemma 3.4]:

$$(\theta;0,0,0),\quad (0;\theta,0,0),\quad (0;\omega^a,\omega^b,\omega^c),\quad (\omega^a;0,\omega^b,-\omega^c),$$

(together with all scalar multiples) where $\theta = \omega = 2$; a, b, c take any integral values; and the last three coordinates may be permuted cyclically. It suffices here to observe that $(1;0,0,0)$, $(1;0,1,6)$ and $(2;0,1,6)$ all lie in this orbit and give three collinear points in $PG(3,7)$. So no cap arises here.

References

[1] M. Aschbacher, *Sporadic groups*, Cambridge Tract in Mathematics, Cambridge University Press, Cambridge, 1994

[2] A. Beutelspacher, $21-6=15$: A connection between two distinguished geometries, Amer. Math. Monthly 93, (1986), 29-41.

[3] C.C. Chevalley, *The algebraic theory of spinors*, New York, 1954.

[4] A. Cossidente, O.H. King, Caps and cap partitions in Galois projective spaces, Europ. J. of Combinatorics (1998) 19, 787-799.

[5] D.A. Foulser, M.J. Kallaher, Solvable, flag–transitive rank 3 collineation groups, Geom. Dedicata, 7, (1978), 111-130.

[6] G. Glauberman, On the Suzuki groups and the outer automorphisms of S_6, in: *Groups, difference sets, and the monster*, Proceedings of a special research quarter at the Ohio State University, Spring 1993, Eds. K.T. Arasu et al. W. de Gruyter, Berlin, New York, 1996.

[7] R. Hill, Rank 3 permutation groups with a regular normal subgroup, Ph.D. Thesis, Nottingham 1971.

[8] R. Hill Caps and groups, in: *Teorie Combinatorie*, volume II, Accad. Naz. dei Lincei, Rome, 1976, (Rome 1973), 389-394.

[9] J.W.P. Hirschfeld, *Finite Projective Spaces of Three Dimensions*, Oxford University Press, Oxford, 1985.

[10] J.W.P. Hirschfeld and J.A. Thas, *General Galois Geometries*, Oxford University Press, Oxford, 1991.

[11] P.B. Kleidman, M.W. Liebeck, *The subgroup structure of the finite classical groups*, London Math. Soc. Lecture Notes Series 129, Cabridge University Press, Cambridge 1990.

[12] M.W. Liebeck, The affine permutation groups of rank three, Proc. London. Math. Soc. (3) 54 (1987), 477-516.

[13] B. Mortimer, The modular permutation representations of the known doubly transitive group, Proc. London. Math. Soc. 41, (1980), 1-20.

[14] T. Szőnyi, On cyclic caps in projective spaces, Designs, codes and Cryptography, 8, (1996), 327-332.

[15] J. Tits, Ovoides et groupes de Suzuki, Arch. Math., 13, (1962), 187-198.

A. Cossidente,
Dipartimento di Matematica,
Università della Basilicata,
via N.Sauro 85, 85100 Potenza, Italy.
e-mail: cossidente@unibas.it

O.H. King,
Department of Mathematics,
The University of Newcastle,
Newcastle Upon Tyne,
NE1 7RU, United Kingdom.
e-mail: o.h.kink@ncl.ac.uk