Regular Spreads and Chain Geometries

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Abstract

Using that the set of all reguli in a projective space can be considered as the chain set of a certain chain geometry, we give a new proof of a theorem due to Grundhöfer that characterizes the translation planes associated to regular spreads.

A theorem due to T. Grundhöfer [5] says that a spread over a field K with $|K| > 2$ is regular exactly if the associated translation plane is a Moufang plane and K is contained in the center of its coordinate alternative field. In Section 4 we are going to prove this theorem by means of chain geometry.

In the first section we collect some known facts on translation planes, spreads, and reguli. In Sections 2 and 3 we deal with chain geometries and related notions:

The set of all those subspaces of some vector space V that possess an isomorphic complement (and thus may be part of a spread) is identified with the projective line $\mathbb{P}(R)$ over the endomorphism ring R of any of these subspaces.

If the ground field K is commutative, the projective line $\mathbb{P}(R)$ is the point set of the chain geometry $\Sigma(K, R)$. Moreover, in this case there exist reguli in the projective space $\mathbb{P}(K, V)$. These reguli correspond to the chains in $\Sigma(K, R)$.

A regular spread can now be considered as a subspace of the chain geometry $\Sigma(K, R)$, coordinatized by a so-called Jordan system in the K-algebra R. This Jordan system is used in Section 4 in order to prove Grundhöfer's theorem.

1 Spreads and Translation Planes.

The interrelations between spreads and translation planes are well known, see, e.g., [11], [14]. They go back to J. André's paper [1]. For the reader's convenience, we shall recall the required notions and results.

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Let V be a (left) vector space over a not necessarily commutative field K. We call two subspaces X, Y of V *complementary*, if $V = X \oplus Y$. A set $\mathcal{S}$ of at least three pairwise complementary subspaces of V is called a *spread* of V, if V is covered by $\mathcal{S}$, i.e., if $V = \bigcup \mathcal{S}$.

If V admits a spread, then of course every $v \in V \setminus \{0\}$ is contained in exactly one $S \in \mathcal{S}$. Any two elements of $\mathcal{S}$ are isomorphic, and V is isomorphic to the vector space $S \times S$ for any $S \in \mathcal{S}$.

Some authors prefer the projective point of view: The elements of a spread $\mathcal{S}$ in V are considered as projective subspaces of the projective space $\mathbb{P}(K, V)$ associated to V. Then each point of $\mathbb{P}(K, V)$ is contained in exactly one subspace belonging to $\mathcal{S}$.

With a spread $\mathcal{S}$ in V one associates a translation plane $\mathbb{A}(\mathcal{S})$ in the following way: The point set of $\mathbb{A}(\mathcal{S})$ is the vector space V, and the line set is the set $\mathcal{S}+V := \{S + v \mid S \in \mathcal{S}, v \in V\}$ of all cosets of the elements of $\mathcal{S}$. The incidence structure $\mathbb{A}(\mathcal{S}) = (V, \mathcal{S} + V)$ is an affine plane with $S + v \parallel T + w \iff S = T$. The group $(V, +)$ acts on $\mathbb{A}(\mathcal{S})$ as a transitive group of translations. So $\mathbb{A}(\mathcal{S})$ is indeed a translation plane.

Each translation plane $\mathbb{A}$ can be described with the help of a spread. One can introduce on the point set of $\mathbb{A}$ the structure of a vector space V, and then find a spread $\mathcal{S}$ in V such that $\mathbb{A} \cong \mathbb{A}(\mathcal{S})$. There are several possibilities to arrive at the vector space and the spread. Since we are going to use the *coordinate quasifield* Q of $\mathbb{A}$ anyway, we shall describe V and $\mathcal{S}$ in terms of Q.

First of all, the translation plane $\mathbb{A}$ is coordinatized by a quasifield Q. This coordinatization is carried out with respect to a given affine basis, so in fact there are many coordinate quasifields for $\mathbb{A}$. The details of the coordinatization and the exact definition of a quasifield can be found in [10] or [14]. Roughly speaking, a (right) *quasifield* is a structure $(Q, +, \circ)$ that does *not necessarily* satisfy the associative law of multiplication and the left distributive law $a \circ (b + c) = a \circ b + a \circ c$, whereas all other laws valid in a field are fulfilled. Coordinatizing $\mathbb{A}$ by Q means that $\mathbb{A}$ is shown to be isomorphic with the translation plane $\mathbb{A}(Q)$ with point set $Q \times Q$ and lines described by certain linear equations over Q.

Let Q be a quasifield. Its *kernel* $K(Q) := \{k \in Q \mid \forall a, b \in Q : k \circ (a + b) = k \circ a + k \circ b,\ k \circ (a \circ b) = (k \circ a) \circ b\}$ is a not necessarily commutative field. The scalar multiplication $kq := k \circ q$ (for $k \in K(Q), q \in Q$) makes Q a left vector space over $K(Q)$. We shall also need the *center* of Q, this is the commutative field $C(Q) := \{k \in K(Q) \mid \forall q \in Q : k \circ q = q \circ k\}$.

Now consider the translation plane $\mathbb{A} = \mathbb{A}(Q)$, and let K be any field contained in $K(Q)$. Then Q and thus also the point set $Q \times Q$ of $\mathbb{A}(Q)$ are left vector spaces over K. The sets $S(q) := \{(x \circ q, x) \mid x \in Q\}$ (with $q \in Q$) and $S(\infty) := Q \times \{0\}$ are subspaces of the vector space $Q \times Q$, and the set $\mathcal{S}(Q) := \{S(q) \mid q \in Q \cup \{\infty\}\}$ is a spread in $Q \times Q$. The elements of $\mathcal{S}(Q)$ are exactly the lines through the point $(0, 0)$ in $\mathbb{A}$. Hence $\mathbb{A} \cong \mathbb{A}(\mathcal{S}(Q))$ for this spread $\mathcal{S}(Q)$.

If one starts with a translation plane $\mathbb{A}(\mathcal{S})$ over a spread $\mathcal{S}$, then one can always coordinatize it by a quasifield Q with respect to an appropriate basis in such a way that the spread $\mathcal{S}$ equals $\mathcal{S}(Q)$ (see [11, pp.13/14], [14, p.22]).

The translation plane $\mathbb{A}$ is an affine Moufang plane exactly if one (and thus each) of its coordinate quasifields is an *alternative field*, i.e., satisfies the left distributive law and the so-called alternative laws: $\forall a, b \in Q : (a \circ a) \circ b = a \circ (a \circ b),\ b \circ (a \circ a) = (b \circ a) \circ a$ (see [10, p.154]).

We want to consider *regular* spreads. For this reason we need the notion of a regulus. Now we take up the projective point of view.

A set $\mathcal{R}$ of at least three pairwise complementary subspaces of the projective space $\mathbb{P}(K, V)$ is called a *regulus*, if the following two conditions hold:

(R1) Any line of $\mathbb{P}(K, V)$, meeting three elements of $\mathcal{R}$, meets every element of $\mathcal{R}$. Such a line is called a *transversal* of $\mathcal{R}$.

(R2) Each point on a transversal of $\mathcal{R}$ belongs to an element of $\mathcal{R}$.

The classical example of a regulus is a complete family of generators of a hyperbolic quadric in the three-dimensional projective space over some commutative field K. The transversals of this regulus are exactly the lines of the second family of generators.

We recall the following important facts on the existence of reguli (see [5, Prop.]):

1.1 Remark. *Let V be a vector space over K.*

(1) If K is not commutative and $\dim V > 2$, then there are no reguli in $\mathbb{P}(K, V)$.

(2) If K is commutative and X, Y, Z are pairwise complementary subspaces of $\mathbb{P}(K, V)$, then there is exactly one regulus $\mathcal{R}(X, Y, Z)$ containing X, Y, Z.

(3) If $\dim V = 2$, then the whole point set of $\mathbb{P}(K, V)$ is one regulus.

Let now $\mathcal{S}$ be a spread — considered as a set of projective subspaces — of $\mathbb{P}(K, V)$. Moreover, in view of 1.1, let K be commutative. The spread $\mathcal{S}$ is called *regular*, if for any three $X, Y, Z \in \mathcal{S}$ the whole regulus $\mathcal{R}(X, Y, Z)$ belongs to $\mathcal{S}$. Note that the case $\dim V = 2$ (and K not necessarily commutative) is not interesting for us because then there is exactly one regulus in $\mathbb{P}(K, V)$, and this regulus at the same time is the only spread in $\mathbb{P}(K, V)$.

If $\mathcal{S}$ is a regular spread, and $\mathfrak{R}(\mathcal{S})$ denotes the set of all reguli in $\mathcal{S}$, then the incidence structure $(\mathcal{S}, \mathfrak{R}(\mathcal{S}))$ turns out to be a so-called *chain space* (see Sections 2 and 3). This will be used in Section 4 in order to give a new proof for the following theorem due to T. Grundhöfer [5, Satz 3] (parts of which already appeared in the papers [4] by R.H. Bruck and R.C. Bose and [6] by A. Herzer):

1.2 Theorem (Grundhöfer 1981). *Let $\mathcal{S}$ be a spread in some vector space over the field K with $|K| > 2$. Let $\mathbb{A} = \mathbb{A}(\mathcal{S})$ be the associated translation plane, and let Q be a coordinate quasifield of $\mathbb{A}$ such that $Q \neq K \leq K(Q)$ and $\mathcal{S} = \mathcal{S}(Q)$. Then the following statements are equivalent:*

(1) The spread $\mathcal{S}$ is regular.

(2) The translation plane $\mathbb{A}$ is an affine Moufang plane, and the field K is contained in the center of the alternative field Q.

2 The Chain Geometry of All Reguli.

Chain geometries are incidence structures defined algebraically using associative algebras. In this section we show that reguli can be considered as chains of a certain chain geometry. A regular spread then turns out to be a subspace of this chain geometry; this will be studied in Section 3, and will then be used in Section 4 in order to prove Grundhöfer's theorem. However, the results of this section are also interesting for their own sake.

We define the chain geometry over an associative algebra as in [9]. Compare also [2], where the commutative case is considered.

First, let R be any ring with 1. By R^* we denote its group of units, and by $GL_2(R)$ we denote the group of invertible 2×2 matrices with entries in R.

The *projective line* over R is the set

$$\mathbb{P}(R) := \left\{ R(a,b) \mid a,b \in R,\ \exists c,d \in R : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R) \right\}.$$

This is the orbit of the submodule $R(1,0)$ of the left R-module $R \times R$ under the natural right action of $GL_2(R)$.

A coordinate-free description is the following: The projective line $\mathbb{P}(R)$ is the set of all free cyclic submodules of $R \times R$ that possess a free cyclic complement.

Note that for $R(a,b) \in \mathbb{P}(R)$ and $r \in R$ the cyclic submodule $R(ra, rb)$ equals $R(a,b)$ exactly if $r \in R^*$.

On $\mathbb{P}(R)$ we have the symmetric, anti-reflexive relation *distant*, denoted by $\triangle$, and defined by

$$R(a,b) \triangle R(c,d) :\Longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R).$$

Hence, two elements of $\mathbb{P}(R)$ are distant exactly if $R \times R$ is the direct sum of them.

In the special case that R is a field, we have that $\mathbb{P}(R)$ is the ordinary projective line over R, and $\triangle$ is the relation $\neq$.

By definition, the group $GL_2(R)$ acts on $\mathbb{P}(R)$ and leaves the relation $\triangle$ invariant. As usual, the induced group of permutations of $\mathbb{P}(R)$ is denoted by $PGL_2(R)$. One can easily verify that $PGL_2(R)$ acts transitively on the set of triples of pairwise distant points of $\mathbb{P}(R)$.

In order to define the chain geometry $\Sigma(K,R)$ we need that the ring R in addition is an algebra over some field K. We always suppose that K is a subfield of R (and hence a subfield of the center of R).

The point set of $\Sigma(K,R)$ is the projective line $\mathbb{P}(R)$. The projective line $\mathbb{P}(K)$ over K is embedded into $\mathbb{P}(R)$ via $K(k,l) \mapsto R(k,l)$. The image of $\mathbb{P}(K)$ under this embedding is denoted by $\mathcal{C}_0$, it is the *standard chain* of our chain geometry. Its orbit $\mathfrak{C}(K,R) := \mathcal{C}_0^{PGL_2(R)}$ is the set of chains of $\Sigma(K,R)$. So a *chain* of $\Sigma(K,R)$ is a subset $\mathcal{C}_0^\gamma$ of $\mathbb{P}(R)$, where $\gamma \in PGL_2(R)$.

Altogether, the *chain geometry* over the K-algebra R is the incidence structure $\Sigma(K,R) := (\mathbb{P}(R), \mathfrak{C}(K,R))$. Obviously, the group $PGL_2(R)$ consists of *automorphisms* of $\Sigma(K,R)$.

One can easily prove that the chain geometry $\Sigma(K,R)$ has the following properties (see, e.g., [9]):

2.1 Remark. *Let R be a K-algebra.*

(1) Two different points of $\mathbb{P}(R)$ are distant, exactly if they are joined by a chain.

(2) Any three pairwise distant points $p, q, r \in \mathbb{P}(R)$ are joined by exactly one chain $\mathcal{C}(p, q, r) \in \mathfrak{C}(K, R)$.

Property (2) is one of the axioms of a *chain space* (see [9]). If $R \neq K$, then $\Sigma(K, R)$ actually is a chain space; since we do not need this here, we do not state the other axioms.

Statement 2.1(2) is similar to statement 1.1(2) on reguli. We want to consider the reguli in a projective space $\mathbb{P}(K, V)$ as chains of a certain chain geometry. But which are the subspaces of $\mathbb{P}(K, V)$ that have to be considered as the points of the chain geometry?

Since every point of a chain geometry lies on a chain, and hence is contained in a triple of pairwise distant points, we consider those subspaces X of $\mathbb{P}(K, V)$ that belong to a triple (X, Y, Z) of pairwise complementary subspaces.

We formalize this in the following way (where we take up the vector space point of view rather than the projective one):

Let V be a left vector space over a field K. For the time being, we allow K to be non-commutative, because we want to prove our first theorem in the most possible generality. Later, when we turn to reguli, we shall restrict ourselves to commutative K.

We assume that V has three different subspaces X, Y, Z with $V = X \oplus Y = X \oplus Z = Y \oplus Z$. This means that $V \cong X \times X$, so in the finite-dimensional case $\dim V$ must be even.

The set of subspaces we are interested in is

$$\mathcal{G} := \{W \leq V \mid W \cong V/W\}.$$

Obviously the subspaces X, Y, Z from above all belong to $\mathcal{G}$. On the other hand, any $W \in \mathcal{G}$ is part of a triple of pairwise complementary subspaces of V: Let $V = W \oplus W'$, and let $\varphi : W \to W'$ be an isomorphism. Then $W'' := \{w + w^\varphi \mid w \in W\}$ is complementary to W and to W'.

If $\dim V = 2n$, then the set $\mathcal{G}$ consists exactly of the n-dimensional subspaces of V.

From now on we always suppose that $V = U \times U$ for a suitable vector space U. Then the elements of $\mathcal{G}$ are those $W \leq V$ that are isomorphic to U and possess a complement isomorphic to U.

Let $R := \mathrm{End}_K(U)$ be the ring of all K-linear mappings $U \to U$. To a pair $(\alpha, \beta) \in R \times R$ we associate the subspace

$$U^{(\alpha,\beta)} := \left\{u^{(\alpha,\beta)} := (u^\alpha, u^\beta) \mid u \in U\right\}$$

of V. In Theorem **2.4** below we shall assert that $\Phi : R(\alpha, \beta) \mapsto U^{(\alpha,\beta)}$ maps the projective line $\mathbb{P}(R)$ onto $\mathcal{G}$. Note that of course Φ is well defined on $\mathbb{P}(R)$ because $U^{(\alpha,\beta)} = U^{(\rho\alpha,\rho\beta)}$ for any $\rho \in R^* = \mathrm{Aut}_K(U)$.

First we consider the ring $\mathrm{M}(2 \times 2, R)$ of all 2×2 matrices with entries in R. It acts K-linearly on the vector space V via

$$(u, w)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := (u^\alpha + w^\gamma, u^\beta + w^\delta) = u^{(\alpha,\beta)} + w^{(\gamma,\delta)}$$

(where $(u, w) \in U \times U = V$, $\alpha, \beta, \gamma, \delta \in R$). On the other hand, every linear mapping $\psi : V \to V$ can be described in this way by a unique matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{M}(2 \times 2, R)$ (see, e.g., [13, p.643], where right instead of left modules are considered).

This action of $\mathrm{M}(2 \times 2, R)$ on V is crucial for our subsequent considerations. We summarize the statements above and some of their important consequences in a lemma.

2.2 Lemma. *The ring $M(2 \times 2, R)$ is isomorphic to the ring $End_K(V)$ of endomorphisms of the vector space V via*

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \left(\psi_M : (u, w) \mapsto (u, w)^M = u^{(\alpha,\beta)} + w^{(\gamma,\delta)}\right).$$

In particular, the following statements hold:

(1) The matrix $M \in M(2 \times 2, R)$ is right invertible exactly if ψ_M is an injection.

(2) The matrix M is left invertible exactly if ψ_M is a surjection.

(3) The groups $GL_2(R)$ and $Aut_K(V)$ are isomorphic.

This makes the vector space V an $\mathrm{M}(2 \times 2, R)$-module.

The set $\mathcal{G}$ under consideration consists of certain subspaces of V that are isomorphic to U. Since every linear mapping $U \to V$ has the form (α, β) as above (with $\alpha, \beta \in R$), the subspaces $W \le V$ isomorphic to U are exactly the images $U^{(\alpha,\beta)}$ under injections $(\alpha, \beta) : U \to V$.

We are interested in direct decompositions $V = W \oplus W'$ where W and W' both are of this type. The following lemma will help to find them.

2.3 Lemma. *Let $\psi = \psi_M \in End_K(V)$ with $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M(2 \times 2, R)$. Then*

(1) ψ is surjective $\iff$ $U^{(\alpha,\beta)} + U^{(\gamma,\delta)} = V$

(2) ψ is injective $\iff$ (α, β), (γ, δ) are injective and $U^{(\alpha,\beta)} \cap U^{(\gamma,\delta)} = \{0\}$

Proof. Recall that for $(u, w) \in U \times U = V$ we have $(u, w)^\psi = u^{(\alpha,\beta)} + w^{(\gamma,\delta)}$. Hence (1) is clear. Moreover, $(u, w) \in \ker \psi$ is equivalent with $u^{(\alpha,\beta)} = -w^{(\gamma,\delta)}$ ($\in U^{(\alpha,\beta)} \cap U^{(\gamma,\delta)}$). This yields (2), because obviously injectivity of ψ implies injectivity of (α, β) and of (γ, δ). ∎

Now we are able to identify the set $\mathcal{G}$ with the projective line $\mathbb{P}(R)$. It should be emphasized once again that we do not require that K is commutative.

2.4 Theorem. *Let Φ be defined by $\Phi : R(\alpha,\beta) \mapsto U^{(\alpha,\beta)}$. Then the following statements hold:*

(1) The mapping Φ maps the projective line $\mathbb{P}(R)$ bijectively onto the set $\mathcal{G}$.

(2) Two points $p, q \in \mathbb{P}(R)$ are distant exactly if the subspaces $p^\Phi, q^\Phi \in \mathcal{G}$ are complementary.

(3) The group actions $(GL_2(R), \mathbb{P}(R))$ and $(Aut_K(V), \mathcal{G})$ are equivalent (via Φ).

Proof. Consider a point $p = R(\alpha,\beta) \in \mathbb{P}(R)$. Then $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(R)$ for suitable elements $\gamma, \delta \in R$, which also means that the point $q = R(\gamma,\delta) \in \mathbb{P}(R)$ is distant to p. Lemma 2.3 (together with Lemma 2.2) yields that $U^{(\alpha,\beta)} = p^\Phi$ and $U^{(\gamma,\delta)} = q^\Phi$ are complementary and both isomorphic to U, thus they belong to $\mathcal{G}$.

Next we show that Φ is a bijection:

Consider any $W \in \mathcal{G}$ with complement $W' \in \mathcal{G}$. Then $W = U^{(\alpha,\beta)}$, $W' = U^{(\gamma,\delta)}$ for injections $(\alpha,\beta), (\gamma,\delta) : U \to V$. With 2.3 and 2.2 we conclude that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(R)$, which means $R(\alpha,\beta) \in \mathbb{P}(R)$, and hence $W = R(\alpha,\beta)^\Phi \in \mathbb{P}(R)^\Phi$.

Assume that $W = R(\alpha,\beta)^\Phi = R(\alpha',\beta')^\Phi$ for $R(\alpha,\beta), R(\alpha',\beta') \in \mathbb{P}(R)$. Then $(\alpha,\beta), (\alpha',\beta')$ both are bijections $U \to W$, and $\rho := (\alpha,\beta)(\alpha',\beta')^{-1} \in \mathrm{Aut}_K(U) = R^*$. Thus $R(\alpha,\beta) = R(\alpha',\beta')$.

Statement (3) now follows directly from the definition of the action of $GL_2(R)$ on V. ∎

We want to remark here, that Herzer proved a similar theorem in [7]. He considered the case that K is commutative, U is finite-dimensional, and A is a K-algebra such that U is a faithful A-module (so A can be embedded into $\mathrm{End}_K(U)$). The special case that $A = \mathrm{End}_K(U)$ is presented in [9, Example 4.5(4)]. Of course, finite-dimensionality simplifies the calculations, e.g., two subspaces then are complementary exactly if they intersect trivially.

In Theorem 2.4 we did not need commutativity of K. But now we turn to reguli, and in view of 1.1 it is convenient to let from now on K be commutative. Then for every $k \in K$ the mapping $k \cdot \mathrm{id} : U \to U : u \mapsto ku$ is K-linear. So we can embed K into $R = \mathrm{End}_K(U)$ via $k \mapsto k \cdot \mathrm{id}$, and R becomes a K-algebra (because all $k = k \cdot \mathrm{id}$ are central in R).

We want to show that the chain geometry $\Sigma(K,R)$ over this algebra is isomorphic (via Φ) to the incidence structure $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$, where $\mathfrak{R}(\mathcal{G})$ consists of all reguli in $\mathcal{G}$. Note that now we identify a subspace of V with the associated projective subspace of $\mathbb{P}(K,V)$. Thus, a subset of $\mathcal{G}$ may be a regulus. Note, moreover, that $\mathfrak{R}(\mathcal{G})$ consists of all reguli in $\mathbb{P}(K,V)$, because elements of a regulus must necessarily belong to $\mathcal{G}$.

Thanks to 2.2 we are allowed to identify $GL_2(R)$ with $\mathrm{Aut}_K(V)$. Hence $GL_2(R)$ induces collineations of $\mathbb{P}(K,V)$, and thus maps reguli to reguli and transversals to transversals. The actions of $GL_2(R)$ on $\mathbb{P}(R)$ and on $\mathcal{G}$ are equivalent by 2.4. All this will now be used several times.

First we consider the standard chain $\mathcal{C}_o = \{R(k,l) \mid (0,0) \neq (k,l) \in K^2\}$ and its image under Φ.

2.5 Lemma. *The set $\mathcal{C}_0^\Phi \subseteq \mathcal{G}$ is a regulus.*

Proof. The elements of $\mathcal{C}_0^\Phi$ are the following: the subspace $R(1,0)^\Phi = U^{(1,0)} = U \times \{0\}$, and all subspaces $R(k,1)^\Phi = U^{(k,1)} = \{(ku,u) \mid u \in U\}$ (with $k \in K$). By 2.4 any two of them are complementary (since any two points of $\mathcal{C}_0$ are distant).

We have to show that $\mathcal{C}_0^\Phi$ satisfies conditions (R1) and (R2) of a regulus.

(R1): Let T be a two-dimensional subspace of V (i.e., a line in $\mathbb{P}(K,V)$) meeting three elements of $\mathcal{C}_0^\Phi$ (here of course "meet" means intersect nontrivially). Since the group $GL_2(K) \le GL_2(R) = \mathrm{Aut}_K(V)$ acts triply transitively on $\mathcal{C}_0$ and preserves transversality, we may assume that T meets the elements $U \times \{0\}$, $\{0\} \times U$, and $\{(u,u) \mid u \in U\}$ of $\mathcal{C}_0^\Phi$. This implies that $T = K(u,0) \oplus K(0,u)$ for some $u \in U \setminus \{0\}$. So $T \cap U^{(k,1)} = K(ku,u) \neq \{0\}$ for any $U^{(k,1)} \in \mathcal{C}_0$.

(R2): Let now T be any transversal of $\mathcal{C}_0^\Phi$. By the above, $T = K(u,0) \oplus K(0,u)$ for some $u \in U \setminus \{0\}$. Let P be a one-dimensional subspace of T (i.e., a "projective point" on T). Then $P = K(ku,lu)$ for a suitable pair $(k,l) \in K^2 \setminus \{(0,0)\}$, and so P is contained in $U^{(k,l)} \in \mathcal{C}_0^\Phi$. ∎

Since the set $\mathfrak{C}(K,R)$ of chains is the orbit of the standard chain $\mathcal{C}_0$ under $GL_2(R)$, and $GL_2(R)$ maps reguli to reguli, the lemma above directly implies the following.

2.6 Corollary. *For any chain $\mathcal{C} \in \mathfrak{C}(K,R)$, the image $\mathcal{C}^\Phi \subseteq \mathcal{G}$ is a regulus.*

Now we know that $\mathfrak{C}(K,R)^\Phi \subseteq \mathfrak{R}(\mathcal{G})$. Our aim is to show that the two sets are equal. This can be done by using a geometric standard argument:

2.7 Lemma. *Every regulus $\mathcal{R} \in \mathfrak{R}(\mathcal{G})$ belongs to $\mathfrak{C}(K,R)^\Phi$.*

Proof. Consider $\mathcal{R} \in \mathfrak{R}(\mathcal{G})$. Then $\mathcal{R} = \mathcal{R}(X,Y,Z)$ for suitable pairwise complementary $X,Y,Z \in \mathcal{G}$. By 2.4, there are pairwise distant points $x,y,z \in \mathbb{P}(R)$ with $x^\Phi = X$, $y^\Phi = Y$, $z^\Phi = Z$. Let $\mathcal{C} = \mathcal{C}(x,y,z) \in \mathfrak{C}(K,R)$ be the unique chain through x,y,z. Then $\mathcal{C}^\Phi$ is a regulus containing X,Y,Z, and hence equals $\mathcal{R}$. ∎

Lemma 2.7 and Corollary 2.6 together imply that the following theorem holds.

2.8 Theorem. *The bijection*

$$\Phi : \mathbb{P}(R) \to \mathcal{G} : R(\alpha,\beta) \mapsto U^{(\alpha,\beta)}$$

maps the chain set $\mathfrak{C}(K,R)$ of $\Sigma(K,R)$ onto the set $\mathfrak{R}(\mathcal{G})$ of all reguli in $\mathbb{P}(K,V)$.

Thus, the incidence structures $\Sigma(K,R)$ and $(\mathcal{G},\mathfrak{R}(\mathcal{G}))$ are isomorphic (via Φ).

A theorem similar to ours for the finite-dimensional case can be found in [7].

As mentioned above, $\Sigma(K,R)$ satisfies the axioms of a chain space if $K \neq R$. This means that $(\mathcal{G},\mathfrak{R}(\mathcal{G}))$ is a chain space if $\dim U > 1$, i.e., if $\dim V > 2$. The case $\dim V = 2$ is not interesting because then the elements of $\mathcal{G}$ are all points on the projective line $\mathbb{P}(K,V)$, and they make up one regulus (see 1.1(3)).

The main results of this section can be summarized as follows: For any vector space U over the *not necessarily commutative* field K, the projective line $(\mathbb{P}(R),\triangle)$ over $R = \mathrm{End}_K(U)$ is isomorphic to the set $\mathcal{G} = \{W \le U \times U = V \mid W \cong$

$V/W\}$ endowed with the relation "complementary". If K is commutative, then this isomorphism maps the chain set $\mathfrak{C}(K,R)$ onto the set $\mathfrak{R}(\mathcal{G})$ of all reguli contained in $\mathcal{G}$.

In the next section we turn back to regular spreads. By the isomorphism established above they can be considered as subspaces of $\Sigma(K,R)$ which are easy to handle algebraically.

3 Subspaces and Jordan Systems.

We consider an arbitrary chain geometry $\Sigma(K,R) = (\mathbb{P}(R), \mathfrak{C}(K,R))$ over a K-algebra R. A subset $\mathbb{S}$ of the point set $\mathbb{P}(R)$ is called a *subspace* of $\Sigma(K,R)$, if for any three pairwise distant points $p,q,r \in \mathbb{S}$ the unique chain $\mathcal{C}(p,q,r) \in \mathfrak{C}(K,R)$ is entirely contained in $\mathbb{S}$.

Let $\mathfrak{C}(\mathbb{S})$ be the set of chains contained in the subspace $\mathbb{S}$. Then also the induced structure incidence $(\mathbb{S}, \mathfrak{C}(\mathbb{S}))$ will be called a *subspace* of $\Sigma(K,R)$.

Note that what we call a subspace here is called *weak subspace* in [8], whereas a subspace there has to fulfil an additional condition guaranteeing that $(\mathbb{S}, \mathfrak{C}(\mathbb{S}))$ satisfies the axioms of a chain space. Our definition of a subspace is due to H.-J. Kroll [12].

A subspace $\mathbb{S}$ of $\Sigma(K,R)$ is said to be *non-trivial*, if it contains at least three pairwise distant points, and *connected*, if for any two points $p,q \in \mathbb{S}$ there is a finite sequence $p_0, p_1, \ldots, p_n \in \mathbb{S}$ such that $p_0 = p, p_n = q$ and for any $i \in \{1,\ldots,n\}$ the points p_{i-1} and p_i are joined by a chain in $\mathfrak{C}(\mathbb{S})$.

Now we consider the incidence structure $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$ as defined in Section 2. By Theorem 2.8 it is isomorphic to the chain geometry $\Sigma(K,R)$ over the K-algebra $R = \mathrm{End}_K(U)$. So the notions just introduced can all be applied to subsets of its point set $\mathcal{G}$.

Note that any vector space V that admits a spread $\mathcal{S}$ must have the form as considered in Section 2 (i.e., one can assume $V = U \times U$), and the spread $\mathcal{S}$ must be a subset of the set $\mathcal{G}$. Thus, any spread in a vector space over a commutative field can be considered as a subset of the point set $\mathcal{G}$ of the incidence structure $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$.

Keeping in mind that the "chains" of $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$ are the reguli in $\mathcal{G}$, one can see directly that the spread $\mathcal{S}$ is regular exactly if it is a subspace of $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$.

Generalizing the notion of a regular spread, we call a subset $\mathcal{H} \subseteq \mathcal{G}$ *regular*, if it is a subspace of $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$, i.e., if for any three pairwise complementary $X,Y,Z \in \mathcal{H}$ the regulus $\mathcal{R}(X,Y,Z)$ is contained in $\mathcal{H}$.

By 2.8, each regular subset of $\mathcal{G}$, in particular each regular spread, corresponds (via Φ) to a subspace of the chain geometry $\Sigma(K,R)$.

Before turning to regular spreads, we first recall a description — due to Herzer [8] — of the subspaces of chain geometries $\Sigma(K,R)$ over arbitrary algebras. This description is possible at least if the algebra satisfies a certain condition and if the subspace is non-trivial and connected.

We need two algebraic notions (compare [3] and [8]).

3.1 Definition. *Let R be an algebra over the field K.*

Let J be a vector subspace of R with $1 \in J$, and let $J^* := J \cap R^*$. For $a \in J$ let $e(a) := \{k \in K \mid a - k \in R^*\}$.

(1) *We call J a Jordan system in R, if for each $a \in J^*$ also a^{-1} belongs to J.*

(2) *We call J strong in R, if for each $a \in J$ the inequality $|e(a)| > |K \setminus e(a)|$ is satisfied.*

An example of a Jordan system that is not a subalgebra is the set J of all symmetric matrices in the matrix algebra $\mathrm{M}(n \times n, K)$. Another example important for us will be presented below.

In the matrix algebra $R = \mathrm{M}(n \times n, K)$ the set $K \setminus e(a)$ consists exactly of the eigenvalues of the matrix $a \in R$. So R is strong exactly if K contains more than $2n$ elements, and in this case of course each Jordan system in R is strong as well.

Now we can state the announced description of the subspaces of $\Sigma(K, R)$, due to Herzer [8].

3.2 Theorem. *Let R be a K-algebra and let $\Sigma(K, R)$ be the associated chain geometry.*

(1) *If J is a strong Jordan system in R, then*

$$\mathbb{P}(J) := \{R(1 + ab, a) \mid a, b \in J\}$$

is a non-trivial connected subspace of $\Sigma(K, R)$.

(2) *Let R be strong, and let $\mathbb{S} \subseteq \mathbb{P}(R)$ be any non-trivial connected subspace of $\Sigma(K, R)$. Then there exist an automorphism $\gamma \in PGL_2(R)$ and a Jordan system J in R, such that $\mathbb{S}^\gamma = \mathbb{P}(J)$.*

Statement (1) will be used below to find an example of a regular spread. Statement (2), however, cannot be used directly for describing the subspaces of $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$, because it is not clear whether the associated algebra $R = \mathrm{End}_K(U)$ is strong.

We continue by giving an example of a strong Jordan system in a certain algebra of type $\mathrm{End}_K(U)$.

3.3 Example. *Let $A = (A, +, \circ)$ be an alternative field, and let K be a subfield of the center $C(A)$. Then A is a vector space over K, and the ring $R = End_K(A)$ of all K-linear mappings $A \to A$ is a K-algebra.*

Consider the set $J = \{\rho_a : A \to A : x \mapsto x \circ a \mid a \in A\}$ of all right multiplications in A. Since A is distributive and K is central in A, this set is a vector subspace of R. Moreover, the element $1_R = id_A = \rho_1$ belongs to J. Finally, the right inversive law $a \circ b = 1 \Longrightarrow (x \circ a) \circ b = x$ valid for all $a, b, x \in A$ (see [10, Thm. 6.17]) implies that the inverse of an invertible right multiplication is again a right multiplication. Altogether, J is a Jordan system in R.

As J^* equals $J \setminus \{0\}$, one has $|K \setminus e(\rho_a)| \leq 1$ for every $\rho_a \in J$. So in case $|K| > 2$ the set J is even a strong Jordan system in R and thus gives rise to a subspace $\mathbb{P}(J)$ of $\Sigma(K, R)$.

This subspace has the form $\mathbb{P}(J) = \{R(\rho_a, 1) \mid a \in A\} \cup \{R(1, 0)\}$, because for $\rho_b, \rho_c \in J$ with $\rho_c \neq 0$ the point $R(1 + \rho_c\rho_b, \rho_c) \in \mathbb{P}(J)$ equals $R(\rho_a, 1)$ for $a = c^{-1} + b$.

Now we show that the subspace $\mathbb{P}(J)$ corresponds to a regular spread in $V = A \times A$, via the isomorphism Φ introduced in Section 2 (where we substitute U by A), and that this spread belongs to the translation plane $\mathbb{A}(A)$ over the alternative field A (as described in Section 1):

3.4 Lemma. *Let A be an alternative field, let K a subfield of $C(A)$ with $|K| > 2$, and let $J \subseteq R = End_K(A)$ be the strong Jordan system of all right multiplications as above. Then the image $\mathbb{P}(J)^\Phi$ is a regular spread in $A \times A$, namely, the spread $\mathcal{S}(A) = \{S(a) \mid a \in A \cup \{\infty\}\}$ associated to the affine Moufang plane over A.*

Proof. The image is a subspace of $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$, so it is necessarily regular. Moreover, the $R(1,0)^\Phi = A \times \{0\} = S(\infty)$, and $R(\rho_a, 0)^\Phi = \{(x \circ a, x) \mid x \in A\} = S(a)$. So $\mathbb{P}(J)^\Phi = \mathcal{S}(A)$ and thus is a spread. ∎

The set J of linear mappings $A \to A$ considered above is a *spread set* ([11]) associated to the spread $\mathcal{S} = \mathcal{S}(A)$. In [11] it is shown how every spread can be described by using a (not uniquely determined) spread set, which is a set of certain linear mappings. The spread sets associated to a spread $\mathcal{S}$ are called *representations* of $\mathcal{S}$ by Bruck and Bose in [4]. They use these representations in order to prove their version of Grundhöfer's theorem [4, Thm. 11.1].

Next we are going to show that in case $|K| > 2$ every regular spread corresponds to a subspace $\mathbb{P}(J)$ over some Jordan system J in the K-algebra $R = \mathrm{End}_K(U)$. This is not a direct consequence of 3.2(2) because R need not be a strong algebra: As indicated above, the matrix algebra $\mathrm{M}(n \times n, K)$ is not strong if $|K| \leq 2n$.

Nevertheless, we obtain the desired description, by imitating Herzer's proof of 3.2(2) for those subspaces we are interested in.

3.5 Lemma. *Let $\mathbb{S}$ be a subspace of the chain geometry $\Sigma(K, R)$ over the K-algebra R with $|K| > 2$. If $\mathbb{S}$ satisfies the condition*

$$\forall p, q \in \mathbb{S} : p \neq q \iff p \vartriangle q$$

and contains the points $R(1,0), R(0,1), R(1,1)$, then there is a Jordan system J in R with $J^* = J \setminus \{0\}$, such that $\mathbb{S} = \mathbb{P}(J) = \{R(1,0)\} \cup \{R(a,1) \mid a \in J\}$.

Proof. As any two different points of $\mathbb{S}$ are distant, $\mathbb{S} = \{R(1,0)\} \cup \{R(a,1) \mid a \in J\}$ for some subset $J \subseteq R$ with $0, 1 \in J$ and $J \setminus \{0\} = J \cap R^* =: J^*$. It remains to show that J is a Jordan system.

We will identify the set of points of $\mathbb{P}(R)$ distant to the point $\infty := R(1,0)$ with the set R via $R(x,1) \leftrightarrow x$. For a chain $\mathcal{C}$ containing ∞ then $\mathcal{C} \setminus \{\infty\}$ is an affine line $a + Kb$ with $a \in R$, $b \in R^*$ (see [9]). Moreover, the subspace $\mathbb{S}$ thus equals $J \cup \{\infty\}$.

— J is closed with respect to scalar multiplication: For any $x \in J \setminus \{0\}$ we know that $x \vartriangle 0$, so $x, 0, \infty \in \mathbb{S}$ are pairwise distant and hence joined by a chain $\mathcal{C} \subseteq \mathbb{S}$. Moreover, $\mathcal{C} = Kx \cup \{\infty\}$, and hence $Kx \subseteq J$.

— J is closed with respect to addition: It suffices to show that for linearly independent $x, y \in J$ the set $K(x+y)$ contains an element $z \in J \setminus \{0\}$. Because of $|K| > 2$ there are at least two points $x_1, x_2 \in Kx \setminus \{0\}$. At most one of the lines L_1, L_2 joining y and x_1, x_2, respectively, is parallel to $K(x+y)$. Hence one line, say L_1, meets $K(x+y)$ in a point $z \neq 0$ (because L_1 lies in the affine plane spanned by the points 0,x, and y). Moreover, $L_1 = \mathcal{C}(\infty, y, x_1) \setminus \{\infty\}$ is contained in J, so $0 \neq z \in K(x+y) \cap J$, as desired.

— J is closed with respect to inversion: Consider the automorphism ι of $\Sigma(K, R)$ induced by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(R)$. The image $\mathbb{S}^\iota = \{R(0,1)\} \cup \{R(1,a) \mid a \in J\} =$

$\{R(0,1)\} \cup \{R(a^{-1},1) \mid a \in J^*\} \cup \{R(1,0)\}$ is also a subspace of $\Sigma(K,R)$. It contains $R(1,0), R(0,1), R(1,1)$ and satisfies the condition that any two different points are distant. Hence, by the above, also the set $J' := \{0\} \cup \{a^{-1} \mid a \in J^*\}$ is a vector subspace of R. Now let $x \in J \setminus \{0,1\}$. The following sequence of implications shows that $x^{-1} \in J$:

$x \in J \setminus \{0,1\} \Longrightarrow 1-x \in J \setminus \{0,1\} \Longrightarrow (1-x)^{-1} \in J' \setminus \{0,1\} \Longrightarrow 1-(1-x)^{-1} \in J' \setminus \{0,1\} \Longrightarrow x^{-1} = (1-(1-x)^{-1})^{-1} \in J$. ■

This lemma is not only useful for the description of regular spreads, but it is also interesting from the point of view of chain geometry: It describes the non-trivial subspaces of $\Sigma(K,R)$ consisting of pairwise distant points. By the lemma, such a subspace is isomorphic to a subspace $\mathbb{P}(J)$ over a Jordan system J in R with $J = J^* \cup \{0\}$, via an automorphism $\gamma \in PGL_2(R)$ of $\Sigma(K,R)$ (because $PGL_2(R)$ acts transitively on the set of triples of pairwise distant points of $\mathbb{P}(R)$). A chain space in which any two different points are distant is called a *Möbius space* in [9]. So the subspaces considered in the lemma are non-trivial *Möbius subspaces* of $\Sigma(K,R)$.

Now we turn to subspaces of $(\mathcal{G}, \mathfrak{R}(\mathcal{G}))$. The Φ-images of the subspaces satisfying the assumptions of the lemma are the regular *partial* spreads in $\mathcal{G}$ (i.e., sets $\mathcal{H} \subseteq \mathcal{G}$ whose elements are pairwise complementary) containing the subspaces $U \times \{0\}$, $\{0\} \times U$, and $\{(u,u) \mid u \in U\}$.

Hence for regular spreads we have the following.

3.6 Corollary. *Let $\mathcal{S} \subseteq \mathcal{G}$ be a regular spread containing $U \times \{0\}$, $\{0\} \times U$, and $\{(u,u) \mid u \in U\}$. If $|K| > 2$, then $\mathcal{S} = \mathbb{P}(J)^\Phi$ for some Jordan system J in $R = End_K(U)$ with $J = J^* \cup \{0\}$.*

4 A New Proof of Grundhöfer's Theorem.

Now we are able to prove Grundhöfer's theorem (Thm. 1.2) on the translation planes associated to regular spreads.

Lemma 3.4 shows how a regular spread $\mathcal{S}$ can be derived from the Jordan system of all right multiplications in an alternative field A, and that $\mathcal{S}$ equals the spread $\mathcal{S}(A)$ which leads to the Moufang plane $\mathbb{A}(A)$ with coordinate alternative field A. This is one of the two directions of Theorem 1.2:

4.1 Remark. *Let $\mathcal{S}$ be a spread in some vector space over the field K. Let the associated translation plane $\mathbb{A} = \mathbb{A}(\mathcal{S})$ be an affine Moufang plane with coordinate alternative field A such that $K \leq C(A)$ and $\mathcal{S} = \mathcal{S}(A)$. Then $\mathcal{S}$ is a regular spread.*

Note that as to the proof of the second direction of his theorem, Grundhöfer in [5] quotes Herzer [6], who, in turn, quotes Bruck and Bose [4]. Our methods below are quite similar to the ones used in [4]. However, we use the concepts of chain geometries and Jordan systems.

The second direction of Grundhöfer's theorem only holds if $|K| > 2$. Otherwise every spread is regular because then the reguli are exactly the sets consisting of three pairwise complementary subspaces.

Grundhöfer does not assume that K is commutative. But he uses that $K \neq Q$ and hence that $\dim V = \dim(Q \times Q) > 2$. In view of 1.1(1), regularity of the spread then implies commutativity of K. So we may restrict ourselves to the commutative case.

We consider the following situation:

Let $\mathcal{S}$ be a regular spread in a vector space V over the commutative field K with $|K| > 2$. Let $\mathbb{A} = \mathbb{A}(\mathcal{S})$ be the associated translation plane, and let $Q = (Q, +, \circ)$ be a coordinate quasifield of $\mathbb{A}$ with $K \leq K(Q)$ such that $V = Q \times Q$ and $\mathcal{S} = \mathcal{S}(Q)$.

By Corollary 3.6 the spread $\mathcal{S}$ equals the Φ-image of a subspace $\mathbb{P}(J)$ over some Jordan system J in $R = \mathrm{End}_K(Q)$. Moreover, $J = J^* \cup \{0\}$ and hence $\mathbb{P}(J) = \{R(1,0)\} \cup \{R(\alpha,1) \mid \alpha \in J\}$. So the spread $\mathcal{S}$ turns out to be the set $\mathbb{P}(J)^\Phi = \{Q^{(1,0)}\} \cup \{Q^{(\alpha,1)} \mid \alpha \in J\}$, where $Q^{(1,0)} = Q \times \{0\}$ and $Q^{(\alpha,1)} = \{(x^\alpha, x) \mid x \in Q\}$ (for $\alpha \in J$). On the other hand, $\mathcal{S} = \mathcal{S}(Q) = \{S(q) \mid q \in Q \cup \{\infty\}\}$ (see Section 1), where $S(\infty) = Q \times \{0\}$ and $S(q) = \{(x \circ q, x) \mid x \in Q\}$ (for $q \in Q$).

Comparing these two descriptions of $\mathcal{S}$, we see that J consists exactly of the right multiplications in Q (which are K-linear because of the right distributive law and because $K \leq K(Q)$). This observation and some of its consequences are stated in the next lemma.

4.2 Lemma. *Let $\mathcal{S} = \mathcal{S}(Q)$ be a regular spread as above. Then the following statements hold:*

(1) The Jordan system J associated to $\mathcal{S}$ is the set $J = \{\rho_q : x \mapsto x \circ q \mid q \in Q\}$.

(2) The mapping $\rho : Q \to J : q \mapsto \rho_q$ is a linear bijection.

(3) For every $a \in Q^* := Q \setminus \{0\}$ the mapping ρ_a is invertible with $(\rho_a)^{-1} = \rho_b$, where $b \in Q$ satisfies the equations $a \circ b = 1 = b \circ a$.

Proof. (2): Obviously $\rho : Q \to J$ is surjective. It is also injective because $\rho_a = \rho_b$ implies $a = 1 \circ a = 1^{\rho_a} = 1^{\rho_b} = b$.

The other assertions follow from the fact that J is a Jordan system:

Let $a, b \in Q$. Then $\rho_a, \rho_b \in J$, and hence also $\rho_a + \rho_b \in J$, i.e., $\rho_a + \rho_b = \rho_c$ for some $c \in Q$. Applying this to $1 \in Q$ we obtain $c = 1^{\rho_c} = 1^{\rho_a + \rho_b} = a + b$, so $\rho_a + \rho_b = \rho_{a+b}$.

The rest of (2) and also (3) can be shown by similar arguments. ∎

Here again the set J is a spread set associated to the spread $\mathcal{S}$ (as in the case of the spread $\mathcal{S} = \mathcal{S}(A)$ in 3.4, see above).

Our considerations in the lemma above and in the next corollary are quite similar to those of Bruck and Bose in [4, Thm. 11.1]; they also use properties of the set of right multiplications in Q (considered as a spread set or "representation" of the spread) in order to deduce properties of the quasifield Q.

4.3 Corollary. *Let $\mathcal{S} = \mathcal{S}(Q)$ be a regular spread as above. Then the following statements hold:*

(1) The field K is contained in the center $C(Q)$.

(2) The quasifield Q is left distributive and hence distributive.

(3) The quasifield Q satisfies the right inversive law, i.e., for all $a, b, x \in Q$ the implication $a \circ b = 1 \Longrightarrow (x \circ a) \circ b = x$ holds.

Proof. (1): Consider $k \in K$, $x \in Q$. By Lemma 4.2, we have $k \cdot \mathrm{id} = k \cdot \rho_1 = \rho_k$, hence $k \circ x = kx = x^{k \cdot \mathrm{id}} = x^{\rho_k} = x \circ k$.

(2): Let $a, b, x \in Q$. By 4.2 we have $\rho_a + \rho_b = \rho_{a+b}$, hence $x \circ a + x \circ b = x^{\rho_a + \rho_b} = x^{\rho_{a+b}} = x \circ (a + b)$.

(3) follows similarly, using 4.2(3) and the fact that for any $a \in Q^*$ the element $b \in Q$ with $a \circ b = 1$ is uniquely determined. ∎

Up to now we have shown that the quasifield Q belonging to the regular spread $\mathcal{S} = \mathcal{S}(Q)$ in $Q \times Q$ over K with $|K| > 2$ is distributive and satisfies the right inversive law, and that, moreover, K is central in Q.

Now the Skornyakov-San Soucie Theorem (see [10, Thm. 6.16]) says that a distributive quasifield with right inversive law is an alternative field. So the translation plane $\mathbb{A}(Q)$ associated to our regular spread is a Moufang plane.

This is the second direction of Grundhöfer's theorem. We repeat it in our final remark:

4.4 Remark. *Let $\mathcal{S}$ be a regular spread in some vector space over the commutative field K with $|K| > 2$. Let $\mathbb{A} = \mathbb{A}(\mathcal{S})$ be the associated translation plane, and let Q be a coordinate quasifield of $\mathbb{A}$ such that $K \leq K(Q)$ and $\mathcal{S} = \mathcal{S}(Q)$. Then Q is an alternative field, $\mathbb{A}$ is an affine Moufang plane, and the field K is contained in the center of Q.*

Recall that the assumption $K \neq Q$ in Grundhöfer's theorem 1.2 implies commutativity of K (if $\mathcal{S}$ is regular). Of course in case $K = Q$ the plane $\mathbb{A} = \mathbb{A}(K)$ is an affine Moufang plane anyway.

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