Amenability of certain Banach algebras with application to measure algebras on foundation semigroups

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A well-known result of G. Willis asserts that for a locally compact group G if the group algebra $L^1(G)$ is separable then $L^1(G)$ is amenable if and only if $I_0(L^1(G))$ is the unique maximal ideal of $\mathcal{F}_a$, where

$$I_0(L^1(G)) = \{f \in L^1(G) : \int_G f = 0\},$$

$$\mathcal{F}_a = \{J_\mu : \mu \text{ is a probability measure on } G\},$$

and

$$J_\mu = \{f - f * \mu : f \in L^1(G)\}^-,$$

(see theorem 1.2 of [**11**]). It is also proved in proposition 1.3 of [**11**] that if $L^1(G)$ is separable and amenable, then there is a discrete probability measure μ on G such that $I_0(L^1(G)) = J_\mu$.

The aim of the present paper is to extend the first result to the general setting of separable Lau algebras and the second result to an extensive class of topological semigroups, namely foundation semigroups. It should be noted that $L^1(G)$, the Fourier algebra $A(G)$, the Fourier-Stiltjes algebra $B(G)$ of a locally compact group G, and the measure algebra $M_a(S)$ of a topological semigroup S are elementary examples of Lau algebras. The class of foundation semigroups is extensive, and includes all discrete semigroups, all locally non-locally-null subsemigroups of locally compact groups and those subsemigroups related to those considered by Rothman [**9**]. For many other examples, see [**10**, Appendix B].

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Preliminaries

Let A be a complex Banach algebra and X be a Banach $A-$bimodule, and X^* be the dual Banach $A-$bimodule, in which the module multiplications are given by

$$(af)(x) = f(xa), \quad (fa)(x) = f(ax) \quad (a \in A, \quad f \in X^*, \quad x \in X)$$

A bounded $X-$derivation is a bounded linear mapping D of A into X such that

$$D(ab) = (Da)b + a(Db) \quad (a, b \in A).$$

The set of bounded $X-$derivations is denoted by $Z^1(A, X)$: it is a linear subspace of $BL(A, X)$, the space of all bounded linear operators of A into X. Given $x \in X$, let δ_x^A be the mapping of A into X given by

$$\delta_x^A(a) = ax - xa \quad (a \in A).$$

It is clear that δ_x^A belongs to $Z^1(A, X)$. We call δ_x^A an *inner X derivation* and denote by $B^1(A, X)$ the set of all inner $X-$derivations. $B^1(A, X)$ is a linear subspace of $Z^1(A, X)$, and we denote by $H^1(A, X)$ the difference space of $Z^1(A, X)$ modulo $B^1(A, X)$, $H^1(A, X)$ is called the *first cohomology group* of A with coefficients in X. A Banach algebra A is said to be *amenable* if $H^1(A, X^*) = 0$ for every Banach $A-$bimodule X.

Recall that a Lau algebra is any pair (A, B) consisting of a Banach algebra A and a von Neumann algebra B for which $A = B_*$ (and $A^* = B$) and the unit u_B of B is a multiplicative linear functional on A (c.f. [**6**] and [**7**]). Such a Lau algebra is also denoted simply by A although, B is not necessarily unique. A Lau algebra A is called *left-amenable* if, for every Banach $A-$module X such that

$$a.x = \langle a, u_B \rangle x$$

whenever $a \in A$ and $x \in X$, one has $H_1(A, X^*) = 0$. We denote the set of all positive elements of A of norm one by $P_1(A)$. Any positive functional M on A^* of norm one is called a *mean* on A^*. A mean M on A^* is called *right invariant* if $\langle f\varphi, M \rangle = \langle f, M \rangle$ whenever $f \in A^*$ and $\varphi \in P_1(A)$. It is well known that a Lau algebra A is left amenable, if and only if A^* admits a right invariant mean (see proposition 3.5 of [**6**]). For a Lau algebra A we denote by $I_0(A)$ the set $\{a \in A : u_B(a) = 0\}$. For every $b \in P_1(A)$ we denote by J_b the norm closure of $\{a - ab : a \in A\}$. Then J_b is a closed left ideal in A. It is also clear that $J_b \subseteq I_0(A)$ for every $b \in P_1(A)$.

Throughout, S denotes a locally compact, Hausdroff topological semigroup. Recall (see, for example [**1**], [**2**], [**4**], [**5**]), that $M_a(S)$ or $\tilde{L}(S)$ denote the space of all measures $\mu \in M(S)$ (the Banach algebra of all bounded regular complex measures) on S for which the mappings
$x \longmapsto \delta_x * |\mu|$ and $x \longmapsto |\mu| * \delta_x$ (where δ_x denotes the point mass at x for $x \in S$) from S into $M(S)$ are weakly continuous. Note that the measure algebra $M_a(S)$ defines a two-sided closed $L-$ideal of $M(S)$ (see, [**1**]). A semigroup S is called a *foundation semigroup* if $\bigcup\{\text{supp}(\mu) : \mu \in M_a(S)\}$ is dense in S. Note that in the case S is a foundation semigroup with identity, for every $\mu \in M_a(S)$ both mappings $x \longmapsto \delta_x * |\mu|$ and $x \longmapsto \delta_x * |\mu|$ from S into $M_a(S)$ are norm continuous (c.f. [**2**]).

Amenability of separable Lau algebras

The aim of the present section is to prove a general theorem on the amenability of separable Lau algebras where in the particular case that A is the group algebra $L^1(G)$ of a separable locally compact group (or a $\sigma-$compact group) G, it gives a generalization of both theorem 1.10 of [**8**] and theorem 1.2 of [**11**].

Application of this result to foundation semigroups provided us with a generalization of theorem 1.14 of [**8**] and proposition 1.3 of [**11**].

We commence with the following lemma whose proof is omitted, since it can be proved in the same direction of lemma 1.1 of [**11**].

Lemma 1. *Let A be a Lau algebra and $\mathcal{F}$ be a norm closed, convex subsemigroup of $P_1(A)$. Let X be a separable closed subspace of A such that*

(i) $J_b \subseteq X$ for every $b \in \mathcal{F}$;

and

(ii) for every $\varepsilon > 0$ and $x \in X$ there is $b \in \mathcal{F}$ such that $d(x, J_b) = \inf\{\|x - y\| : y \in J_b\} < \varepsilon$.

Then there is $b_0 \in \mathcal{F}$ such that $X = J_{b_0}$.

The following theorem which generalizes theorem 1.2 of [**11**] is needed for the proof of the main result.

Theorem 2. *Let (A, B) be a separable Lau algebra. Then the following are valid:*

(i) if $\mathcal{J} = \{J_b : b \in P_1(A)\}$; then every ideal in $\mathcal{J}$ is contained in a maximal one;

and

(ii) if A has a bounded left approximate identity and $\mathcal{J}$ has a unique maximal ideal, then A is left amenable;

(iii) if A is left amenable, then $\mathcal{J}$ has a unique maximal ideal.

In both cases (ii) and (iii) the unique maximal ideal of $\mathcal{J}$ is $I_0(A)$.

Proof. The proof of (i) is similar to that of part (a) of theorem 1.2 of [**11**].

(ii). Suppose that A has a bounded left approximate identity and $\mathcal{J}$ has a unique maximal ideal, J_b say, and let x be in $I_0(A)$. By Cohen's factorization theorem (theorem 32.26 of [**3**]) we can write $x = ay'$ for some $a \in A$ and $y' \in I_0(A)$. If we decompose $y' = ({y'}_1 - {y'}_2) + i({y'}_3 - {y'}_4)$ with ${y'}_i$ positive for $1 \leq i \leq 4$, then

$$0 = u_B(y') = u_B({y'}_1) - u_B({y'}_2) + i(u_B({y'}_3) - u_B({y'}_4)),$$

where u_B denotes the multiplicative identity functional on A. Hence $u_B({y'}_1) = u_B({y'}_2)$ and $u_B({y'}_3) = u_B({y'}_4)$. Since $u_B({y'}_i) = \|{y'}_i\|(1 \leq i \leq 4)$, it follows that $\|{y'}_1\| = \|{y'}_2\|$ and $\|{y'}_3\| = \|{y'}_4\|$. Putting $c_i = \|{y'}_i\|$ and $y_i = {y'}_i/c_i(1 \leq i \leq 4)$ we obtain

$$x = c_1(a - ay_1) + (-c_1)(a - ay_2) + (ic_2)(a - ay_3) + (-ic_2)(a - ay_4).$$

Let J_b denote the unique maximal ideal in $\mathcal{J}$. Now each $(a - ay_i) \in J_{y_i}$ $(1 \leq i \leq 4)$. Since by (i) each J_{y_i} is contained in a maximal ideal in $\mathcal{J}$, it follows that x belongs to J_b. Thus $I_0(A) \subseteq J_b$. It is clear that $J_b \subseteq I_0(A)$. So $J_b = I_0(A)$. Let $y \in I_0(A)$, then

$y \in J_b$. Thus $\lim_{n\to\infty} \|y(\frac{1}{n}\sum_{k=1}^{n} b^k)\| = 0$. Since for every $n \in \mathbb{N}$, $\frac{1}{n}\sum_{k=1}^{n} b^k \in P_1(A)$, from proposition 3.6 of [**7**] it follows that A is left amenable.

(iii). Suppose now that A is left amenable. Given $x \in I_0(A)$ and $\varepsilon > 0$, by (iii) of proposition 3.6 of [**7**] there exists $b \in P_1(A)$ such that $\|xb\| < \varepsilon$. Thus $d(x, J_b) < \varepsilon$. Hence the conditions of lemma 1 are satisfied if we take $\mathcal{F} = \mathcal{J}$ and $X = I_0(A)$. So there is a $b_0 \in P_1(A)$ with $I_0(A) = J_{b_0}$. Thus $I_0(A)$ is the unique maximal ideal in $\mathcal{J}$.

Before turning to the next result we first need to prove one more lemma.

Lemma 3. Let S be a foundation semigroup with identity. Then for every $\varepsilon > 0$, μ and $\nu \in M_a(S)$, there exists $c_1, c_2, \ldots, c_N$ in $\mathbb{C}$ and $x_1, x_2, \ldots, x_N$ in S such that $\sum_{n=1}^{N} c_n = \mu(S)$ and

$$\|\nu * \mu - \sum_{n=1}^{N} \nu * c_n\delta_{x_n}\| < \varepsilon.$$

Proof. Given $\varepsilon > 0$, choose a compact subset K of S such that $|\mu|\,(S\setminus K) < \varepsilon'$. Where $\varepsilon' = \frac{\varepsilon}{2\|\nu\|+\|\mu\|}$. Since the mapping $x \longmapsto \nu * \delta_x$ is norm continuous and K is compact, there are finitely many points $x_n \in K$, say $1 \le n \le N$, and open neighbourhoods U_n of x_n covering K such that $\|\nu * \delta_x - \nu * \delta_{x_n}\| < \varepsilon'$ for $x \in U_n$. Define now a partition of S as follows. Let $A_1 = K \cap U_1$, $A_n = (K \setminus \bigcup_{1\le j<n} A_j) \cap U_n$ for $1 < n \le N$ if $N > 1$. Put $A_0 = S \setminus K$ and $x_0 = e$ (the identity of S). The sets A_n $(0 \le n \le N)$ are Borel sets, mutually disjoint, and their union is S. Putting $c_n = \mu(A_n)$, then for every $F \in M_a(S)^*$ by lemma 2.5 of [**4**] we have

$$\begin{aligned}
\mid F(\nu*\mu) - F(\sum_{n=0}^{N} \nu * c_n\delta_{x_n}) \mid &= \mid F(\nu*\mu) - \sum_{n=0}^{N} c_n F(\nu*\delta_{x_n}) \mid \\
&= \mid \sum_{n=0}^{N} \int_{A_n} [F(\nu*\delta_x) - F(\nu*\delta_{x_n}) d\mu(x)] \mid \\
&\le \int_{A_0} \mid F(\nu*\delta_x - \nu*\delta_{x_n}) \mid d\,|\mu|\,(x) \\
&+ \sum_{n=1}^{N} \int_{A_n} \mid F(\nu*\delta_x - \nu*\delta_{x_n}) \mid d\,|\mu|\,(x) \\
&\le \|F\| 2\|\nu\|\,|\mu|\,(S\setminus K) + \\
&\sum_{n=1}^{N} \|F\| \int_{A_n} \|\nu*\delta_x - \nu*\delta_{x_n}\| d\,|\mu|\,(x) \\
&\le 2\|F\|\|\nu\|\varepsilon' + \sum_{n=1}^{N} \|F\|\varepsilon'\,|\mu|\,(A_n) \\
&< \|F\|(2\|\nu\| + \|\mu\|)\varepsilon'.
\end{aligned}$$

Since the above inequalities hold for every $F \in M_a(S)^*$, we conclude that

$$\|\nu * \mu - \sum_{n=0}^{N} \nu * c_n\delta_{x_n}\| \le \varepsilon.$$

The following theorem which generalizes proposition 1.3 and corollary 1.14 of [**8**] is the main result of the paper.

Theorem 4. *Let S be a foundation semigroup with identity. If $M_a(S)$ is separable and left amenable, then there exists a discrete probability measure μ on S such that $I_0(M_a(S)) = J_\mu$.*

Proof. Let $\nu \in I_0(M_a(S))$ and $\varepsilon > 0$. Then by corollary 4.7 of [**6**] there exists a probability measure μ' in $P_1(M_a(S))$ such that $\|\nu * \mu'\| < \varepsilon/2$. By theorem 3 there exists a discrete probability measure μ on S such that $\|\nu * \mu - \nu * \mu'\| < \varepsilon/2$. Thus, $\|\nu * \mu\| < \varepsilon$. So the hypothesis of lemma 1 are satisfied whenever X is replaced by $I_0(M_a(S))$ and $\mathcal{F}$ by $P(M_d(S))$ (the set of all discrete probability measures on S). Therefore there is a discrete probability measure μ on S such that $I_0(M_a(S)) = J_\mu$.

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