The use of Adomian decomposition method for solving a specific nonlinear partial differential equations

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Abstract

In this paper, by considering the Adomian decomposition method, explicit solutions are calculated for partial differential equations with initial conditions. The method does not need linearization, weak nonlinearly assumptions or perturbation theory. The decomposition series analytic solution of the problem is quickly obtained by observing the existence of the self-cancelling "noise" terms where sum of components vanishes in the limit.

1 Introduction

The theory of nonlinear problem has recently undergone much study. We do not attempt to characterize the general form of nonlinear equations [1]. Rather, we solve a specific equation in the following nonlinear problem by using the Adomian decomposition method [2-4]. By solving this type of problems, we do not use conventional transformations which transform a nonlinear problem to an evolution equation and the reduced to a bilinear form. Some times transformation of the nonlinear problem might produce an even more complicated problem. Nonlinear phenomena play a crucial role in applied mathematics and physics. The nonlinear problems are solved easily and elegantly without linearizing the problem by using the Adomian's decomposition method.

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We now describe how the decomposition method can be used to construct the solution for the partial differential equation [1], with initial condition

$$u_t = -e^u u_x + k(u_x)^2 + ku_{xx} + e^{-u}R(x,t); \quad u(x,0) = f(x), \tag{1}$$

where $R(x,t)$ is a given function.

In this paper, we generated an appropriate Adomian's polynomials for nonlinear terms of equation (1), that will be handled more easily, quickly, and elegantly by implementing the Adomian's decomposition method rather than the traditional methods.

2 The method of solution

Let us consider the nonlinear problem (1) in an operator form

$$L_t(u) = -(Ku) + k(Mu) + kL_{xx}(u) + (Nu)R(x,t) \tag{2}$$

where the notations $Ku = e^u u_x$, $Mu = (u_x)^2$, and $Nu = e^{-u}$ symbolize the nonlinear terms. The notation $\text{Ł}_t = \frac{\partial}{\partial t}$ and $\text{Ł}_{xx} = \frac{\partial^2}{\partial x^2}$ symbolize the linear differential operators. Assuming that the inverse operator Ł_t^{-1} exists and it can conveniently be taken as the definite integral with respect to t from 0 to t, i.e., $\text{Ł}_t^{-1} = \int_0^t (.)dt$. Thus, applying the inverse operator Ł_t^{-1} to (2) yields

$$L_t^{-1}L_t(u) = -L_t^{-1}(Ku) + kL_t^{-1}(Mu) + kL_t^{-1}L_{xx}(u) + L_t^{-1}((Nu)R(x,t)). \tag{3}$$

Therefore, it follows that

$$u(x,t) - u(x,0) = -L_t^{-1}(Ku) + kL_t^{-1}(Mu) + kL_t^{-1}L_{xx}(u) + L_t^{-1}((Nu)R(x,t)). \tag{4}$$

We next decompose the unknown function $u(x,t)$ by a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{5}$$

with u_0 identified as $u(x,0)$. An important part of the method is to express the Adomian's polynomials; thus $Ku = e^u u_x = \sum_{n=0}^{\infty} A_n$, $Mu = (u_x)^2 = \sum_{n=0}^{\infty} B_n$, and $Nu = e^{-u} = \sum_{n=0}^{\infty} C_n$ where the A_n, B_n, and C_n are the appropriate Adomian's polynomials generated as follows:

$$\begin{aligned}
A_0 &= (u_0)_x e^{u_0} \\
A_1 &= (u_0)_x u_1 e^{u_0} + (u_1)_x e^{u_0} \\
A_2 &= (u_0)_x(\frac{u_1^2}{2} + u_2)e^{u_0} + u_1(u_1)_x e^{u_0} + (u_2)_x e^{u_0} \\
A_3 &= (u_0)_x(\frac{u_1^3}{6} + u_1u_2 + u_3)e^{u_0} + (u_1)_x(\frac{u_1^2}{2} + u_2)e^{u_0} + (u_2)_x u_1 e^{u_0} + (u_3)_x e^{u_0} \\
&\vdots
\end{aligned} \tag{6}$$

A_n polynomials can be derived from the general form of Adomian polynomial given in [2] and an alternative approach can be find in [5],

$$\begin{aligned}
B_0 &= (u_{0_x})^2 \\
B_1 &= 2(u_0)_x(u_1)_x \\
B_2 &= (u_{1_x})^2 + 2(u_0)_x)(u_2)_x \\
B_3 &= 2(u_0)_x(u_3)_x + 2(u_1)_x(u_2)_x \\
&\vdots
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
C_0 &= e^{-u_0} \\
C_1 &= -u_1 e^{-u_0} \\
C_2 &= (\frac{u_1^2}{2} - u_2)e^{-u_0} \\
C_3 &= (-\frac{u_1^3}{6} + u_1u_2 - u_3)e^{-u_0} \\
&\vdots
\end{aligned} \tag{8}$$

The remaining components $u_n(x,t)$, $n \geq 1$, can be completely determined such that each term is computed by using the previous term. Since u_0 is known, then

$$\begin{aligned}
u_1 &= -L_t^{-1}(A_0) + kL_t^{-1}(B_0) + kL_t^{-1}L_{xx}(u_0) + L_t^{-1}(C_0R(x,t)) \\
u_2 &= -L_t^{-1}(A_1) + kL_t^{-1}(B_1) + kL_t^{-1}L_{xx}(u_1) + L_t^{-1}(C_1R(x,t)) \\
&\vdots \\
u_n &= -L_t^{-1}(A_{n-1}) + kL_t^{-1}(B_{n-1}) + kL_t^{-1}L_{xx}(u_{n-1}) + L_t^{-1}(C_{n-1}R(x,t)).
\end{aligned} \tag{9}$$

Recently, Wazwaz [6] proposed that the construction of the zeroth component of the decomposition series can be define in a slightly different way. In [6], the author assumed that if the zeroth component is $u_0 = g$, the function g is possible to divide into two parts such as g_1 and g_2, then one can formulate the recursive algorithm u_0 and (9) general term in a form of the modified recursive scheme as follows:

$$u_0 = g_1, \tag{10}$$

$$u_1 = g_2 - L_t^{-1}(A_0) + kL_t^{-1}(B_0) + kL_t^{-1}L_{xx}(u_0) + L_t^{-1}(C_0R(x,t)), \tag{11}$$

$$u_{n+1} = -L_t^{-1}(A_n) + kL_t^{-1}(B_n) + kL_t^{-1}L_{xx}(u_n) + L_t^{-1}(C_nR(x,t)), \quad n \geq 1. \tag{12}$$

This type of modification is giving more flexibility to the modified decomposition method in order to solve complicate nonlinear differential equations. In many case the modified scheme avoids the unnecessary computations, especially in calculation of the Adomian polynomials. Furthermore, sometimes we do not need to evaluate the so-called Adomian polynomials or if we need to evaluate these polynomials the computation will be reduced very considerably by using the modified recursive scheme. For more details of the modified decomposition method, one can see references [6]. Illustration purpose we will consider both homogeneous and inhomogeneous nonlinear equations in the following section.

For numerical comparisons purposes, based on the decomposition method, one can constructed the solution $u(x,t)$ as

$$\lim_{n\to\infty} \phi_n = u(x,t) \tag{13}$$

where

$$\phi_n(x,t) = \sum_{k=0}^{n} u_k(x,t), \quad n \geq 0 \tag{14}$$

and the recurrence relation is given as in (9) or (10)-(12). Furthermore, the decomposition series (5) solutions are generally converge very rapidly in real physical problems [2]. The convergence of the decomposition series have investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered by Cherruault [7] and Rèpaci [8]. In [7], Cherruault proposed a new definition of the technique and then he insisted that it will become possible to prove the convergence of the decomposition method. Rèpaci [8] shown a convergence of this method based upon a suitable connection with fixed point techniques. This is essentially the same conclusion derived by Cherruault [7]. These results have been improved by Cherruault and Adomian [9], who proposed a new convergence proof of Adomian's technique based on properties of convergent series. They obtained some results about the speed of convergence of this method providing us to solve linear and nonlinear functional equations.

Adomian and Rach [10] and Wazwaz [11] have investigate the phenomena of the self-cancelling "noise" terms where sum of noise terms vanishes in the limit. An important observation was made that "noise" terms appear for inhomogeneous cases only. Further, it was formally justified that if terms in u_0 are cancelled by terms in u_1, even though u_1 includes further terms, then the remaining non cancelled terms in u_1 are cancelled by terms in u_2, and so on. Finally, the exact solution of the equation is readily found for the inhomogeneous case by determining the first two or three terms of the solution $u(x,t)$ and by keeping only the non cancelled terms of u_0.

The solution $u(x,t)$ must satisfy the requirements imposed by the initial conditions. The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques.

To give a clear overview of the methodology, the following homogeneous nonlinear example will be consider and the solution of which is to be obtained subject to the initial condition

$$u_t = -e^u u_x + (u_x)^2 + u_{xx} + e^{-u}(1+x+t), \quad u(x,0) = ln(x). \tag{15}$$

To solve this equation, we simply take the equation in an operator form exactly in the same manner as the form of equation (4). The zeroth component is $u_0 = ln(x)$. The components u_1, u_2, u_3 are obtained in succession by using (9). Hence, we find

$$\begin{aligned} u_1 &= -L_t^{-1}(A_0) + L_t^{-1}(B_0) + L_t^{-1}L_{xx}(u_0) + L_t^{-1}(C_0(1+x+t)) \\ &= \frac{t}{x} + \frac{t^2}{2x}, \qquad (16) \\ u_2 &= -L_t^{-1}(A_1) + L_t^{-1}(B_1) + L_t^{-1}L_{xx}(u_1) + L_t^{-1}(C_1(1+x+t)) \\ &= -\frac{t^2}{2x} - \frac{t^2}{2x^2} - \frac{t^3}{6x} - \frac{t^3}{2x^2} - \frac{t^4}{8x^2}, \qquad (17) \\ u_3 &= -L_t^{-1}(A_2) + L_t^{-1}(B_2) + L_t^{-1}L_{xx}(u_2) + L_t^{-1}(C_2(1+x+t)) \\ &= \frac{t^3}{6x} + \frac{t^3}{2x^2} + \frac{t^3}{6x} + \frac{t^4}{6x} + \frac{t^4}{4x^3} + \ldots, \qquad (18) \end{aligned}$$

and so on, in this manner the rest of components of the decomposition series were obtained using *Matematica*. Substituting (16)-(18) and the other calculated terms into (5) gives the solution $u(x,t)$ in a series form and in a close form solution by

$$u(x,t) = \ln(x) + \frac{t}{x} - \frac{t^2}{2\,x^2} + \frac{t^3}{3\,x^3} - \frac{t^4}{4\,x^4} + \frac{t^5}{5\,x^5} - \frac{t^6}{6\,x^6} + \ldots \qquad (19)$$

or $u(x,t) = ln(x+t)$ which can be verified through substitution.

As an example of an application of the self cancelling phenomena [6,10,11], let us seek the explicit solution of an inhomogeneous nonlinear equation (1), with initial condition

$$u_t = F(x,t) - e^u u_x + k(u_x)^2 + k u_{xx} + e^{-u}R(x,t); \quad u(x,0) = 1, \qquad (20)$$

where $F(x,t) = -2t + x^2(1-4t^2) - e^{-(1+tx^2)} + 2txe^{1+tx^2}$ is a right hand side given function, $R(x,t) = 1$ is a given function, and $k = 1$ is a constant.

To obtain the decomposition solution subject to initial condition given, we first used (20) in an operator form in the same manner as form (4) and then we used (10)-(12) to determine the individual terms of the decomposition series, we get immediately

$$\begin{aligned} u_0 &= 1 + tx^2, \qquad (21) \\ u_1 &= -t^2 - \frac{4t^3x^2}{3} + \frac{1}{x^3}\{xe^{-(1+tx^2)} - \frac{(x-2e^2)}{e} + 2e(-1+tx^2)e^{tx^2}\} - \\ &\quad -L_t^{-1}(A_0) + L_t^{-1}(B_0) + L_t^{-1}L_{xx}(u_0) + L_t^{-1}(C_0) \\ &= \frac{1}{ex^2} - \frac{2e^2}{x^3} - \frac{(-2e+x)}{ex^3}, \qquad (22) \\ u_{n+1} &= 0, \quad n \geq 1. \qquad (23) \end{aligned}$$

It is obvious that the "noise" terms appear between the components of u_1, and these are all cancelled. As seen equation (22), the closed form of the solution can be find very easily by proper selection of g_1 and g_2. In the case of right choice of these functions, the modified technique accelerate the convergence of the decomposition series solution by computing just u_0 and u_1 terms of the series. The term u_0 provides the exact solution as $u(x,t) = 1 + tx^2$ and this can be justifies through substitution.

It may be concluded that, the Adomian methodology is very powerful and efficient in finding exact solutions for wide classes of problems. With regard to this application, the decomposition method outlined in the above analysis shows many of the equations of physics appear to be solvable analytically without linearization, perturbation or discretization. It is also worth noting that the advantage of the decomposition methodology is that it displays a fast convergence of the solution. It may be achieved by observing the self-cancelling "noise" terms. In addition, the numerical results obtained by this method have illustrated a high degree of accuracy as discussed in [12,13].

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