Osculating subspaces, normal rational curves and generalized strange curves

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1 Introduction

Let $C \subset \mathbb{P}^n$ be an integral non–degenerate projective curve. Fix an integer t with $1 \leq t \leq n-1$. In this paper, we study properties of the family of all osculating t–subspaces at general points of C. We work over an algebraically closed field $\mathbb{K}$ and set $p := \text{char}\ (\mathbb{K}) \geq 0$. We are mainly interested in the case in which $\mathbb{K}$ is the algebraic closure of a finite field $GF(q)$, because in this case, some concepts may be translated in the set–up of finite geometries (see the use of [4] in Section 3).

A *normal rational curve* C_n in $\mathbb{P}^n$ is a non–degenerate smooth rational curve of degree n. The curve C_n is unique up to projective transformations, and it is projectively equivalent to the curve with parametric equations $\{P(t) = (t^n, t^{n-1}, \dots, t, 1), t \in \mathbb{K} \cup \{\infty\}\}$, where $t = \infty$ gives the point $(1, 0, 0, \dots, 0)$.

Many results on normal rational curves can be found in [7, Sec. 27.5].

In Section 2, we give two characterizations of the normal rational curves in terms of the linear span of families of osculating subspaces (see Theorems 2.4, 2.5 and Remark 2.6).

In Section 2, we give the definition of $(a, b; t)$–strange curve in $\mathbb{P}^n$, a natural generalization of the notion of strange curve (see Definition 3.1). Furthermore, we study curves $C \subset \mathbb{P}^n$ whose general osculating planes mutually intersect. The main result of this Section (Theorem 3.7) uses several nice results from [4].

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2 Characterizations of normal rational curves

Let $C \subset \mathbb{P}^n$ be an integral non–degenerate curve, and $\pi : X \to C$ the normalization. Let $f : X \to \mathbb{P}^n$ be the composition of π with the inclusion of C in $\mathbb{P}^n$.

The *order sequence* of f at a point P of X is defined to be the set $b_i(P,C)_{0\le i\le n}$ of intersection multiplicities at the point P of C with the hyperplanes of $\mathbb{P}^n$. Almost all points of C have the same order sequence which is called the order sequence of C and is denoted by $b_i(C)$, $0 \le i \le n$, where $b_0(C) < b_1(C) < \cdots < b_n(C)$. If $Q \in C_{reg}$, we will often write $b_i(C,Q)$ instead of $b_i(C, f^{-1}(Q))$.

In characteristic 0, the order sequence is $\{0, 1, \ldots, n\}$. For this reason, C is said to have *classical orders* if $b_i(C) = i$, $i = 0, \ldots, n$.

For further results on order sequence see [9], [10], [11]

For every $Q \in C_{reg}$, we will denote by $W(C,Q,t)$ the osculating t–space of C at Q. For every closed subscheme Z of $\mathbb{P}^n$, $\langle Z\rangle$ will denote its linear span in $\mathbb{P}^n$, i.e. the minimal linear subspace of $\mathbb{P}^n$ containing the scheme Z.

Definition 2.1. *Fix an integer j with $0 \le j \le n$. We will say that f (or C) has no ramification of level $\le j$, if for every integer i, with $0 \le i \le j$, and every $P \in X$, we have $b_i(C,P) = b_i(C)$.*

Remark 2.2. Fix a point $Q \in C_{reg}$. For every integer $d > 0$, dQ is a Cartier divisor of C [6]. The Cartier divisor $(b_i(C,Q) + 1)Q$ is the connected component, supported by Q, of the scheme–theoretic intersection of C with the osculating t–subspace $W(C,Q,t)$.

Now, assume $C \subset \mathbb{P}^n$ linearly normal, i.e. assume that C is non–degenerate and $h^0(C, \mathcal{O}_C(1)) = n+1$, see [1]. For every integer d, the Cartier divisor dQ of C is contained in $W(C,Q,t)$, but not in $W(C,Q,t-1)$ if and only if $h^0(C, \mathcal{O}_C(1)(-dQ)) = n - t$. In particular, we have $b_i(C_n, Q) = i$, for every $Q \in C_n$.

If $Q_1, \ldots, Q_s$ are distinct points of C_{reg} and $a_1, \ldots, a_s$ are non–negative integers, the linear span of the scheme $\bigcup_{i=1}^s a_iQ_i$ of C has dimension $n+1-h^0(C, \mathcal{O}_C(1)(-\sum_{i=1}^s a_iQ))$. In particular, if $C = C_n$, we have dim $(\langle \bigcup_{i=1}^s a_iQ_i\rangle)$ min $\{n, \sum_{i=1}^s a_i - 1\}$, for all positive integers s, $a_1 \ldots, a_s$ and all choices of the points $Q_1, \ldots Q_s \in C_n$.

Notice that the last sentence of the previous remark, guarantees that two osculating linear spaces of dimension $[(n-1)/2]$, $n \ge 3$, of C_n, do not intersect, and hence it is stronger then [7, Lemma 27.5.2 ii)]. Alternatively, every hyperplane through the i–dimensional osculating space at a point P of C_n intersects C_n at the point P with multiplicity bigger that i. Hence, two $[(n-1)/2]$–dimensional osculating spaces cannot intersect, or else they define a hyperplane intersecting C_n in at least (counting multiplicities) $2((n-1)/2 + 1) > n$ points.

If $p > 0$, the intersection of all osculating hyperplanes of C_n may be not empty; indeed, the dimension of the intersection is a known function of n and p, see [5] for more details. For instance, if $p = 2$, a smooth plane conic C_2 has a nucleus, i.e. there is a point N such that each tangent line to C_2 contains N.

We recall, [6] that a curve $C \subset \mathbb{P}^n$ is said to be *strange* if there exists a point N of $\mathbb{P}^n$ (called center or strange point) such that all tangent lines at smooth points of C contain N. The curve C_n is strange if and only if $p = n = 2$.

Now, we will give three characterizations of normal rational curves. To put our main result into a proper context, we recall the following well known fact.

Theorem 2.3. *Let $C \subset \mathbb{P}^n$ be an integral non–degenerate curve and let $\pi : X \to C$ be the normalization. Let $f : X \to C$ be the composition of π with the inclusion of C in $\mathbb{P}^n$. Assume that f has no osculating hyperplane (or lower dimensional subspace) in the sense of [9], i.e. assume that f has no ramification of level $\leq n-1$. Then C is the normal rational curve.*

Proof. Set $d :=\deg(C)$ and $g := p_a(X)$. By the Brill–Segre formula ([8],[9, Theorem 9]) the weighted number w of osculating hyperplanes to C is $(2g-2)(\sum_{i=0}^{n}(b_i(C))+(n+1)d$. Since $w = 0$ by assumption and $b_i \geq i$ for every i, we have $g = 0$ and $(n+1)d = 2(\sum_{i=0}^{n}(b_i(C))$

If the order sequence of C is the classical one, i.e. if $b_i = i$, for every i, we obtain $d = n$, as wanted. In the general case, we need to work more. Since the theorem is true if and only if $d = n$, we may assume $d > n$ and look for a contradiction.

Since $g = 0$, C is a linear projection of a normal rational curve C_d of $\mathbb{P}^d$ from a linear subspace M of $\mathbb{P}^d$, with $\dim(M)= d-n-1$, and $M \cap C_d = \emptyset$.

Fix $P \in C_d$ and let $W(C_d, P, n)$ be the n–dimensional osculating subspace to C_d at P, i.e. the linear subspace of $\mathbb{P}^d$ spanned by the Cartier divisor $(n+1)P$ of C_d (see Remark 2.2). Since C_d is non–degenerate and $P \in W(C_d, P, n)$ for every P, the union of all subspaces $W(C_d, P, n)$ is a closed $(n+1)$–dimensional subvariety Z of $\mathbb{P}^d$. Since $\dim(M)= d-n-1$, we have $Z \cap M \neq \emptyset$, i.e. there exists $P \in C_d$ with $W(C_d, P, n) \cap M \neq \emptyset$. If $W(C_d, P, n) \cap M \neq \emptyset$, then the image of $W(C_d, P, n)$ in $\mathbb{P}^n$ is contained in a osculating hyperplane of C, contradicting our assumption. ∎

Theorem 2.3, together with Remark 2.2, give the following characterization of normal rational curves.

Theorem 2.4. *Let $C \subset \mathbb{P}^n$ be an integral non–degenerate curve. Then C is a normal rational curve if and only if it has no ramification of level $\leq n-1$.*

Now, we can prove the main result of this Section.

Theorem 2.5. *Let $C \subset \mathbb{P}^n$ be a smooth non–degenerate curve with classical order sequence, i.e. $b_j(C) = j$, for every j. Fix an integer $s \geq 1$, s points $P_i \in C$, $1 \leq i \leq s$, and integers $a_i \geq 0$ such that $\sum_{i=1}^{s} a_i \leq n+1-s$. Set $M := \langle \bigcup_{i=1}^{s} W(C, P_i, a_i)\rangle$ and assume that $x :=$ dim $(M) = \sum_{1=1}^{s} a_i + s - 1$. Assume that C has no ramification of level $\leq n-x-1$ and that M does not contain any $W(C, P_i, a_i+1)$. Moreover, assume that for every $P \in C \setminus \{P_1, \ldots, P_s\}$, we have $W(\mathcal{C}, P, n-x-1) \cap M = \emptyset$. Then $\mathcal{C}$ is a normal rational curve.*

Proof. Let $u := \mathbb{P}^n \backslash M \to \mathbb{P}^{n-x-1}$ be the linear projection from M. Let $D \subset \mathbb{P}^{n-x-1}$ be the linear projection of C from M. Since $n-x-1 \geq 2$, D is a curve.

Our assumptions imply $C \cap M = \{P_1, \ldots, P_s\}$ and that the points of $D \setminus (u(C \setminus \{P_1, \ldots P_s\}))$ correspond to the images of suitable osculating spaces to C at the points $P_1, \ldots, P_s$.

Since M does not contain any $W(C, P_i, a_i+1)$, the curve C is smooth without any ramification of order $\leq n-x-1$, the points of $D \setminus (u(C \setminus \{P_1, \ldots P_s\})$ are not hyperosculating points of D. We have deg $(D) =$ deg $(C) - (\sum_{i=1}^{s} a_i + s) =$ deg $(C) + x + 1$ and our assumptions imply that the scheme–theoretic intersection $C \cap M$ has length $\sum_{i=1}^{s} a_i + s$. Since for every point $P \in \{P_1, \ldots, P_s\}$, we have

$W(\mathcal{C}, n-x-1, P) \cap M = \emptyset$, and $b_j(C,P) = b_j$, for every $j \leq x$, no point of the normalization $(u(C \setminus \{P_1, \dots P_s\})$ is a hyperosculating point of the normalization of D. Thus D has no ramification point, and by Theorem 2.3, D is a normal rational curve. Since deg(C) =deg$(\mathcal{D}) + x + 1$, we have deg$(C) = n$, i.e. C is a normal rational curve. ∎

Remark 2.6. By Remark 2.2, a normal rational curve $C_n \subset \mathbb{P}^n$ satisfies all the assumptions of Theorem 2.5. Hence C_n is, up to a projective transformation, the only curve satisfying all the assumptions of Theorem 2.5. Thus Theorem 2.5 gives a characterization of normal rational curves, which generalizes the one given in [3] for space curves.

3 Generalizations of strange curves

Motivated by the paper [4], we introduce the following definitions.

Definition 3.1. *Fix integers n, a, b, t, with $n \geq 3$, $a \geq b \geq 0$ and $b < t < n$. Let $C \subset \mathbb{P}^n$ be an integral non–degenerate curve.*

We will say that C is $(a, b; t)$–strange, if there exists a linear subspace M of $\mathbb{P}^n$, with dim$(M) = a$, such that for a general point $P \in C$, the osculating linear subspace of dimension t, $W(C, P, t)$ to C at P we have dim$(M \cap W(C, P, t)) \geq b$.

Definition 3.2. *Fix integers n, b, t with $n \geq 3$, $0 < b < t < n$. Let $C \subset \mathbb{P}^n$ be an integral non–degenerate curve. We will say that C is $(b; t)$–Klein if for general points $P, Q \in C$, the t–osculating subspaces, $W(C, P, t)$ and $W(C, Q, t)$ of C at P and Q, respectively, intersect in a linear space of dimension $\geq b$.*

Since in characteristic zero, the order sequence $\{b_i(C)\}_{0 \leq i \leq n}$ (in the sense of [9] or [10]) of an integral non–degenerate curve $C \subset \mathbb{P}^n$ is classical, i.e. $b_i(C) = i$, for every $0 \leq i \leq n$, the notion of $(a, b; t)$–strange curve is interesting only in positive characteristic. Notice that every $(b, b; t)$–strange curve is $(b; t)$–Klein. Since a non–degenerate curve $C \subset \mathbb{P}^n$ is not contained in a hyperplane, C cannot be $(t; t)$–Klein or $(a, t; t)$–strange, with $a \leq n-1$. Since for any linear subspaces V and W of $\mathbb{P}^n$, with dim (V)+ dim $(W) \geq n$, we have $V \cap W \neq \emptyset$, and dim $(V \cap W) \geq$ dim (V)+ dim $(W) - n$, every non–degenerate curve in $\mathbb{P}^n$ is $(b+n-t, b; t)$–strange and $(2n-t; t)$–Klein if $t \geq b+n$. It does not follows formally from the definition, but it is easy to check that a curve is strange (i.e. $(0, 0; 1)$–strange) if and only if it is $(0; 1)$–Klein. Hence, the previous definitions are not new for $t = 1$.

We will use the general classification theorem proved in [4] to analyze the case $t = 2$. First we need to introduce some more notation.

Definition 3.3. *Let $C \subset \mathbb{P}^n$ be a non–degenerate $(b; t)$–Klein curve, and assume that C is not $(b+1; t)$–Klein.*

Let V be the open subset of $C_{reg} \times C_{reg}$, such that for all pairs $(P, Q) \in V$, we have dim$(W(C, t, P) \cap W(C, t, Q)) = b$.

The t–base space $B(C, t)$ of the osculating t–subspaces of C, is the linear span of all $W(C, P, t) \cap W(C, Q, t)$, for $(P, Q) \in V$. Notice that, $B(C, t)$ does not change if instead of V we take any open subset V' of V with $V' \neq \emptyset$.

For a general point $P \in C_{reg}$, with $(P, R) \in V$, for some $R \in C_{reg}$, let $W(C, P, t, *)$ be the linear span of the subspaces $W(C, P, t) \cap W(C, Q, t)$, with Q such that $(P, Q) \in V$. The subspace $W(C, P, t, *)$ does not depend on the choice of V or, if we take a Zariski dense open subset V' of V instead of V, and the integer $d := dim\,(W(C, P, t, *))$ does not depend on the choice of the general point P. We will call d the small t–base dimension of the osculating t–subspaces of C.

The above terminology was motivated by [4, Definition 1.3].

Proposition 3.4. *Let $C \subset \mathbb{P}^n$ be a non–degenerate $(b; t)$–Klein curve. If $b < t$, assume that $\mathcal{C}$ is not $(b + 1; t)$–Klein. Let s be the small t–base dimension of C. Then C is $(s, b; t)$–strange.*

Proof. Fix a general point $P \in C_{reg}$. By definition, $W(\mathcal{C}, P, t, *)$ is a proper s–dimensional linear subspace of $W(C, P, t)$, and for a general point $Q \in C_{reg}$, we have $W(C, Q, t) \cap W(C, P, t) \subset W(C, Q, t) \cap W(C, P, t, *)$. Thus, we may take the linear subspace $W(C, P, t, *)$ to check the $(s, b; t)$–strangeness of C. ∎

Remark 3.5. Let $C \subset \mathbb{P}^n$ be a non–degenerate $(b; t)$–Klein curve which is not $(b+1; t)$–Klein, and such that the base space $B(C, t)$ is a linear subspace of dimension $a < n$. Then it follows from Proposition 3.4 that C is $(a, b; t)$–strange.

Remark 3.6. Let $C \subset \mathbb{P}^n$ be a non–degenerate $(1; 2)$–Klein curve. Fix two general points P, P' of C. Set $D := W(C, P, 2) \cap W(C, P', 2)$. By assumption D is a line, and $\dim(\langle W(C, P, 2) \cup W(C, P', 2)\rangle) = 3$.

If for a general point $Q \in C$, we have $D \subset W(C, Q, 2)$, then C is $(1, 1; 2)$–strange, with D as its strange subspace.

If $W(C, Q, 2)$ does not contains D for a general Q, then $W(C, Q, 2)$ contains two different lines of the 3–dimensional linear space $\langle W(C, P, 2) \cup W(C, P', 2)\rangle$. Hence C is contained in $\langle W(C, P, 2) \cup W(C, P', 2)\rangle$, i.e. $n = 3$. Thus, if $n > 3$, then C must be $(1, 1; 2)$–strange.

As a by–product of the general classification theorem proved in [4], we obtain the following result.

Theorem 3.7. *Let $C \subset \mathbb{P}^n$, $n \geq 5$, be a non–degenerate $(0; 2)$–Klein curve which is neither $(4, 1; 2)$–strange not $(0, 0; 2)$–strange. Then $n = 5$.*

Proof.

Let S be an infinite set of lines of $\mathbb{P}^{n-1}$ such that for all $D, D' \in S$, we have $D \cap D' \neq \emptyset$. It is a standard and well known exercise that there is a point $P \in \mathbb{P}^{n-1}$, with $P \in D$, for every $D \in S$, or there is a plane $\Pi \subset \mathbb{P}^{n-1}$, with $D \subset \Pi$, for every $D \in S$. Hence, taking a general hyperplane of $\mathbb{P}^n$, we reduce to the case in which for general $(Q, Q') \in C \times C$, the linear subspace $W(C, Q, 2) \cap W(C, Q', 2)$ is a point and not a line.

Hence, we are in the set–up of [4] for the infinite set of all subspaces $\{W(C, Q, 2)\}$, Q general in C.

Set $m := \dim\,(B(C, 2))$. By [4, §6], we have $m \leq 6$. We may exclude the case $m = 6$ by the general classification theorem proved therein, because our ground field is infinite, and hence we may find infinitely many osculating planes to C such that each pair of these planes intersect.

Assume $m = 5$. Since C is non–degenerate and irreducible, and the ground field is infinite, [4, Lemma 4.5] implies $n = 5$. Hence, it is enough to prove that $m \geq 5$. We have $m > 0$ because C is not $(0, 0; 2)$–strange.

Assume $1 \leq m \leq 4$ and take general points $Q, Q', Q'' \in C$. Since C is not $(4, 1; 2)$–strange, we have $\dim(B(C,2) \cap W(C,2,Q)) = \dim(B(C,2) \cap B(C,2,Q')) = \dim(B(C,2) \cap B(C,2,Q'')) = 0$. Since $W(C,Q,2) \cap W(C,Q',2)$, $W(C,Q,2) \cap W(C,Q'',2)$, $W(C,Q',2) \cap W(C,Q'',2)$ are contained in $B(C,2)$ by definition of 2–base space of C, we obtain that C is $(0,0;2)$–strange with $B(C,2) \cap W(C,2,Q)$ as common point of sufficiently general osculating planes.

Since C is assumed to be not $(0,0;2)$–strange, we obtain a contradiction to the assumption $1 \leq m \leq 4$, proving the theorem. ∎

To exclude the case $m = 4$ (the only difficult one with $m < 5$) in the proof of the previous theorem one can use [4, Lemma 3.3 and Theorem 3.6].

This observation may be useful if we make different assumptions on the curve C.

Every non–degenerate curve in $\mathbb{P}^4$ is obviously $(0; 2)$–Klein. We want to discuss why we were not able to use [4, Theorem 5.1] to show that if $m = n = 5$, then the union of the osculating planes to C is contained in a smooth quadric hypersurface, and in particular C is contained in a smooth quadric hypersurface.

First of all, to apply [4, Theorem 5.1] we need to assume $p > 0$. This is always the case under the assumptions of Theorem 3.7, because the projection of C into $\mathbb{P}^2$ from a general osculating plane is a plane curve with non–classical order sequence.

Secondly, to apply [4, Theorem 5.1], we need to work with the algebraic closure of a finite field. This is by far the more interesting case, but we could even reduce the general case to this case by a specialization argument.

The key problem for us was the following one. Assume that $\mathbb{K}$ is the algebraic closure of $GF(p)$. Then C is defined over some field $GF(q)$, with $q = p^e$. However, by the Hasse–Weil theorem, the curve C has roughly around q points defined over $GF(q)$, for large q, and around q osculating planes defined over $GF(q)$. This is far less than $3(q^2 + q + 1)$ planes needed to apply [4, Theorem 5.1].

Now, we give a recipe to construct several examples of $(0, 0; t)$–strange curves, just starting from one example satisfying a mild condition (see Construction 3.8).

To motivate that construction and show that all examples arise in this way, we first consider the general case of $(b, b; t)$–strange curves, with $b \geq 0$.

(3.8) Fix integers b, t, n, with $0 \leq b \leq t - 2 \leq n - 3$.

Let $C \subset \mathbb{P}^n$ be an integral non–degenerate curve.

Assume that C is $(b, b; t)$–strange with respect to a linear subspace M of $\mathbb{P}^n$, with $\dim(M) = b$.

Let $\pi : \mathbb{P}^n \setminus M \to \mathbb{P}^{n-b-1}$ be the linear projection from M. Let Z be the closure of $\pi(C \setminus (C \cap M))$. Thus $Z \subset \mathbb{P}^{n-1-b}$ is an integral non–degenerate curve.

Let $\{b_i(C)\}_{0 \leq i \leq n}$ (resp. $\{b_i(Z)\}_{0 \leq i \leq n}$ be the order sequence of $\mathcal{C}$ (resp. Z).

Since a general osculating t–subspace of C contains M, we have $b_i(Z) \geq b_{i+b+1}(C)$, for every integer i, with $t - b - 1 \leq i \leq n - b - 1$.

Let $\pi' := C \setminus (C \cap M) \to Z$ be the regular map induced by it. If C is not $(b, 0; 1)$–strange with respect to Z, then π' is separable. Notice that C is contained in the cone with vertex M and base Z.

Now, if $b = 0$, we reverse the process. Starting from a suitable curve $Z \subset \mathbb{P}^{n-1}$ we construct a family of $(0, 0; t)$–strange curves of $\mathbb{P}^n$ contained in a 2–dimensional cone with Z as base.

Construction 3.9.

Fix integers n and t, with $n > t \geq 2$. Let $Z \subset \mathbb{P}^{n-1}$ be an integral non–degenerate curve. See $\mathbb{P}^{n-1}$ as a hyperplane, H, of $\mathbb{P}^n$. Fix $M \in (\mathbb{P}^n \setminus H)$, and let T be the cone with basis Z and vertex M. Call $\pi : T \setminus \{M\} \to Z$ the projection. Let $Y \subset T$ an integral non–degenerate curve. Since Y is not a line through M, the closure of $\pi(Y \setminus M)$ is Z. Fix a general point $Q \in Z$. Hence $b_i(Z, Q) = b_i(Z)$, for every i.

Let U be the cone with basis the Cartier divisor $(b_{t-1}(Z)+1)Q$ of Z and vertex M. Clearly U_{red} is the line $\langle\{Q, M\}\rangle$, i.e. U is a multiple line with $\deg(U) = b_{t-1}(Z)+1$. Since $b_{t-1}(Z, Q) = b_{t-1}(Z)$, the Cartier divisor $(b_{t-1}(Z)+1)Q$ of Z spans the $(t-1)$–dimensional osculating subspace $W(Z, Q, t-1)$ to Z at Q.

Since M is not contained in the hyperplane spanned by Z, we have $\dim(\langle U\rangle) = t$. Fix a point $Q' \in Y$, with $Q' \neq M$ and $\pi(Q') = Q$.

By the generality of Q, we have $Q' \in Y_{reg}$, and $b_i(Y, Q') = b_i(Y)$, for every i. Notice that $b_{j-1}(Z) \leq b_j(Z)$, for every j, with $1 \leq j \leq n-1$.

Since the closure of $\pi(Y \setminus M)$ is Z and $\pi(U \setminus M) = W(Z, Q, t-1)$, it follows that $\langle U\rangle$ contains a length $b_{t+1}(Z)$ subscheme of Y with $\{Q'\}$ as support.

Assume also $b_t(Z) = b_{t-1}(Z)+1$. Then either $b_t(Y) = b_t(Z)$, or $b_t(Y) = b_{t-1}(Z)$, and the latter case occurs if and only if a general $(t-1)$–dimensional osculating subspace to Y contains the point M.

Now, we assume to be in the set–up of 3.8 with as C any non–degenerate curve Y' in T. Hence M is contained in all general t–osculating subspaces of Y'. From Y', we obtain by projection Z, and hence the cone T and all curves Y contained in T. If Y' is not $(0, 0; t-1)$–strange, then for any such Y, we have $b_t(Y) = b_t(Z)$. this assumption is not very restrictive because it just means that we take as t the first integer x such that Y' is $(0, 0; x)$–strange.

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