# A Nonmeasurable Partition of the Reals

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**Abstract**

It is shown that there are Vitali type Lebesgue nonmeasurable subsets V of, say, the real unit interval with outer measure of V being equal to any preassigned positive real $\leq 1$ and with inner measure of $V$ being always equal to 0.

The present paper is in the setting of the real numbers which is denoted by $R$. All notions of measure are in the sense of Lebesgue [1, p.62]. As usual, $m^*(S)$, $m_*(S)$ and $m(S)$ respectively, stand for the *outer measure,* the *inner measure,* and the *measure* of a subset $S$ of $R$.

Vitali's construction [2, p.22] of a nonmeasurable subset $V$ of the closed-open unit interval $[0,1)$ denoted by $I$, is very often stated in the literature. However, no mention of the value of the outer measure $m^*(V)$ of any $V$ is given.

Below we show that for every positive real number $r \leq 1$, there exists a Vitali nonmeasurable subset $V$ of the unit interval $[0,1)$ such that $m^*(V) = r$ and always $m_*(V) = 0$.

First however, we prove the following:

**LEMMA 1.** *Let $A$ be a subset of a nonempty closed-open interval $[a,b)$ such that $A$ has at least one point in common with every closed subset of positive measure of $[a,b)$. Then*

(1) $m^*(A) = b - a$

*Proof.* Assume on the contrary that $m^*(A) < b - a$. Then $A$ can be covered by an open set $E$ with $m(E) < b - a$. Clearly $[a,b) - E$ has a closed subset of positive measure of $[a,b)$ which has no point in common with $A$, contradicting our assumption. Thus the Lemma is proved. ∎

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Using the Axiom of Choice, we let $W$ denote a well ordering of $[0,1)$.

In what follows any order among the elements of subsets of $[0,1)$ is made in connection with $W$. Thus, every subset of $[0,1)$ is well ordered.

We recall [3, p.6] that every uncountable closed subset of $R$ is of the continuum cardinality $\aleph = \overline{\overline{R}}$ and therefore:

(2) every closed subset $C_u$ of positive measure of $[a,b)$ is of cardinality $\aleph$

We recall also that the set of all closed subsets of the nonempty interval $[a,b)$ has cardinality $\aleph$ and therefore:

(3) the set $C$ of all closed subsets $C_u$ of positive measure of the nonempty interval $[a,b)$ forms a family indexed by $\aleph$, i.e., $C = (C_u)_{u\in\aleph}$

where without loss of generality we let

(4) $a \in C_0$

Below $Q$ stands for the subset of all rational numbers in $I$ and for every real number $r$ of $I$, we let

(5) $r + Q \;=\; \{r + q : q \in Q\}$

From now on we assume that $[a,b)$ is a nonempty interval of $I = [0,1)$, i.e.,

(*) $[a,b) \;\subseteq\; [0,1)$ with $b > a$

**THEOREM 1.** *The interval [0,1) is the union of continuum many pairwise disjoint subsets $S_u$ of [0,1) such that any closed subset $C_u$ mentioned in (3) has at least one point in common with one of the $S_u$'s.*

*Proof.* For ordinals $u$ elements of $\aleph$, based on (3), we define

(6) $S_0 = c_0 + Q\,(mod\,1)$ with $c_0 = a$

and

(7) $S_u = c_u + Q \;(\text{mod } 1)$ where $c_u$ is the first element (with respect to $W$) of $C_u - (\cup(S_i)_{i<u})$ for every nonzero $u < \aleph$

The above is possible since for every $u < \aleph$ by (*) and by (2) we have $\overline{\overline{C}}_u = \aleph$ and $\overline{\overline{\cup(S_i)}}_{i<u} < \aleph$ in asmuch as every $S_u$ is countable and $\overline{\overline{u}} < \aleph$.

From (6) and (7) it follows that

(8) if $u < v$ then $S_u \cap S_v = \emptyset$

since otherwise $c_u + q_1 = c_v + q_2$ (mod 1) for rationals $q_1$ and $q_2$ in $Q$ which contradicts (7). Hence,

(9) $(S_u)_{u<\aleph}$ are pairwise disjoint subsets of $I$

If $[0,1) - (\cup(S_u)_{u<\aleph}) \neq \emptyset$, then we continue as in (6) and (7) by defining

(10) $S_\aleph = k_0 + Q \pmod 1$ where $k_0$ is the first element of $[0,1) - (\cup(S_u)_{u<\aleph})$

and for ordinals $t = 1, 2, \ldots$ we let

(11) $S_{\aleph+t} = k_t + Q \pmod 1$ where $k_t$ is the first element of
$[0,1) - (\cup(S_u)_{u<\aleph} \cup (\cup(S_{\aleph+i})_{i<t}))$

provided the expression in (11) is nonempty. Clearly, the above process must stop for some $t = w$ (since $\overline{\overline{[0,1)}} = \aleph$). Obviously,

(12) $I = \cup(S_u)_{u<\aleph+w}$

As in the case of (8) here also it can be seen readily that

(13) if $u < v < \aleph + w$ then $S_u \cap S_v = \emptyset$

Therefore, by (12) and (13) we see that $I$ is the union of pairwise disjoint subsets $S_u$ of $I$. Clearly from (4), (6), (7) it follows that every $u < \aleph$ the closed subset $C_u$ mentioned in (3), has at least one point, namely $c_u$, in common with $S_u$. This need not be the case for the remaining $S_u$'s appearing in $(S_u)_{\aleph \leq u < \aleph+w}$. Thus Theorem 1 is proved. ∎

**LEMMA 2.** *Let*

*(14)* $V = \{c_u : u < \aleph\} \cup \{k_t : \aleph \leq t < \aleph + w\}$

*where the $c_u$ 's are as given in (4), (6), (7) and the $k_t$'s are as given in (10), (11). Then $V$ is a nonmeasurable subset of $[a,b)$ and*

*(15)* $m^*(V) = b - a$

*Proof.* From (4), (6), (7) it follows that $V$ has one point, namely $c_u$ in common with every closed subset $C_u$ of positive measure of $[a,b)$. Thus by (1), we see that $m^*(V) = b - a$. On the other hand, because of (1), (13), (14), we also have

(16) $(V+p)_{mod1} \cap (V+q)_{mod1} = \emptyset$ and $m^*(V) = m^*(V+q)_{mod1} = b - a$
for distinct rationals $p$ and $q$ in $Q$ ∎

**REMARK 1.** *We observe that our definition of $V$ given in (14) resembles Vitali's construction [2, p.22]. However, with the significant difference that in our case $V$ has one point in common with every closed subset of positive measure of $[a,b)$ from which it follows that $m^*(V) = b - a$ which may not be the case in the Vitali's construction.*

**LEMMA 3.** *Let $(q_i)_{i\in\omega}$ be an enumeration of the rationals in $Q$. Then the interval $[0,1)$ is a countable union of pairwise disjoint nonmeasureable subsets*

$(V+q_i)_{mod1}$ with $i \in \omega$ where each $(V+q_i)_{mod1}$ is congruent by translation to $V$ as given in (14).

Moreover,

(17) $\quad [0,1) = \bigcup_{i\in\omega}(V+q_i)_{mod1} \quad$ with $\quad m^*(V+q_i)_{mod1} = b-a$

and $\quad m_*(V+q_i)_{mod1} = 0$

*Proof.* From (6), (7), (11) it follows that

(18) $\quad S_u = \{c_u+q_0,\ c_u+q_1,\ c_u+q_2,\ \ldots\ \} \quad$ for $\ u < \aleph$

$S_u = \{k_u+q_0,\ k_u+q_1,\ k_u+q_2,\ \ldots\ \} \quad$ for $\ \aleph \le u < \aleph + w$

where in (18) all the sums are (mod 1). Then the first two equalities in (17) readily follows from (12), (14), (16), (18). On the other hand for every $i \in \omega$ we have $m_*(V+q_i) = 0$ This is because otherwise every $V+q_i$ would contain a closed subset of positive measure which by (16) and the first equality in (17) would imply that $m[0,1)$ is infinite, which is a contradiction. Thus, Lemma 3 is proved. ■

Finally, we have:

**THEOREM 2.** *The set $R$ is a disjoint union of countably many congruent by translation nonmeasurable subsets each of which is of outer measure b - a and of inner measure 0.*

*Proof.* Clearly

(19) $\quad R = [0,1) \ \cup\ [-1,0) \ \cup\ [1,2) \ \cup\ [-2,-1) \ \cup\ [2,3)\ \ldots$

But then the proof of Theorem 2 follows from Lemma 3 applied to each of the terms of the union given in (19). ■

REMARK 2. Based on Theorem 2 easy proofs of some known important statements can be given. For instance: *Every subset H of positive outer measure of $R$ has a nonmeasurable subset. This is because $H$ must have a subset $M$ of positive outer measure in common with at least one of the terms in (19), say with, $[0,1)$. But then by (17) we see that $M$, in its turn, must have a subset $N$ of positive outer measure in common with at least one of the $(V+q_i)_{mod1}$. Thus, the subset $N$ of $H$ cannot be measurable for otherwise it would imply that the interval $[0,1)$ is of infinite measure.*

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