On the Positive Periodic Solutions of the Nonlinear Duffing Equations with Delay and Variable Coefficients *

Yuji Liu Weigao Ge

Abstract

We consider the existence and nonexistence of the positive periodic solutions of the non-autonomous nonlinear Duffing equation with delay and variable coefficients

$$x''(t) - p(t)x'(t) + q(t)x(t) = \lambda h(t)f(t, x(t-\tau(t))) + r(t),$$

It is shown that the equation has positive periodic solutions under certain conditions and no positive $T-$periodic solution under some other conditions by using a fixed theorem in cones.

1 Introduction

Consider the existence and nonexistence of the positive periodic solutions of the non-autonomous nonlinear delay Duffing equation

$$x''(t) - p(t)x'(t) + q(t)x(t) = \lambda h(t)f(t, x(t-\tau(t))) + r(t), \tag{1}$$

where $\lambda > 0$ is a parameter. $p:\ R \to R^+$, $q:\ R \to R^+$, $h:\ R \to R^+$, $r:\ R \to R$, $\tau:\ R \to R$ and $f:\ R \times R^+ \to R$ are continuous and $T-$periodic about the variable t. $R = (-\infty, +\infty)$ and $R^+ = [0, +\infty)$.

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Recently, there is extensive literature related to the existence and uniqueness of the periodic solutions(almost periodic solutions, or bounded solutions) of second order differential equations with or without delay. To identify a few, we refer the reader to [1-13] and the references cited there.

Zeng in [1] studied the existence of almost periodic solutions of the equation

$$x''(t) - x(t) + x^3(t) = f(t), \tag{2}$$

where f is a almost periodic function. Very recently, Wang in [2] investigated the same problem for the following equation

$$x''(t) - x(t) \pm x^m(t-r) = f(t) \tag{3}$$

by using the changing of variable and fixed point method and method of approximation. In [3], the authors gave the periodic solutions of the equation

$$x''(t) + Cx'(t) + g(t, x(t)) = e(t) \tag{4}$$

by constructing method. In [14], Zima studied the existence of positive solutions of the problem

$$x''(t) - k^2 x(t) + f(t, x(t)) = 0, \quad x(0) = \lim_{t\to+\infty} x(t) = 0.$$

To the best of our knowledge, the existence and nonexistence of the positive periodic solutions for second order nonlinear delay differential equations, especially for Duffing equations, has not been investigated till now.

In this paper, we studied the existence and nonexistence of the positive periodic solutions of the equation (1). Our aims are to establish the existence and non-existence criterion of positive $T-$periodic solutions of (1). Since (1) is non-autonomous and has variable time lag and coefficients, the methods used in the papers mentioned above are not effective. Our method in this paper is different from those mentioned above.

In this paper, we give the assumptions as follows. These assumptions will be used in the main results.

(H_1) $p, q \in C(R, R^+)$ are T-periodic functions with $\int_0^T p(s)ds > o$ and $\int_0^T q(s)ds > 0$.

(H_2) $h \in C(R, R^+)$ is T-periodic with $\int_0^T h(s)ds > 0$.

(H_3) $\tau \in C(R, R)$ is T-periodic.

(H_4) $r \in C(R, R)$ is T-periodic.

(H_5) $r(t) = 0$ for all $t \in R$.

(H_6) There is $M > 0$ such that $f(t, x) \geq -M$ for $(t, x) \in R \times R^+$.

(H_7) $\lambda > 0$ is a parameter, $T > 0$ a constant.

(H_8) $f: R \times R^+ \to R^+$ is continuous and is T-periodic about t.

(H_9) $\lim_{x\to+\infty} f(t, x)/x = N \in (0, +\infty]$ uniformly on $[0, T]$.

(H_{10}) $\lim_{x\to+\infty} f(t, x)/x = l \in [0, +\infty]$ and $\lim_{x\to 0} f(x)/x = L \in [0, +\infty]$ uniformly on [0,T].

This paper is organized as follows: in section 2, we give some useful lemmas. Our main results will be given in section 3.

Finally, for easy reference we state here the fixed point theorem [15,p.94] which is employed in this paper.

LEMMA K[15]**.** *Let X be a Banach space,and K be a cone of X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega_1} \subseteq \Omega_2$, and let $A : K \bigcap (\overline{\Omega_2} - \Omega_1) \to K$ be a completely continuous operator such that*

(1). $||Ax|| \leq ||x||$ *for every* $x \in K \bigcap \partial\Omega_1$ *and* $||Ax|| \geq ||x||$ *for every* $x \in K \bigcap \partial\Omega_2$ *or*

(2). $||Ax|| \geq ||x||$ *for every* $x \in K \bigcap \partial\Omega_1$ *and* $||Ax|| \leq ||x||$ *for every* $x \in K \bigcap \partial\Omega_2$ *Then A has at least one fixed point in $K \bigcap (\overline{\Omega_2} - \Omega_1)$.*

2 Lemmas

To begin with, we consider the following equation

$$x'(t) - a(t)x(t) = -\lambda g(x(t)) - m(t), \tag{5}$$

where a is a non-negative continuous T-periodic function with $\int_0^T a(s)ds > 0$, m a continuous positive T-periodic function, $\lambda > 0$ a parameter, $g : \ R^+ \to R^+$ a continuous function.

Lemma 1. *Suppose*

$$\lim_{x \to +\infty} \frac{g(x)}{x} = N \in (0, +\infty]. \tag{6}$$

Then (5) has at least one positive T-periodic solution if $A < B$ and $\lambda \in (A, B)$. A and B are defined as follows.

$$A = \frac{2[-\exp(-\int_0^T a(u)du) + 1]}{N\sigma T \exp(-\int_0^T a(u)du)}, \quad B = \frac{R_0[-\exp(-\int_0^T a(u)du) + 1]}{\sigma M_1 T},$$

where

$$\begin{array}{ll} R_0 = ||R(t)||, & \sigma = \exp(-\int_0^T a(u)du), \\ M_1 = \max_{x \in [0, R_0(\sigma+1)/\sigma]} h(x), & h(x) = \begin{cases} g(x), & x \in [0, +\infty), \\ g(0), & x \in (-\infty, 0), \end{cases} \\ R(t) = \int_t^{t+T} G_1(t,s)m(s)ds, & G_1(t,s) = \frac{\exp(-\int_t^s a(u)du)}{-\exp(-\int_0^T a(u)du)+1}. \end{array}$$

Proof. Suppose that $y : \ R \to [0, +\infty)$ is continuous T-periodic function and $x(t)$ is a solution of the equation

$$x'(t) = a(t)x(t) - y(t). \tag{7}$$

One gets

$$(x(t)e^{-\int_0^t a(s)ds})' = -e^{-\int_0^t a(s)ds} y(t),$$

after integration from t to $t + T$, we get

$$x(t) = \lambda \int_t^{t+T} G_1(t,s)y(s)ds,$$

where

$$G_1(t,s)=\frac{\exp(-\int_t^s a(u)du)}{-\exp(-\int_0^T a(u)du)+1}\quad for\ t\le s\le t+T. \tag{8}$$

Since

$$G_1(t,s)\le\frac{1}{-\exp(-\int_0^T a(u)du)+1}$$

and

$$G_1(t,s)\ge\frac{\exp(-\int_0^T a(u)du)}{-\exp(-\int_0^T a(u)du)+1},$$

we have

$$x(t)=\lambda\int_t^{t+T}G_1(t,s)y(s)ds\ge\frac{\lambda\exp(-\int_0^T a(u)du)}{-\exp(-\int_0^T a(u)du)+1}\int_t^{t+T}y(s)ds$$

and

$$x(t)=\lambda\int_t^{t+T}G_1(t,s)y(s)ds\le\lambda\frac{1}{-\exp(-\int_0^T a(u)du)+1}\int_t^{t+T}y(s)ds.$$

Hence

$$x(t)\ge\sigma||x||,\quad t\in R.$$

Now, let X be the set of all real continuous $T-$periodic functions endowed with the usual linear structure as well as the norm $||x||=\max_{t\in[0,T]}|x(t)|$. Then X is a Banach space.

Let

$$K=\{x\in X:\quad x(t)\ge\sigma||x||,\quad t\in[0,T]\}.$$

Then K is a cone of space X. It is easy to know that $R(t)$ satisfies

$$x'(t)=-a(t)x(t)+m(t).$$

Next, consider the following equation

$$y'(t)=a(t)y(t)-\lambda g\left(y(t)-R(t)\right). \tag{9}$$

We find that (5) has a positive T-periodic solution $x(t)$ if and only if $y(t)=x(t)-R(t)$ is a T-periodic solution of (9) and $y(t)+R(t)>0$ for all $t\in[0,T]$. Let the operator F be defined as follows.

$$Fy(t)=\lambda\int_t^{t+T}G_1(t,s)g\left(y(s)+R(s)\right)ds$$

for $y\in X$. It is easy to see that F is completely continuous and $FK\subset K$ by the periodic assumptions. Since $\lambda<B$, we choose $\alpha>1$ such that $\lambda\le\frac{B}{\alpha}$. Set $\Omega_1=\{\ x\in X:\quad ||x||<R_0/\sigma\ \}$. Then one finds $y(t)+R(t)\le||y||+||R||=R_0(\sigma+1)/\sigma$, and $y(t)+R(t)\ge\sigma||y||-||R||=0$ if $y\in K\cap\partial\Omega_1$. Thus

$$\begin{aligned}Fy(t)&\le\lambda M_1\int_t^{t+T}G_1(t,s)ds\\&\le\lambda M_1\frac{T}{-\exp(-\int_0^T a(u)du)+1}\\&<\frac{R_0}{\alpha\sigma}<||y||.\end{aligned}$$

i.e., $||Fy|| < ||y||$ for all $y \in K \cap \partial\Omega_1$. Now, choosing $\epsilon > 0$ such that

$$\frac{\lambda(N-\epsilon)\sigma T\exp(-\int_0^T a(u)du)}{2[-\exp(-\int_0^T a(u)du)+1]} \geq 1$$

by $\lambda > A$. Again, one can choose $H > R_0/\sigma > R_0$ such that

$$\frac{h(y)}{y} = \frac{g(y)}{y} > N - \epsilon \ \ for\ y \geq H.$$

Set $\Omega_2 = \{\ x \in X :\ \ ||x|| < (H+R_0)/\sigma\ \}$. If $y \in K \bigcap \partial\Omega_2$, we find

$$y(t)+R(t) \geq \sigma||y|| - R_0 \geq \sigma\frac{H+R_0}{\sigma} - R_0 = H,\ ||y|| = \frac{H+R_0}{\sigma}.$$

Then

$$\begin{aligned} Fy(t) &\geq \lambda\int_t^{t+T} G_1(t,s)(N-\epsilon)(y(s)+R(s))ds \\ &\geq \lambda(N-\epsilon)\int_t^{t+T} G_1(t,s)(\sigma||y||-R_0)ds \\ &\geq \lambda H(N-\epsilon)\int_0^T \frac{\exp(-\int_0^T a(u)du)}{-\exp(-\int_0^T a(u)du)+1}ds \\ &= \lambda(N-\epsilon)H\frac{T\exp(-\int_0^T a(u)du)}{-\exp(-\int_0^T a(u)du)+1} \\ &> \lambda(N-\epsilon)\frac{H+R_0}{2}\frac{T\exp(-\int_0^T a(u)du)}{-\exp(-\int_0^T a(u)du)+1} \\ &\geq \frac{H+R_0}{\sigma} = ||y||. \end{aligned}$$

i.e., $||Fy|| \geq ||y||$ for $y \in K \cap \partial\Omega_2$. Hence F has at least one fixed point y by applying Theorem K such that $R_0/\sigma < ||y|| \leq (H+R_0)/\sigma$ that is a T-periodic solution of (9). We claim that $||y|| > \frac{R_0}{\sigma}$, for the contradiction, there is t_0 so that $\frac{R_0}{\sigma} = y(t_0) = Fy(t_0) < \frac{R_0}{\alpha\sigma}$, which is condition. On the other hand,

$$y(t)+R(t) \geq \sigma||y|| - R_0 > \sigma\frac{R_0}{\sigma} - R_0 = 0.$$

So, $x(t) = y(t) + R(t)$ is a positive T-periodic solution of equation (5).

Remark: If $N = +\infty$, then (5) has at least one positive T-periodic solution if $0 < \lambda < B$.

Lemma 2. *Suppose that* (H_1) *holds and*

$$\frac{R_1[-\exp(-\int_0^T p(u)du)+1]}{M_1 T} > 1. \tag{10}$$

Then there are continuous T-periodic functions a and b such that $b(t) > 0,\ \int_0^T a(u)du > 0$ *and*

$$a(t)+b(t) = p(t), \quad -b'(t)+a(t)b(t) = q(t)\ for\ all\ t \in R, \tag{11}$$

where

$$R_1 = ||R_1(t)||, \quad R_1(t) = \int_t^{t+T} G(t,s)q(s)ds,$$
$$M_1 = \frac{R_1^2(\sigma+1)^2}{\sigma^2}, \quad \sigma = \exp(-\int_0^T p(s)ds).$$

Proof. From (11), one gets

$$b'(t) - p(t)b(t) = -b^2(t) - q(t). \tag{12}$$

Let $g(x) = x^2$, $a(t) = p(t)$, $m(t) = q(t)$ and $\lambda = 1$. Then $N = \lim_{x\to+\infty} g(x)/x = +\infty$. By Lemma 1, if

$$0 < \lambda = 1 < \frac{R_1[-\exp(-\int_0^T p(u)du) + 1]}{\sigma M_1 T},$$

then (12) has at least one positive continuous T-periodic solution $b(t)$. Now, since $\int_0^T q(s)ds > 0$ and $-b'(t)/b(t) + a(t) = q(t)/b(t)$, we get

$$\int_0^T a(u)du = \int_0^T \frac{q(u)}{b(u)}du > 0.$$

Remark: Condition (10) in Lemma 2 is a sharp condition. If p, q, a and b are real numbers, (11) becomes the following $a + b = p$ and $ab = q$. It is well known that the condition that guarantee a and b is $p^2 \geq 4q$. On the other hand, it is easy to know that $R_1 = \frac{q}{p}$, $M_1 = [R_1(\sigma+1)/\sigma]^2$ and $\sigma = e^{-pT}$. Thus

$$\lim_{T\to 0} \frac{R_1[-\exp(-\int_0^T p(u)du) + 1]}{M_1 T} = \lim_{T\to 0} \frac{p}{q}\frac{-e^{-pT}+1}{T}\left(\frac{e^{-pT}}{e^{-pT}+1}\right)^2 = \frac{p^2}{4q}.$$

This is $p^2 \geq 4q$.

Next, assume (H_1) and y a T-periodic function, we consider the equation

$$x''(t) - p(t)x'(t) + q(t)x(t) = y(t). \tag{13}$$

When (10) holds, $y(t) \geq 0$, by Lemma 2, there are non-negative continuous T-periodic functions a and b such that (11) holds. We transform (13) into the following equation

$$x''(t) - a(t)x'(t) - b'(t)x(t) - b(t)x'(t) + a(t)b(t)x(t) = y(t). \tag{14}$$

Then

$$(x'(t)e^{-\int_0^t a(u)du})' - (b(t)x(t)e^{-\int_0^t a(u)du})' = e^{-\int_0^t a(u)du}y(t).$$

Integrating it from t to $t+T$, we get

$$x'(t) - b(t)x(t) = -\int_t^{t+T} \frac{\exp(-\int_t^s a(u)du)}{-\exp(-\int_0^T a(u)du) + 1} y(s)ds.$$

Similarly, we get

$$x(t) = \int_t^{t+T} \frac{\exp(-\int_t^s b(u)du)}{-\exp(-\int_0^T b(u)du) + 1}\left[\int_s^{s+T} \frac{\exp(-\int_s^v a(u)du)}{-\exp(-\int_0^T a(u)du) + 1} y(v)dv\right] ds.$$

By exchanging the integrating order, one gets

$$x(t) = \int_t^{t+T} G(t,s)y(s)ds, \tag{15}$$

where $G(t,s)$ is defined as follows.

$$G(t,s) = \frac{\int_t^s \exp[-\int_t^u a(v)dv - \int_u^s b(v)dv]du + \int_s^{t+T} \exp[-\int_t^u a(v)dv - \int_u^{s+T} b(v)dv]du}{[-\exp(-\int_0^T a(v)dv) + 1][-\exp(-\int_0^T b(v)dv) + 1]}$$

for $s \in [t, t+T]$. On the other hand, From (11), we get

$$\int_0^T a(u)du + \int_0^T b(u)du = \int_0^T p(u)du, \quad \int_0^T a(u)du = \int_0^T \frac{q(u)}{b(u)}du. \tag{16}$$

Since b and q are continuous and $b(t) > 0$, choosing $0 < \frac{T}{n} < \frac{2T}{n} < \cdots < \frac{nT}{n} = T$ and $\xi_i \in (\frac{(i-1)T}{n}, \frac{iT}{n})$, one gets

$$\int_0^T b(u)du = \lim_{n\to+\infty} \sum_{i=1}^n b(\xi_i)\frac{T}{n}, \quad \int_0^T \frac{q(u)}{b(u)}du = \lim_{n\to+\infty} \sum_{i=1}^n \frac{q(\xi_i)}{b(\xi_i)}\frac{T}{n}.$$

So

$$\begin{aligned}
\int_0^T a(u)du \int_0^T b(u)du &= \int_0^T b(u)du \int_0^T \frac{q(u)}{b(u)}du \\
&= \lim_{n\to+\infty} \sum_{i=1}^n b(\xi_i)\frac{T}{n} \lim_{n\to+\infty} \sum_{i=1}^n \frac{q(\xi_i)}{b(\xi_i)}\frac{T}{n} \\
&\geq \frac{T^2}{n^2} \lim_{n\to+\infty} \left[n^2 \sqrt[n]{\prod_{i=1}^n b(\xi_i)} \sqrt[n]{\prod_{i=1}^n \frac{q(\xi_i)}{b(\xi_i)}} \right] \\
&= T^2 \lim_{n\to+\infty} \sqrt[n]{\prod_{i=1}^n q(\xi_i)} \\
&= T^2 \exp(\frac{1}{T} \int_0^T \ln q(u)du).
\end{aligned}$$

Together with (15), if

$$[\int_0^T p(u)du]^2 \geq 4T^2 \exp\left(\frac{1}{T} \int_0^T \ln q(u)du \right), \tag{17}$$

it follows that

$$\min\{\int_0^T a(u)du, \ \int_0^T b(u)du\} \geq \frac{\int_0^T p(u)du - \sqrt{[\int_0^T p(u)du]^2 - 4T^2 \exp(\frac{1}{T}\int_0^T \ln q(u)du)}}{2} := n_0$$

and

$$\max\{\int_0^T a(u)du, \ \int_0^T b(u)du\} \leq \frac{\int_0^T p(u)du + \sqrt{[\int_0^T p(u)du]^2 - 4T^2 \exp(\frac{1}{T}\int_0^T \ln q(u)du)}}{2} := m_0.$$

So. it is easy to know that

$$\begin{aligned} G(t,s) &\le \frac{\int_t^s du + \int_s^{t+T} du}{[-\exp(-\int_0^T a(v)dv)+1][-\exp(-\int_0^T b(v)dv)+1]} \\ &= \frac{T}{[-\exp(-\int_0^T a(v)dv)+1][-\exp(-\int_0^T b(v)dv)+1]} \\ &\le \frac{T}{[-e^{-n_0}+1]^2} \end{aligned}$$

and

$$\begin{aligned} & G(t,s) \\ \ge\ & \frac{\int_t^s \exp\left[-\int_t^{t+T} a(v)dv - \int_t^{t+T} b(v)dv\right] du}{\left[-\exp\left(-\int_0^T a(v)dv\right)+1\right]\left[-\exp\left(-\int_0^T b(v)dv\right)+1\right]} \\ & + \frac{\int_s^{t+T} \exp\left[-\int_t^{t+T} a(v)dv - \int_t^{t+T} b(v)dv\right] du}{\left[-\exp\left(-\int_0^T a(v)dv\right)+1\right]\left[-\exp\left(-\int_0^T b(v)dv\right)+1\right]} \\ \ge\ & \frac{T\exp(-\int_0^T p(s)ds)}{[-e^{-m_0}+1]^2}. \end{aligned}$$

Then we get the following.

Lemma 3. *Suppose that y is a T-periodic function with $y(t) \ge 0$ for all t, (H_1), (10) and (17) hold. Then the solution of (13) satisfies*

$$x(t) \ge \mu ||x||\ for\ all\ t \in R,$$

where

$$\mu = \frac{(-e^{-n_0}+1)^2}{\exp(\int_0^T p(v)dv)[-e^{-m_0}+1]^2}.$$

Proof. Let $x(t)$ be a T-periodic solution of (13). We get

$$x(t) = \int_t^{t+T} G(t,s)y(s)ds \le \frac{T}{(-e^{-n_0}+1)^2}\int_t^{t+T} y(s)ds$$

and

$$x(t) \ge \frac{T\exp(-\int_0^T a(u)du)}{(-e^{-m_0}+1)^2}\int_t^{t+T} y(s)ds.$$

So $x(t) \ge \mu||x||$ for every $t \in R$. The proof is complete.

Remark: If p and q are real numbers, (17) becomes $p^2 \ge 4q$.

3 Main results and proofs

Theorem 1. *Suppose that $(H_1)-(H_4)$, $(H_7)-(H_9)$, (10) and (17) hold. Then equation (1) has at least one positive $T-$periodic solution if $a < B$ and $\lambda \in (A,B)$. A and B are defined as follows.*

$$A = \frac{2[-e^{-n_0}+1]^2}{N\mu T\exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}$$

and

$$B = \frac{R_0(-e^{-n_0}+1)^2}{\mu M_1 T \int_0^T h(s)ds},$$

where $G(t,s)$ and μ are defined in Lemma 3 and m_0 and n_0 in section 2,

$$\begin{array}{ll} R_0 = ||R(t)||, & R(t) = \int_t^{t+T} G(t,s)r(s)ds, \\ M_1 = \max\limits_{(t,x)\in[0,T]\times[0,R_0(\mu+1)/\mu]} g(t,x), & g(t,x) = \begin{cases} f(t,x), & (t,x)\in[0,T]\times[0,+\infty), \\ f(t,0), & (t,x)\in[0,T]\times(-\infty,0). \end{cases} \end{array}$$

Proof. It is easy to know that $R(t)$ satisfies

$$x''(t) - p(t)x'(t) + q(t)x(t) = r(t).$$

Now, consider the following

$$y''(t) - p(t)y'(t) + q(t)y(t) = \lambda h(t) f(t, y(t-\tau(t)) + R(t-\tau(t))). \tag{18}$$

We find that (1) has a positive T-periodic solution $x(t)$ if and only if $y(t) = x(t) - R(t)$ is a T-periodic solution of (17) and $y(t) + R(t) > 0$ for all $t \in [0,T]$. Let the operator F be defined as follows.

$$Fy(t) = \lambda \int_t^{t+T} G(t,s)h(s)g(s, y(s-\tau(s)) + R(s-\tau(s)))ds$$

for $y \in X$. It is easy to see that F is completely continuous and $FK \subset K$ by the periodic assumptions. Set $\Omega_1 = \{ x \in X : \ ||x|| < R_0/\mu \}$. Then one finds $y(t) + R(t) \leq ||y|| + ||R|| = R_0(\mu+1)/\mu$, if $y \in K \cap \partial\Omega_1$, and $y(t) + R(t) \geq \sigma||y|| - ||R|| = 0$. Choose $\alpha > 1$ so that $\lambda < \frac{B}{\mu}$. Thus

$$\begin{aligned} Fy(t) &\leq \lambda M_1 \int_t^{t+T} G(t,s)h(s)ds \\ &\leq \lambda M_1 \frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2} \\ &< \frac{R_0}{\alpha\mu} < ||y||. \end{aligned}$$

i.e., $||Fy|| < ||y||$ for all $y \in K \cap \partial\Omega_1$. Now, choosing $\epsilon > 0$ such that

$$\frac{\lambda(N-\epsilon)\mu T \exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}{2[-e^{-m_0}+1]^2} \geq 1$$

by $\lambda > A$. Again, by (H_9), one can choose $H > R_0/\mu > R_0$ such that

$$\frac{g(t,y)}{y} = \frac{f(t,y)}{y} > N - \epsilon \ \ \textit{for} \ \ t \in [0,T] \ \textit{and} \ y \geq H.$$

Set $\Omega_2 = \{ x \in X : \ ||x|| < (H+R_0)/\mu \}$. We find

$$y(t) + R(t) \geq \mu||y|| - R_0 \geq \mu\frac{H+R_0}{\mu} - R_0 = H.$$

Then

$$\begin{aligned}
Fy(t) &\geq \lambda \int_t^{t+T} G(t,s)h(s)(N-\epsilon)\left(y(s-\tau(s)) + R(s-\tau(s))\right) ds \\
&\geq \lambda(N-\epsilon)\int_t^{t+T} G(t,s)h(s)(\mu||y|| - R_0)ds \\
&\geq \lambda H(N-\epsilon)\int_0^T \frac{T\exp(-\int_0^T p(s)ds)}{(-e^{-m_0}+1)^2} h(s)ds \\
&= \lambda(N-\epsilon)H\frac{T\exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}{(-e^{-m_0}+1)^2} \\
&> \lambda(N-\epsilon)\frac{H+R_0}{2}\frac{T\exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}{(-e^{-m_0}+1)^2} \\
&\geq \frac{H+R_0}{\mu} = ||y||.
\end{aligned}$$

i.e., $||Fy|| \geq ||y||$ for $y \in K \cap \partial\Omega_2$. Hence F has at least one fixed point y such that $R_0/\sigma < ||y|| \leq (H+R_0)/\mu$ that is a T-periodic solution of (17). We claim that $||y|| > \frac{R_0}{\mu}$. For the contradiction, there is t_0 so that $\frac{R_0}{\mu} = y(t_0) = Ay(t_0) < \frac{R_0}{\alpha\mu}$, which is contradiction. On the other hand,

$$y(t) + R(t) \geq \sigma||y|| - R_0 > \mu\frac{R_0}{\mu} - R_0 = 0.$$

So, $x(t) = y(t) + R(t)$ is a positive T-periodic solution of system (1).

Remark: When $N = +\infty$ in (9), then (1) has positive T-periodic solutions if $(H_1) - (H_4)$, (H_7), (H_8), (10) and (16) hold and $\lambda \in (0, B)$.

Theorem 2. *Assume that $(H_1) - (H_7)$, (H_9), (10) and (16) hold. Then (1) has at least one positive T-periodic solution if $A < B$ and $\lambda \in (A, B)$. A and B are defined as follows.*

$$A = \frac{2(-e^{-m_0}+1)^2}{\mu N T\exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}$$

and

$$B = \min\{\frac{(-e^{-n_0}+1)^2}{M_1 T\int_0^T h(s)ds}, \ \frac{\mu(-e^{-n_0}+1)^2}{MT\int_0^T h(s)ds}\},$$

where $G(t,s)$ and μ are defined in Lemma 3 and m_0 and n_0 in section 2,

$$M_1 = \max_{(t,x)\in[0,T]\times[0,1]} g(t,x), \quad g(t,x) = \begin{cases} f(t,x) + M, & (t,x) \in [0,T]\times[0,+\infty), \\ f(t,0) + M, & (t,x) \in [0,T]\times(-\infty,0). \end{cases}$$

Proof. Set

$$w(t) = \int_t^{t+T} G(t,s)h(s)ds \quad t \in R, \quad z(t) = \lambda M w(t).$$

Then (1) has positive T-periodic solution $x(t)$ if and only if $x + z := y$ is a T-periodic solution of the following equation

$$y''(t) - p(t)y'(t) + q(t)y(t) = \lambda h(t)g(t, y(t-\tau(t)) - z(t-\tau(t))), \tag{19}$$

and $y(t) > z(t)$ for all $t \in R$. Define an operator F as follows.

$$Fy(t) = \lambda \int_t^{t+T} G(t,s)h(s)g(s, y(s-\tau(s)) - z(s-\tau(s)))ds$$

for $y \in X$. We know that F is completely continuous and $FK \subset K$. Let λ be fixed and $A < \lambda < B$. Choosing $\Omega_1 = \{\ x \in X :\ \ ||x|| < 1\}$. One has $y(t-\tau(t)) - z(t-\tau(t))) \leq y(t-\tau(t)) \leq ||y|| = 1$ for $y \in K \cap \partial\Omega_1$. Then

$$\begin{aligned} Fy(t) &\leq \lambda M_1 \int_t^{t+T} G(t,s)h(s)ds \\ &\leq \lambda M_1 \frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2} \\ &\leq 1 = ||y|| \ (\textbf{using } \lambda < \mathbf{B}), \end{aligned}$$

i.e., $||Fy|| \leq ||y||$ for all $y \in K \cap \partial\Omega_1$. Now, choosing $\epsilon > 0$ such that

$$\frac{\lambda(N-\epsilon)\mu T \exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}{2(-e^{-m_0}+1)^2} \geq 1.$$

Let $\overline{R} > 1$ such that

$$\mu - \frac{\lambda M}{\overline{R}} \frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2} \geq \frac{\mu}{2} \tag{20}$$

and

$$\frac{g(t,x)}{x} = \frac{f(t,x)}{x} \geq N - \epsilon \ for \ (t,x) \in [0,T] \times [\frac{\mu\overline{R}}{2}, +\infty).$$

Set $\Omega_2 = \{\ x \in X :\ \ ||x|| < \overline{R}\ \}$. If $y \in K \cap \partial\Omega_2$, we find

$$\begin{aligned} y(t) - z(t) &\geq \mu||y|| - \lambda M w(t) \\ &\geq \mu\overline{R} - \lambda M \frac{T\int_0^T h(u)du}{(-e^{-n_0}+1)^2} \\ &\geq \frac{\overline{R}\mu}{2} \ \ (\textbf{using (20)}). \end{aligned}$$

Hence

$$\begin{aligned} Fy(t) &\geq \lambda(N-\epsilon)\int_t^{t+T} G(t,s)h(s)(y(s-\tau(s)) - z(s-\tau(s)))ds \\ &\geq \frac{\lambda\mu\overline{R}(N-\epsilon)}{2}\int_t^{t+T} G(t,s)h(s)ds \\ &\geq \frac{\lambda\mu\overline{R}(N-\epsilon)}{2}\frac{T\exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}{(-e^{-m_0}+1)^2} \\ &\geq \overline{R} = ||y||. \end{aligned}$$

i.e., $||Fy|| \geq ||y||$ for $y \in K \cap \partial\Omega_2$. Hence F has at least one fixed point y such that $1 \leq ||y|| \leq \overline{R}$. On the other hand,

$$y(t) \geq \mu||y|| \geq \mu > \frac{\lambda M T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2}$$

and

$$w(t) \leq \frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2} \text{ }(\textbf{using } \lambda < \mathbf{B})$$

implies

$$y(t) > \lambda M w(t) = z(t) \ \ for \ \ all \, t \in R.$$

So $x(t) = y(t) - z(t)$ is a positive T-periodic solution of (1).

Remark: When $N = +\infty$ in (H_9), (1) has at least one positive T-periodic solution if $(H_1) - (H_7)$, (10) and (16) hold and $\lambda \in (0, B)$.

Theorem 3. *Assume that $(H_1) - (H_5)$, $(H_7), (H_8), (H_{10})$, (10) and (16) hold. Then (1) has at least one positive $T-$periodic solution if $A < B$ and $\lambda \in (A, B)$. A and B are defined as follows*

$$A = \frac{(-e^{-m_0}+1)^2}{\mu l T \exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}$$

and

$$B = \frac{(-e^{-n_0}+1)^2}{LT\int_0^T h(s)ds}.$$

Proof. Define an operator F as follows.

$$Fy(t) = \lambda \int_t^{t+T} G(t,s)h(s)f(s, y(s-\tau(s)))ds$$

for $y \in X$. We see that A is completely continuous and $FK \subset K$. Choosing $\epsilon > 0$ and $R_2 > 0$ such that

$$\lambda(L+\epsilon)\frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2} \leq 1$$

and $f(t,x)/x \leq L + \epsilon$ for $(t,x) \in [0,T] \times [R_2, +\infty)$. Set $\Omega_1 = \{\ x \in X : \ ||x|| < R_2/\mu\ \}$. If $y \in K \cap \partial\Omega_1$, then $y(t) \geq \mu||y|| = R_2$ and

$$\begin{aligned}
Fy(t) &= \lambda \int_t^{t+T} G(t,s)h(s)f(s, y(s-\tau(s)))ds \\
&\leq \lambda(L+\epsilon)\int_t^{t+T} g(t,s)h(s)y(s-\tau(s))ds \\
&\leq \lambda(L+\epsilon)||y||\int_t^{t+T} G(t,s)h(s)ds \\
&\leq \lambda(L+\epsilon)||y||\frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2} \\
&\leq ||y||.
\end{aligned}$$

i.e., $||Fy|| \leq ||y||$ for all $y \in K \cap \partial\Omega_2$. Now, choosing $\epsilon > 0$ such that

$$\frac{\lambda\mu(l-\epsilon)T\exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}{(-e^{-m_0}+1)^2} \geq 1.$$

Choosing $0 < R_1 < R_2$ such that $f(t,x)/x > l - \epsilon$ for $(t,x) \in [0,T] \times (0, R_1]$. Set $\Omega_1 = \{ \ x \in X : \ \ ||x|| < R_1 \ \}$. For $y \in K \cap \partial\Omega_1$, we find

$$\begin{aligned} Fy(t) &\geq \lambda \int_t^{t+T} G(t,s)h(s)y(s-\tau(s))(l-\epsilon)ds \\ &\geq \lambda(l-\epsilon) \int_t^{t+T} G(t,s)h(s)\mu||y||ds \\ &= \lambda(l-\epsilon)\mu||y|| \int_t^{t+T} G(t,s)h(s)ds \\ &\geq \lambda(l-\epsilon)\mu||y|| \frac{T\exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}{(-e^{-m_0}+1)^2} \\ &\geq ||y||. \end{aligned}$$

i.e., $||Fy|| \geq ||y||$ for $y \in K \cap \partial\Omega_1$. Hence F has at least one fixed point y such that $R_1 \leq ||y|| \leq R_2$ that is a positive periodic solution of (1).

Theorem 4. *Assume that $(H_1)-(H_5)$, $(H_7),(H_8),(H_{10})$, (10) and (16) hold. Then (1) has at least one positive $T-$periodic solution if $A < B$ and $\lambda \in (A,B)$. A and B are defined as follows.*

$$A = \frac{(-e^{-m_0}+1)^2}{\mu T\exp(-\int_0^T p(s)ds)L\int_0^T h(s)ds}$$

and

$$B = \frac{(-e^{-n_0}+1)^2}{lT\int_0^T h(s)ds}.$$

The proof is similar to that of Theorem 3 and then omitted.

Remark: For the case where f is sub-linear(i.e., $l = +\infty$ and $L = 0$) or super-linear(i.e., $l = 0$ and $L = +\infty$), Theorem 3 or 4 is effective respectively.

Theorem 5. *If $(H_1)-(H_3)$, (H_5), (H_7) and (H_8), (10) and (16) hold and*

$$\lambda \frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2} f(t,x) < x \tag{21}$$

for $(t,x) \in [0,T] \times (0,+\infty)$, then (1) has no positive T-periodic solution.

Proof. Assume to the contrary that $y(t)$ is a positive T-periodic solution of (1). There is $t_0 \in [0,T]$ such that $||y|| = y(t_0)$. Thus we have

$$\begin{aligned} ||y|| = y(t_0) &= \lambda \int_{t_0}^{t_0+T} G(t_0,s)h(s)f(s,y(s-\tau(s)))ds \\ &< \left[\frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2}\right]^{-1} \times \int_{t_0}^{t_0+T} G(t_0,s)h(s)y(s-\tau(s))ds \\ &\leq \left[\frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2}\right]^{-1} \int_{t_0}^{t_0+T} G(t_0,s)h(s)ds||y|| \\ &\leq \left[\frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2}\right]^{-1} \left[\frac{T\int_0^T h(s)ds}{(-e^{-n_0}+1)^2}\right] ||y|| \\ &= ||y|| \end{aligned}$$

which is a contradiction.

Theorem 6. *If $(H_1)-(H_3)$ and $(H_5),(H_7),(H_8)$, (10) and (16) hold and*

$$\lambda\frac{\mu T\exp(-\int_0^T p(s)ds)\int_0^T h(s)ds}{(-e^{-m_0}+1)^2}f(t,x) > x \tag{22}$$

for $(t,x)\in[0,T]\times(0,+\infty)$, then (1) has no positive T-periodic solution.
The proof is similar to that of Theorem 5.

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Department of Mathematics,
Hunan Institute of Technology,
Yueyang,414000, P.R.China

Department of Applied Mathematics,
Beijing Institute of Technology,
Beijing 100081, P.R.China