Spectral geometry of non-local topological algebras

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0. Introduction

The classical problem of existence of non-local function algebras was settled in the affirmative by *Eva Kallin* in the early sixties by her well-known example [17], (see also [6, p. 170] and [22, p. 83, Example]. A few years later *R. G. Blumenthal* [3, 4] remarked that *Kallin's* example was simply a particular case of a type of algebras studied by *S. J. Sidney* in his dissertation (see [21]). The previous results were obtained within the standard context of *Banach function algebra theory.*

On the other hand, working within the general framework of Topological Algebras, *not necessarily normed ones* (we refer to *A. Mallios* [18] for the relevant terminology), we have already considered in [12] *the spectrum of Sidney's algebra.* More precisely, we looked at it, as a "*gluing space*" of the spectra of two factor tensor product algebras, whose sum constituted, by definition, the *algebra of Sidney.* In point of fact, it was *Blumenthal* (loc. cit.), who actually defined the spectrum of *Sidney's* algebra, as a gluing space, his result being thus subsumed into ours [12, Theorem 5.2]. Now, continuing herewith our previous work in [12], we further obtain a general *existence theorem for non-local topological algebras* (à la *Blumenthal*; see Theorem 3.2). Furthermore, based on a recent article of *R. D. Mehta* [19], still within the Banach function algebra theory, we consider the *Choquet boundary* of the (generalized) algebra of *Sidney* (cf. Theorem 4.1 in the sequel). Indeed, by changing the hypotheses, appropriately, we are able to have the same boundary in a more concrete form, than that one in [12]. Yet, following in the preceding general set-up *A. Mallios* [18] (see Lemma 4.1 below), we also obtain the *Šilov boundary* of the same *Sidney's algebra*, as above.

Received by the editors September 2002.
Communicated by E. Colebunders.

1. Preliminaries

In all that follows by a *topological algebra* E we mean a topological $\mathbb{C}$-algebra with non-empty *spectrum* $\mathfrak{M}(E)$, endowed with the *Gel'fand topology.* The *Gel'fand map* of E, given by

$$\begin{aligned}\mathcal{G} : E \to \mathcal{C}_c(\mathfrak{M}(E)) : x \mapsto \mathcal{G}(\hat{x}) \equiv \hat{x} : \mathfrak{M}(E) \to \mathbb{C}\\ : f \mapsto \hat{x}(f) := f(x),\end{aligned}$$

defines, through its image, the so called *Gel'fand transform algebra* of E, denoted by $E\hat{}$ and topologized as a *locally m-convex algebra*, when $\mathcal{C}_c(\mathfrak{M}(E))$ carries the topology "c" of compact convergence [18, p. 19, Example 3.1]. By a *spectral algebra*, we mean a topological algebra E satisfying

$$Sp_E(x) = \hat{x}(\mathfrak{M}(E)), \quad x \in E, \tag{1.1}$$

where $Sp_E(x)$ stands for the spectrum of $x \in E$. *This* spectral property *characterizes advertible completeness in a unital locally m-convex algebra* [ibid. p. 104, Corollary 6.4]. Besides E is said to be *hereditarily Weierstrass*, when every $|\hat{x}|$, $x \in E$, attains its supremum at a point of $\mathfrak{M}(E)$, while this property is retained by every closed subset of $\mathfrak{M}(E)$ (take e.g. *a normal Weierstrass space or a countably compact space* [8, p. 11, 1.4]). Now, given a 2-sided ideal I of E, the set

$$h_E(I) = \{f \in \mathfrak{M}(E) : ker(f) \supseteq I\},$$

being, apparently, a closed subset of $\mathfrak{M}(E)$, is called the *hull* of I in $\mathfrak{M}(E)$, realizing the spectrum of E/I, as well as, that one of $E/\overline{I}$. That is, one has

$$\mathfrak{M}(E/I) \underset{\text{homeo}}{\cong} h_E(I) = h_E(\overline{I}) \underset{\text{homeo}}{\cong} \mathfrak{M}(E/\overline{I}). \tag{1.2}$$

(See also [18, p. 330, Definition 1.1, p. 335, Lemma 3.1, p. 339, Theorem 4.1 and p. 347, (4.41)]). Furthermore, one obtains [9, p. 314, Theorem 2.1]

$$\mathfrak{M}(I) \underset{\text{homeo}}{\cong} (h_E(I))^c = (h_E(\overline{I}))^c \underset{\text{homeo}}{\cong} \mathfrak{M}(\overline{I}). \tag{1.3}$$

For a given $B \subseteq \mathfrak{M}(E)$, the *geometric* (or even *E-convex*) *hull of* B is defined by the relation,

$$\begin{aligned}(B)_E \equiv E - hull(B) = \{f \in \mathfrak{M}(E) : |f(x)| \le p_B(\hat{x})\\ \equiv \sup\nolimits_{f\in B} |\hat{x}(f)|,\ x \in E\},\end{aligned} \tag{1.4}$$

viz. one has a closed subset of $\mathfrak{M}(E)$, such that

$$(B)_E = (\overline{B})_E\ . \tag{1.5}$$

The same notion is *inclusion-preserving*, while if $(B)_E = B$, then B is called *E-convex.* We note that *every hull and every zero set of an* $\hat{x}$, $x \in E$, *are* E*-convex* [15]. Considering the *restriction algebra* $E\hat{}|_B$, endowed with the relative topology from $\mathcal{C}_c(B)$, *the continuity of the Gel'fand map of* E *implies the following homeomorphism into* (cf. [9, p. 283, Theorem 1.2])

$$\mathfrak{M}(E\hat{}|_B) \underset{\to\text{homeo}}{\subseteq} (B)_E, \tag{1.6}$$

where the indicated map is given by

$$\theta \equiv {}^t(r \circ \mathcal{G}),$$

with $r : E\hat{} \to E\hat{}|_B$, *the restriction map.* Since $(im\theta)_E = (B)_E$, we say that $\mathfrak{M}(E\hat{}|_B)$ *is "E-convex"* (viz. $\theta(\mathfrak{M}(E\hat{}|_B))$ is so) *iff* θ *is onto.* The surjectivity of θ is also attained, when B is compact, or when $E\hat{}|_B$ is a Q'*-algebra*, in the sense that *every maximal regular 2-sided ideal is closed* (see, for instance, [16] for the terminology applied). The significance, for several applications, of this type of topological algebras, in place of "Q" ones, has been already pointed out by *A. Mallios* in [18, p. 73, Scholium 7.1]. This sort of a topological algebra has also lately named by *M. Abel* [1] a "*Mallios algebra*".

Now, the *Šilov boundary of* E, denoted by $\partial(E)$, is the least *boundary set of* E; that is, the smallest closed subset of $\mathfrak{M}(E)$ on which the Gel'fand transform of every $x \in E$ attains its maximum absolute value [18, p. 189, Definition 2.2]. On the other hand, the *Choquet boundary* $Ch(E)$ *of* E, is that set of continuous characters of E which are represented only by the respective *Dirac measures* [11, (4.1)]. In this regard, we note that *the compactness of* $\mathfrak{M}(E)$ *of a unital topological algebra* E *ensures both the existence of* $\partial(E)$*, as well as, the density of* $Ch(E)$ *in* $\partial(E)$. Given a *continuous algebra morphism* ϕ *between the topological algebras* E *and* F, we know that the corresponding *transpose map* ${}^t\phi : \mathfrak{M}(F) \to \mathfrak{M}(E)$ *preserves the Choquet boundaries*, that is, one has

$$ {}^t\phi(Ch(F)) \subseteq Ch(E), \tag{1.7}$$

while if ${}^t\phi$ *is 1-1 and proper, then one has* ([10, p. 124, Lemma 5.1] and/or [13, Theorem 3.4]),

$$ {}^t\phi(Ch(F)) = Ch(E) \cap {}^t\phi(\mathfrak{M}(F)). \tag{1.8}$$

Yet, it is a standard result that, given two topological algebras E and F, the corresponding tensor product algebra $E \otimes F$, endowed with a "*compatible*" topology τ, has spectrum (see [18, p. 409, Theorem 1.1])

$$\mathfrak{M}(E \underset{\tau}{\otimes} F) \underset{\text{homeo}}{\cong} \mathfrak{M}(E) \times \mathfrak{M}(F). \tag{1.9}$$

Now, by looking at the Šilov boundary of a topological tensor product algebra $E \underset{\tau}{\otimes} F$, this exists, when $\partial(E)$ and $\partial(F)$ exist, and is realized by

$$\partial(E \underset{\tau}{\otimes} F) \underset{\text{homeo}}{\cong} \partial(E) \times \partial(F). \tag{1.10}$$

(ibid., p. 436, Theorem 1.1). Yet, concerning the Choquet boundary of $E \underset{\tau}{\otimes} F$, based on certain characterizations of the Choquet points, given in [14, Theorem 5.1], one obtains,

$$Ch(E \underset{\tau}{\otimes} F) \underset{\text{homeo}}{\cong} Ch(E) \times Ch(F), \tag{1.11}$$

when the spectra $\mathfrak{M}(E)$, $\mathfrak{M}(F)$ are Q*-spaces* and the Gel'fand transform algebras $E\hat{}$, $F\hat{}$, $(E \otimes F)\hat{} = E\hat{} \otimes F\hat{}$ are σ*-complete* (cf. also [9, p. 378, Theorem 2.1]). In this connection, we recall that a completely regular space X is a Q*-space* (*"Hewitt space"*, cf. [23, p. 206, (Q4)]), if every character of the algebra $\mathcal{C}_c(X)$ is continuous. See also [20, pp. 140, 142].

2. Gluing spectra together

Given two "*intersected*" topological algebras E and F (they share an ideal, cf. Lemma 2.1 below), with spectra $\mathfrak{M}(E)$, $\mathfrak{M}(F)$, respectively, related by a continuous injection, one computes the bigger spectrum by attaching the smaller one to the spectrum of a suitable quotient algebra, along the intersection of the latter two spectra; this, in turn, has a special bearing on the *intersection of the algebras* involved (Theorem 2.1, below). This specializes to *Blumenthal's Theorem* [3, p. (2.2), Theorem 2.2] and/or [4, p. 343, Theorem 1.1], formulated for Banach function algebras (see also [22, p. 97, Theorem 9.15]). Here we also applied [12, p. 2629, Theorem 3.1] an *elementary proof, in comparison with that of Blumenthal*, namely, *not depending on the local maximum modulus principle.*

For convenience, we recall the concept of "*gluing two sets together, along their intersection.*"

Definition 2.1. Given two sets X , Y with non-empty intersection $X \cap Y$, *gluing* Y *to* X *along their intersection* $X \cap Y$ (via the natural embedding of $X \cap Y$ in Y), means the quotient space

$$X \cup Y/\sim \, \equiv X \bigcup_{X\cap Y} Y,$$

where the equivalence relation "$\sim$" is defined as follows: Denoting by $\pi : X \cup Y \to X \cup Y/\sim$ the quotient map and $j : X \cap Y \to Y$ the natural embedding, we set

$$\pi(x) := \{x\}, \;\; if \;\; x \in (X \setminus Y) \cup (Y \setminus X),$$

and

$$\pi(x) := X \cap Y, \;\; if \;\; x \in X \cap Y.$$

Clearly, the binary relation π *is the equivalence relation generated by the graph of* j. Besides this equivalence relation renders $X \cup Y/\sim$ a disjoint union of X and Y, since

$$\begin{aligned} X \bigcup_{X\cap Y} Y &= \pi(X \cup Y) \cong X \cup (Y \setminus X) \\ &= X \oplus (Y \setminus X) \cong X \oplus Y. \end{aligned} \tag{2.1}$$

So, if X , Y are topological spaces, in view of (2.1), $X \bigcup_{X\cap Y} Y$ is endowed with the *topology of a disjoint union (: topological sum).*

The following are needed for the main conclusions of this paper, as contained in Sections 3, 4 below. Those proofs of the results stated herewith, that are not already derived from the relevant discussion, can be found in our previous publication in [12]; however, for convenience of reference, we also give here the corresponding statements.

Lemma 2.1.. *Let* E , F *be topological algebras with spectra* $\mathfrak{M}(E)$, $\mathfrak{M}(F)$, *respectively, such that*

$$i : \mathfrak{M}(E) \underset{\to}{\subseteq}_{i} \mathfrak{M}(F),$$

and I *a common ideal of* E , F. *Then, one has*

$$h_E(I) = (h_E(I))_F \cap \mathfrak{M}(E) = h_F(I) \cap \mathfrak{M}(E). \tag{2.2}$$

In particular, if I is a 2-sided ideal, such that on I the relative topology from F is smaller than that of E, that is, the natural injection

$$(I\,,\tau_E)\underset{\rightarrow}{\subsetneq}(I\,,\tau_F)$$

is continuous, then one gets

$$\mathfrak{M}(F)\setminus\mathfrak{M}(E)\subseteq h_F(I),$$

up to a natural embedding (: continuous injection). ∎

Theorem 2.1.. *Let $E\,,F$ be topological algebras, such that*

$$i:\mathfrak{M}(E)\underset{\rightarrow cont.}{\subset}\mathfrak{M}(F).$$

Moreover, let I be a common 2-sided ideal of E and F, having the following (canonical) injection continuous

$$(I\,,\tau_E)\underset{\rightarrow}{\subsetneq}(I\,,\tau_F).$$

Then, one has

$$\mathfrak{M}(F)=\mathfrak{M}(E)\bigcup_{\mathfrak{M}(E/I)}\mathfrak{M}(F/I), \tag{2.3}$$

up to a continuous bijection. This becomes a homeomorphism, when the second member in (2.3) *is compact.* ∎

In this regard, we still obtain that,

$$\mathfrak{M}(E)\cap\mathfrak{M}(F/I)\underset{\text{homeo}}{\cong}\mathfrak{M}(E/I)\equiv B$$

(see [12, (3.23) in Theorem 3.1]).

The preceding Theorem 2.1 supplies a generalization of the particular case that $E=I$, since then, due to (1.2) and (1.3), we have,

$$\mathfrak{M}(F)=h_F(I)\cup(h_F(I))^c\underset{\text{homeo}}{\cong}\mathfrak{M}(F/I)\vee\mathfrak{M}(I).$$

On the other hand, by considering the topological algebra $F\hat{}|_B$, instead of F/I we have, under the continuity of the Gel'fand map of F, that (cf. also (1.4), (1.6)),

$$\begin{aligned}\mathfrak{M}(E/I)\underset{\text{homeo}}{\cong}h_E(I)\equiv B\underset{\rightarrow}{\subsetneq}\mathfrak{M}(F\hat{}|_B)\underset{\rightarrow\text{ homeo}}{\subsetneq}(B)_F\\ \subseteq(B)_F^I=h_F(I)\underset{\text{homeo}}{\cong}\mathfrak{M}(F/I)\subseteq\mathfrak{M}(F).\end{aligned} \tag{2.4}$$

Thus, in view of (2.2), as well, one further obtains,

$$\begin{aligned}B&=\mathfrak{M}(E)\cap\mathfrak{M}(F\hat{}|_B)\underset{\text{homeo}}{\cong}\mathfrak{M}(E)\cap(B)_F\\&=\mathfrak{M}(E)\cap(B)_F^I=\mathfrak{M}(E)\cap h_F(I),\end{aligned} \tag{2.5}$$

while, by (2.5), we still have

$$\begin{aligned}\mathfrak{M}(F/I)\underset{\text{homeo}}{\cong}h_F(I)&=(\mathfrak{M}(E))^c\cup B=(\mathfrak{M}(E))^c\cup\mathfrak{M}(F\hat{}|_B)\\&=(\mathfrak{M}(E))^c\cup(B)_F.\end{aligned}$$

Therefore, by *assuming*

$$\mathfrak{M}(F/I) = \mathfrak{M}(F\hat{}|_B) \text{ or, } \textit{equivalently, } (\mathfrak{M}(E))^c \subseteq \mathfrak{M}(F\hat{}|_B), \tag{2.6}$$

and based on the proof of Theorem 2.1, the space $\mathfrak{M}(F)$ *is obtained by gluing* $\mathfrak{M}(F\hat{}|_B)$ *to* $\mathfrak{M}(E)$ *along their intersection* $B \equiv h_E(I) \underset{\text{homeo}}{\cong} \mathfrak{M}(E/I)$, via the natural embedding (cf. (2.4)) of $B \equiv h_E(I)$ in $\mathfrak{M}(F\hat{}|_B)$. Thus, as an immediate consequence of the preceding, one gets the next result, specializing to *Blumenthal's Theorem*, that initially was given for Banach function algebras in [3, p. (2.2)], [4, p. 343].

Theorem 2.2.. *Let $E\,, F$ be topological algebras with F having a continuous Gel'fand map, such that*

$$i : \mathfrak{M}(E) \underset{\to cont.}{\subseteq} \mathfrak{M}(F).$$

Moreover, let I be a common 2-sided ideal of E and F, with

$$(I\,, \tau_E) \underset{\to}{\subseteq} (I\,, \tau_F)$$

continuous and

$$B \equiv h_E(I),$$

such that

$$\mathfrak{M}(F/I) = \mathfrak{M}(F\hat{}|_B). \tag{2.7}$$

Then, one has

$$\mathfrak{M}(F) = \mathfrak{M}(E) \bigcup_B \mathfrak{M}(F\hat{}|_B), \tag{2.8}$$

up to a continuous bijection, which becomes a homeomorphism, when the second member of (2.8) *is compact.* ■

Referring to the *rel.* (2.7), this *is attained when* $\mathfrak{M}(F\hat{}|_B)$ *is F-convex and*

$$p_B(\hat{x}) \equiv p_{h_E(I)}(\hat{x}) = p_{h_F(I)}(\hat{x}), \;\; x \in F,$$

since then (see also comments after (1.5))

$$\mathfrak{M}(F\hat{}|_B) \cong (B)_F \equiv (h_E(I))_F = (h_F(I))_F = h_F(I) \cong \mathfrak{M}(F/I).$$

Another *condition guaranteeing* (2.7) *is $h_E(I)$ to be a boundary set for* $\mathcal{C}(h_F(I))$; for then

$$h_F(I) = (h_E(I))_{\mathcal{C}(h_F(I))} \subseteq (h_E(I))_F \subseteq h_F(I).$$

On the other hand, the equivalent relation to (2.7), as given by (2.6), is accomplished by employing, within our extended framework, the *local maximum modulus principle*. Concerning this principle for topological (non-normed) algebras, see, for instance, [2, p. 67, Corollary 4.3], [18, p. 258, Lemma 1.3] and [9, p. 328, Corollary 3.1].

As an application of Theorems 2.1 and 2.2, we get now at the corresponding *Theorem of Sidney for topological tensor product algebras*:

Thus, suppose we are given two topological algebras E and F, a subalgebra A of E and a 2-sided ideal J of F. Hence, $E \otimes J$ and $A \otimes F$ are a 2-sided ideal and a subalgebra of $E \otimes F$, respectively. Furthermore, consider the topological algebra

$$\mathfrak{A}_0 = E \underset{\tau}{\otimes} J + A \underset{\tau}{\otimes} F \subseteq E \underset{\tau}{\otimes} F, \tag{2.9}$$

which also has $E \otimes J$ as a 2-sided ideal. Now, we note that (cf. [9, Lemma 3.1], along with (1.2) in the preceding),

$$\begin{aligned} \mathfrak{M}(E \underset{\tau}{\otimes} F / E \otimes J) \underset{\text{homeo}}{\cong} h_{E \underset{\tau}{\otimes} F}(E \otimes J) &= \mathfrak{M}(E) \times h_F(J) \\ &= \mathfrak{M}(E) \times \mathfrak{M}(F/J) \end{aligned}$$

and that, if A is separating in E, the same holds true for $A \underset{\tau}{\otimes} F \subseteq \mathfrak{A}_0$ in $E \underset{\tau}{\otimes} F$, accordingly for $\mathfrak{A}_0$, as well, that is,

$$\mathfrak{M}(E \underset{\tau}{\otimes} F) \underset{\to \text{cont.}}{\subseteq} \mathfrak{M}(\mathfrak{A}_0).$$

Thus, by further applying Theorem 2.1 to the algebras $E \underset{\tau}{\otimes} F$ and $\mathfrak{A}_0$, we immediately have the next.

Theorem 2.3.. *Let E , F be unital topological algebras, A a separating subalgebra of E, containing the constants and J a 2-sided ideal of F. Moreover, let $E \underset{\tau}{\otimes} F$ be the corresponding unital tensor product algebra, endowed with a compatible topology τ, and $\mathfrak{A}_0$ the subalgebra of it given by* (2.9)*. Then, one has*

$$\mathfrak{M}(\mathfrak{A}_0) = (\mathfrak{M}(E) \times \mathfrak{M}(F)) \bigcup_{\mathfrak{M}(E \underset{\tau}{\otimes} F / E \underset{\tau}{\otimes} J)} \mathfrak{M}(\mathfrak{A}_0 / E \underset{\tau}{\otimes} J), \tag{2.10}$$

up to a continuous bijection, which becomes a homeomorphism when the second member in (2.10) *is compact.* ∎

Yet, as an application of Theorem 2.2, we also obtain the next general version of *Sidney's Theorem*, as above, employing *an entirely simple proof*; cf., instead, [21, p. 135, Theorem 3.3], or even [3, p. (2.5), Corollary 2.4]. Thus, one has.

Theorem 2.4.. *Assuming the context of Theorem 2.3, suppose that $\mathfrak{A}_0$ has continuous Gel'fand map $\mathcal{G}_{\mathfrak{A}_0}$ and*

$$\mathfrak{M}(\mathfrak{A}_0 / E \underset{\tau}{\otimes} J) = \mathfrak{M}(\mathfrak{A}_0\hat{}|_{\mathfrak{M}(E) \times h_F(J)}).$$

Then, one gets

$$\mathfrak{M}(\mathfrak{A}_0) = (\mathfrak{M}(E) \times \mathfrak{M}(F)) \bigcup_{\mathfrak{M}(E) \times \mathfrak{M}(F/J)} (\mathfrak{M}(A\hat{}|_{\mathfrak{M}(E)}) \times \mathfrak{M}(F/J)), \tag{2.11}$$

up to a continuous bijection, which becomes a homeomorphism, under the compactness of the second member in (2.11)*. In particular, if $\mathfrak{M}(A\hat{}|_{\mathfrak{M}(E)})$ is A-convex, then it is homeomorphic to $(\mathfrak{M}(E))_A$, while it is further identified with $\mathfrak{M}(A)$, iff*

$$\sup_{f \in \mathfrak{M}(E)} |f(x)| = \sup_{f \in \mathfrak{M}(A)} |f(x)|, \;\; x \in A, \tag{2.12}$$

so that, one then obtains

$$\mathfrak{M}(\mathfrak{A}_0) = (\mathfrak{M}(E) \times \mathfrak{M}(F)) \bigcup_{\mathfrak{M}(E) \times \mathfrak{M}(F/J)} (\mathfrak{M}(A) \times \mathfrak{M}(F/J)). \tag{2.13}$$

∎

In the previous Theorem 2.4 the condition (2.12) is attained under suitable conditions, either for the spectra of the algebras involved, or for the algebras themselves. Thus, (2.12) *is fulfilled when*

$$\overline{\mathfrak{M}(E)} = \mathfrak{M}(A),$$

or in the case E is a *commutative advertibly complete locally m-convex algebra*, with the subalgebra A *being* also *advertibly complete in the relative topology*; indeed, their spectral radii then coincide, viz. one has

$$\sup_{f\in\mathfrak{M}(E)} |f(x)| = r_E(x) = r_A(x) = \sup_{f\in\mathfrak{M}(A)} |f(x)|, \;\; x \in A$$

(cf. [18, p. 99, Theorem 6.1, p. 104, Corollary 6.5]).

Furthermore, since

$$\mathfrak{M}(A) \times \mathfrak{M}(F/J) \subseteq \mathfrak{M}(A) \times \mathfrak{M}(F) \underset{\text{homeo}}{\cong} \mathfrak{M}(A \underset{\tau}{\otimes} F),$$

and, by hypothesis for A,

$$\mathfrak{M}(E) \times \mathfrak{M}(F) \cong \mathfrak{M}(E \underset{\tau}{\otimes} F) \underset{\to\text{cont.}}{\subseteq} \mathfrak{M}(A \underset{\tau}{\otimes} F),$$

we obtain from (2.13) that

$$\mathfrak{M}(\mathfrak{A}_0) \underset{\to\text{cont.}}{\subseteq} \mathfrak{M}(A \underset{\tau}{\otimes} F), \tag{2.14}$$

where $\mathfrak{M}(\mathfrak{A}_0)$ is only continuously identified with the second member of (2.13).

3. Non-local topological algebras

In this section we show that *the algebras constructed in Sidney's Theorem* [21, p. 135, Theorem 3.3] *belong to the special class of non-local (topological) algebras.* In this regard, given a *topological algebra* E, we say that a function $h \in \mathcal{C}_c(\mathfrak{M}(E))$ is *locally in* $E\hat{}$, if there exists an open covering $\{U_i : i \in I\}$ of $\mathfrak{M}(E)$, such that $h|_{U_i} \in E\hat{}|_{U_i}$, $i \in I$; *equivalently*, for every $f \in \mathfrak{M}(E)$, there exists an open neighbourhood U of f in $\mathfrak{M}(E)$, such that $h|_U \in E\hat{}|_U$. Thus, E is said to be a *non-local algebra*, if there exists a function h in $\mathcal{C}(\mathfrak{M}(E))$, which *locally belongs to* $E\hat{}$*, but not globally to it*, viz. $h \notin E\hat{}$. (See also e.g. [18, p. 348, Theorem 5.1].

Now, given a topological algebra E, assume that $\mathfrak{M}(E)$ has a partition defined by two closed subsets X, Y. Furthermore, considering the algebra $\mathfrak{A}_0$, as given by (2.9), suppose that there is $\hat{z} \in F\hat{}\backslash\overline{J}\hat{}$, such that

$$h_F(J) \subseteq ker(\hat{z}). \tag{3.1}$$

Thus, setting

$$V = \mathfrak{M}(\mathfrak{A}_0)\backslash(Y \times \mathfrak{M}(F)) = (X \times \mathfrak{M}(F)) \cup (\mathfrak{M}(A)\backslash\mathfrak{M}(E)) \times h_F(J),$$
$$W = \mathfrak{M}(\mathfrak{A}_0)\backslash(X \times \mathfrak{M}(F) = (Y \times \mathfrak{M}(F)) \cup (\mathfrak{M}(A)\backslash\mathfrak{M}(E)) \times h_F(J),$$

when $\mathfrak{M}(E \underset{\tau}{\otimes} F)$ is closed in $\mathfrak{M}(\mathfrak{A}_0)$, we get an open covering of $\mathfrak{M}(\mathfrak{A}_0)$, through V, W, with

$$V \cap W = (\mathfrak{M}(A)\backslash\mathfrak{M}(E)) \times h_F(J). \tag{3.2}$$

We now define the function

$$\phi = \begin{cases} 0 & \text{on } V \\ \hat{z} & \text{on } W, \end{cases} \tag{3.3}$$

which, due to (3.2) and (3.1), is well-defined and continuous on $\mathfrak{M}(\mathfrak{A}_0)$, i.e. $\phi \in \mathcal{C}_c(\mathfrak{M}(\mathfrak{A}_0))$. Besides, since $\phi|_V = 0$ and $\phi|_W = \hat{z} = 1_{A\hat{}} \otimes \hat{z} \in A\hat{}\otimes F\hat{} \subseteq \mathfrak{A}_0\hat{}$, ϕ is locally in $\mathfrak{A}_0\hat{}$. Here we *assume that A is unital*; this is actually guaranteed by Theorem 3.1 below (based on *Šilov's Idempotent Theorem*), the unit element of A being, in fact, independent of that of E. We show that $\phi \notin \mathfrak{A}_0\hat{}$, through a characterization of the algebra A, based on a previous Banach function algebra result of *R. G. Blumenthal* [3, Lemma 3.2].

The following byproduct of the preceding, although not needed for the sequel, it does have, however, an interest p e r c e, which thus permits its inclusion herein. Namely, one has.

Lemma 3.1.. *Let E, F be topological algebras and $\phi : E \to F$ an onto continuous algebra morphism, with ${}^t\phi : \mathfrak{M}(F) \to \mathfrak{M}(E)$ the respective transpose map. Moreover, consider $B \subseteq \mathfrak{M}(F)$ with ${}^t\phi(B)$ a boundary set for E. Then, B is a boundary set for F, as well.*

Proof. By the hypothesis for ${}^t\phi(B)$, for every $x \in E$ there exists $g_0 \in B$, such that

$$\|\hat{x}\|_{\mathfrak{M}(E)} \equiv \sup_{g \in \mathfrak{M}(E)} |\hat{x}(g)| = |\hat{x}({}^t\phi(g_0))| = |\widehat{\phi(x)}(g_0)|. \tag{3.4}$$

Now, for every $y \in F$, there exists, due to surjectivity of ϕ, $x \in E$, such that $y = \phi(x)$. Thus (cf. also (3.4))

$$\begin{aligned} \|\hat{y}\|_{\mathfrak{M}(F)} &= \|\widehat{\phi(x)}\|_{\mathfrak{M}(F)} = \|\hat{x} \circ {}^t\phi\|_{\mathfrak{M}(F)} = \|\hat{x}\|_{{}^t\phi(\mathfrak{M}(F))} \\ &\leq \|\hat{x}\|_{\mathfrak{M}(E)} = |\widehat{\phi(x)}(g_0)| = |\hat{y}(g_0)|, \end{aligned}$$

for some $g_0 \in B$, which proves the assertion. ■

Theorem 3.1.. *Let E, F be unital topological algebras, A a commutative semi-simple complete locally m-convex separating subalgebra of E and J a 2-sided ideal of F satisfying* (3.1) *for some $\hat{z} \in F\hat{}\setminus\overline{J\hat{}}$. Moreover, let $E \otimes F$ be the corresponding unital tensor product algebra endowed with a compatible topology τ, and $\mathfrak{A}_0$ the subalgebra of it given by* (2.9)*, with continuous Gel'fand map $\mathcal{G}_{\mathfrak{A}_0}$, such that $\mathfrak{M}(E \underset{\tau}{\otimes} F)$ is closed in $\mathfrak{M}(\mathfrak{A}_0)$ and*

$$\mathfrak{M}(\mathfrak{A}_0/E \underset{\tau}{\otimes} J) = \mathfrak{M}(\mathfrak{A}_0\hat{}|_{\mathfrak{M}(E)\times h_F(J)}).$$

Finally, assume that $A\hat{}|_{\mathfrak{M}(E)}$ is closed in $\mathcal{C}_c(\mathfrak{M}(E))$, with spectrum A-convex, such that

$$\sup_{f \in \mathfrak{M}(E)} |f(x)| = \sup_{f \in \mathfrak{M}(A)} |f(x)|, \ x \in A,$$

while $\mathfrak{M}(E)$ is a k-space, having a partition by two closed subsets X, Y. Then, the function ϕ, defined by (3.3)*, is not in $\mathfrak{A}_0\hat{}$, if, and only if, either one of the two following relations holds true, viz.*

$$(X)_A \cup (Y)_A \neq \mathfrak{M}(A), \tag{3.5}$$

or

$$(X)_A \cap (Y)_A \neq \emptyset. \tag{3.6}$$

Proof. If both (3.5) and (3.6) are not valid, by the *Šilov Idempotent Theorem* (cf. [5] and/or [7]), there exists $x \in A$, such that

$$\hat{x}^2 = \hat{x}, \quad (\hat{x})^{-1}(0) = (X)_A, \quad (\hat{x})^{-1}(1) = (Y)_A\,.$$

Then, for $\hat{z} \in F\hat{}\setminus\overline{J\hat{}}$ satisfying (3.1), we have

$$\widehat{x \otimes z} = \begin{cases} 0 & \text{on } (X)_A \times \mathfrak{M}(F) \\ \hat{z} & \text{on } (Y)_A \times \mathfrak{M}(F), \end{cases}$$

so that $\phi = \widehat{x \otimes z}|_{\mathfrak{M}(\mathfrak{A}_0)} \in (A \underset{\tau}{\otimes} F)\hat{}|_{\mathfrak{M}(\mathfrak{A}_0)} \subseteq \mathfrak{A}_0\hat{}$. Hence, either (3.5) or (3.6) is a necessary condition for ϕ not to be in $\mathfrak{A}_0\hat{}$.

Assuming now, that either (3.5) or (3.6) holds true, if we find a measure $\mu \in \mathcal{M}_c(\mathfrak{M}(E \underset{\tau}{\otimes} F)) \cong (\mathcal{C}_c(\mathfrak{M}(E \underset{\tau}{\otimes} F)))'$ (cf. [18, p. 474, Lemma 2.1]), such that

$$\mu \in (\mathfrak{A}_0\hat{})^{\perp} \quad \textit{and} \quad \mu(\phi) \neq 0, \tag{3.7}$$

then, we have, of course, that $\phi \notin \mathfrak{A}_0\hat{}$. Since $\hat{z} \in F\hat{}\setminus\overline{J\hat{}}$, there is (Hahn-Banach) $\nu \in \mathcal{M}_c(\mathfrak{M}(F))$, with

$$\nu \in (\overline{J\hat{}})^{\perp} \quad \textit{and} \quad \nu(\hat{z}\,) \neq 0. \tag{3.8}$$

Now, applying Šilov's Idempotent Theorem to $\mathcal{C}_c(\mathfrak{M}(E))$, one finds $h \in \mathcal{C}_c(\mathfrak{M}(E))$, such that $h^2 = h$, $h^{-1}(0) = X$ and $h^{-1}(1) = Y$. We show that $h \notin A\hat{}|_{\mathfrak{M}(E)} \subseteq E\hat{}$; if $h \in A\hat{}|_{\mathfrak{M}(E)}$, then $h = \hat{\alpha}_0|_{\mathfrak{M}(E)}$, for some $\alpha_0 \in A$, with $\hat{\alpha}_0^2 - \hat{\alpha}_0 = 0\,|_{\mathfrak{M}(E)}$, and since $\mathfrak{M}(A\hat{}|_{\mathfrak{M}(E)}) \cong (\mathfrak{M}(E))_A = \mathfrak{M}(A)$ one obtains (Šilov Idempotent Theorem for $A\hat{}|_{\mathfrak{M}(E)}$)

$$\hat{\alpha}_0^{-1}(0) \cup \hat{\alpha}_0^{-1}(1) = \mathfrak{M}(A). \tag{3.9}$$

By the A-convexity of $\hat{\alpha}_0^{-1}(0)$ and the continuity of $\mathcal{G}_A$, implied by that one of $\mathcal{G}_{\mathfrak{A}_0}$, we get $\mathfrak{M}(A\hat{}|_{\hat{\alpha}_0^{-1}(0)}) \cong \hat{\alpha}_0^{-1}(0)$, and thus

$$(X)_{A\hat{}|_{\hat{\alpha}_0^{-1}(0)}} \subseteq (X)_A \subseteq \hat{\alpha}_0^{-1}(0), \tag{3.10}$$

while $\hat{\alpha}_0^{-1}(0) \cap \mathfrak{M}(E) = X$. Now, since $\alpha_0 \in A$ is idempotent, we have [7]

$$A = \alpha_0 A + (1 - \alpha_0)A,$$

so that if $f \in \hat{\alpha}_0^{-1}(0)$, then, for every $A \ni \alpha = \alpha_0 x + (1 - \alpha_0)y$, we get

$$\begin{aligned} |\hat{\alpha}(f)| = |\hat{y}(f)| &\leq \|\hat{\alpha}\|_{\hat{\alpha}_0^{-1}(0)} = \|(1 - \hat{\alpha}_0)\hat{y}\|_{\hat{\alpha}_0^{-1}(0)} \\ &\leq \|(1 - \hat{\alpha}_0)\hat{y}\|_{\mathfrak{M}(A)} = \|(1 - \hat{\alpha}_0)\hat{y}\|_{\mathfrak{M}(E)} = \|\hat{y}\|_X = \|\hat{\alpha}\|_X, \end{aligned}$$

hence, $f \in (X)_{A\hat{}|_{\hat{\alpha}_0^{-1}(0)}}$. Thus, due to (3.10), one gets

$$(X)_A = \hat{\alpha}_0^{-1}(0),$$

and similarly

$$(Y)_A = \hat{\alpha}_0^{-1}(1),$$

which along with (3.9) contradicts both (3.5) and (3.6). Therefore, $h \notin A\hat{}|_{\mathfrak{M}(E)}$, implying the existence of a measure $\lambda \in \mathcal{M}_c(\mathfrak{M}(E))$, such that

$$\lambda \in (A\hat{}|_{\mathfrak{M}(E)})^{\perp} \quad and \quad \lambda(h) \equiv \int_Y d\lambda \neq 0. \tag{3.11}$$

Finally, by taking the measure $\mu \equiv \lambda \times \nu \in \mathcal{M}_c(\mathfrak{M}(E \underset{\tau}{\otimes} F))$, we have by (3.8) and (3.11) that $\mu(\phi) \neq 0$ and $\mu \in ((E \underset{\tau}{\otimes} J)\hat{})^{\perp}$, $((A \underset{\tau}{\otimes} F)\hat{})^{\perp}$, which implies (3.7), hence, $\phi \notin \mathfrak{A}_0\hat{}$. ■

An immediate consequence of the preceding Theorem 3.1 is now the following result, providing *a method of constructing non-local topological algebras.*

Theorem 3.2.. *Suppose we are given the context of Theorem 3.1 and assume that either one of the following two relations holds true, viz.*

$$(X)_A \cup (Y)_A \neq \mathfrak{M}(A),$$

or

$$(X)_A \cap (Y)_A \neq \emptyset.$$

Then, $\mathfrak{A}_0$ *is a non-local topological algebra.* ■

4. Boundaries of $\mathfrak{A}_0$

We compute in this section certain standard boundaries of $\mathfrak{A}_0$ (cf. (2.9)). In this connection, we have already given in [12] the Šilov, Bishop and Choquet boundaries using the technique of gluing topological spaces together. Here, based on the relation (2.11), we obtain, by a different approach, a *more concrete* form of *Choquet and Šilov boundaries* of $\mathfrak{A}_0$, generalizing also a previous Banach function algebra result of *R. D. Mehta* [19, Theorem 3.1]; the latter was formulated for a certain particular function algebra on a product space. Thus, one has the following.

Theorem 4.1.. *Let* E, F *be unital topological algebras with spectra* $\mathfrak{M}(E)$, $\mathfrak{M}(F)$ *Q-spaces and Gel'fand transform algebras* $E\hat{}$, $F\hat{}$ σ*-complete,* A *a separating subalgebra containing the constants, with spectrum* $\mathfrak{M}(A)$ *a Q-space,* $A\hat{}$ σ*-complete and* J *a 2-sided ideal of* F*. Moreover, let* $E \underset{\tau}{\otimes} F$ *be the corresponding unital tensor product algebra endowed with a compatible topology* τ*, having* $(E \underset{\tau}{\otimes} F)\hat{} = E\hat{} \otimes F\hat{}$ σ*-complete and* $\mathfrak{A}_0$ *the subalgebra of* $E \underset{\tau}{\otimes} F$*, defined by* (2.9)*, with continuous Gel'fand map* $\mathcal{G}_{\mathfrak{A}_0}$*, such that*

$$\mathfrak{M}(\mathfrak{A}_0/E \underset{\tau}{\otimes} J) = \mathfrak{M}(\mathfrak{A}_0\hat{}|_{\mathfrak{M}(E)\times h_F(J)}).$$

Finally, let $\mathfrak{M}(A\hat{}|_{\mathfrak{M}(E)})$ *be A-convex with*

$$\sup_{f\in\mathfrak{M}(E)} |f(x)| = \sup_{f\in\mathfrak{M}(A)} |f(x)|, \quad x \in A.$$

Then, one has

$$Ch(\mathfrak{A}_0) \underset{\leftarrow}{\supseteq} \Big(Ch(E) \times (Ch(F) \backslash \mathfrak{M}(F/J))\Big) \vee \Big(Ch(A) \times \mathfrak{M}(F/J)\Big), \tag{4.1}$$

up to a continuous injection. In particular, if $\mathfrak{M}(E)\,,\mathfrak{M}(J)$ *and the second member in* (4.1) *are compact* (equivalently each factor is compact)*, we obtain the homeomorphisms*

$$Ch(\mathfrak{A}_0) = (Ch(E) \times Ch(J)) \vee (Ch(A) \times \mathfrak{M}(F/J)), \tag{4.2}$$

and

$$\partial(\mathfrak{A}_0) = (\partial(E) \times \partial(J)) \vee (\partial(A) \times \mathfrak{M}(F/J)). \tag{4.3}$$

Proof. By hypothesis for A, we have (cf. also (1.9) and (2.14))

$$\begin{aligned} &\mathfrak{M}(E) \times \mathfrak{M}(F) \cong \mathfrak{M}(E \underset{\tau}{\otimes} F) \underset{\rightarrow}{\subseteq}_{\text{cont.}} \mathfrak{M}(\mathfrak{A}_0) \\ &\underset{\rightarrow}{\subseteq}_{\text{cont.}} \mathfrak{M}(A \underset{\tau}{\otimes} F) \cong \mathfrak{M}(A) \times \mathfrak{M}(F), \end{aligned}$$

so that, by virtue of (1.11), we get

$$Ch(E) \times Ch(F) \underset{\rightarrow}{\subseteq}_{\text{cont.}} Ch(\mathfrak{A}_0) \underset{\rightarrow}{\subseteq}_{\text{cont.}} Ch(A) \times Ch(F). \tag{4.4}$$

Now, since $Ch(F)\backslash\mathfrak{M}(F/J) \subseteq Ch(F)$, one gets, by (4.4),

$$Ch(E) \times \Big(Ch(F)\backslash\mathfrak{M}(F/J)\Big) \subseteq Ch(E) \times Ch(F) \subseteq Ch(\mathfrak{A}_0). \tag{4.5}$$

On the other hand, by taking $h_0 = f_0 \otimes g_0 \in Ch(A) \times h_F(J) \subseteq \mathfrak{M}(A) \times h_F(J) \subseteq \mathfrak{M}(\mathfrak{A}_0) \underset{\rightarrow}{\subseteq} \mathfrak{M}(A) \times \mathfrak{M}(F)$, we show that $h_0 \in Ch(\mathfrak{A}_0)$. So, if W is a neighbourhood of h_0 in $\mathfrak{M}(\mathfrak{A}_0)$, there exist open neighbourhoods U of f_0 in $\mathfrak{M}(A)$ and V of g_0 in $h_F(J)$, with $h_0 = f_0 \otimes g_0 \in U \otimes V \subseteq W$. Since $f_0 \in Ch(A)$, given U, as before, there exists, equivalently [14, Theorem 5.1, 4)], $x \in A$ such that

$$f_0 \in M_{\hat{x}} \subseteq U,$$

where

$$M_{\hat{x}} = \{f \in \mathfrak{M}(A) : |f(x)| = \sup_{h \in \mathfrak{M}(A)} |h(x)|\}.$$

Thus, $x \otimes 1_F \in \mathfrak{A}_0$, with $|h_0(x \otimes 1_F)| = |f_0(x)|$, while

$$\begin{aligned} M_{\widehat{x\otimes 1_F}} = \{h &= (f,g) \in \mathfrak{M}(\mathfrak{A}_0) : |h(x \otimes 1_F)| = |f(x)| \\ &= \sup_{h' \in \mathfrak{M}(\mathfrak{A}_0)} |h'(x \otimes 1_F)| = \sup_{f' \in \mathfrak{M}(A)} |f'(x)|\} = M_{\hat{x}}\,. \end{aligned}$$

Hence, $h_0 \in M_{\widehat{x\otimes 1_F}} = M_{\hat{x}} \subseteq U = U \times \{g_0\} \subseteq U \times V \subseteq W$, implying that $h_0 \in Ch(\mathfrak{A}_0)$, so that

$$Ch(A) \times h_F(J) \subseteq Ch(\mathfrak{A}_0). \tag{4.6}$$

Now, given that (4.5) and (4.6) have empty intersection, we get (4.1) up to a continuous injection. On the other hand, assuming the compactness of $\mathfrak{M}(E)\,,\mathfrak{M}(J)$ and due to (1.8), we have

$$Ch(E) = Ch(A) \cap \mathfrak{M}(E),$$

as well as (cf. also (1.2), (1.3)),

$$Ch(J) = Ch(F) \cap \mathfrak{M}(J) = Ch(F)\backslash\mathfrak{M}(F/J).$$

Furthermore, assuming the compactness of the second member in (4.1), then (4.2) would have been proved, up to a homeomorphism, have we shown

$$\Big((Ch(E)\times(Ch(F)\backslash h_F(J)))\cup(Ch(A)\times h_F(J))\Big)^c\subseteq Ch(\mathfrak{A}_0)^c\,. \tag{4.7}$$

Hence, one gets (4.3), as well, by taking closures to both sides of (4.2).

Now, since the first member in (4.7) is equal to the union of the following sets,

$$\begin{aligned}
K&\equiv(\mathfrak{M}(E)\cap Ch(A)^c)\times(Ch(F)\cup h_F(J))^c,\\
L&\equiv(\mathfrak{M}(E)\cap Ch(A)^c)\times\Big((Ch(F)^c\cup h_F(J))\cap h_F(J)\Big),\\
M&\equiv Ch(E)\times(Ch(F)\cup h_F(J))^c,\\
N&\equiv(\mathfrak{M}(E)\cap Ch(A)^c)\times(Ch(F)\cap h_F(J)^c),\\
S&\equiv(\mathfrak{M}(E)\cup Ch(A)^c)\times\Big((Ch(F)^c\cup h_F(J))\cap h_F(J)\Big),
\end{aligned}$$

we prove e.g. that $N\subseteq Ch(\mathfrak{A}_0)^c$, that, in turn, implies (4.7). So, given $(f,g)\in N$ and a representing measure $\mu_{(f,g)}$ of it, that is $\mu_{(f,g)}\in\mathcal{M}_c^+(\mathfrak{M}(\mathfrak{A}_0))$ (:positive measures) with $\mu_{(f,g)}=\delta_{(f,g)}|_{\mathfrak{A}_0{}^\wedge}$, let $\mu_{(f,g)}=\delta_{(f,g)}$. Then, $\mu_{(f,g)}\circ{}^ti\in\mathcal{M}_c^+(\mathfrak{M}(A\underset{\tau}{\otimes}F))$, where ${}^ti:\mathcal{C}(\mathfrak{M}(A\underset{\tau}{\otimes}F))\to\mathcal{C}(\mathfrak{M}(\mathfrak{A}_0))$, such that $(\mu_{(f,g)}\circ{}^ti)(h)=\mu_{(f,g)}(h\circ i)=\delta_{(f,g)}(h\circ i)=h(i(f,g))=\delta_{i(f,g)}(h)$, for every $h\in\mathcal{C}(\mathfrak{M}(A\underset{\tau}{\otimes}F))$. Hence, $i(f,g)\equiv(f,g)\in Ch(A\underset{\tau}{\otimes}F)\cong Ch(A)\times Ch(F)$, a contradiction, since $f\notin Ch(A)$. Thus $\mu_{(f,g)}\neq\delta_{(f,g)}$, implying that $(f,g)\notin Ch(\mathfrak{A}_0)$, that is the assertion. ■

Scholium 4.1.- In the case J is a *primary ideal of* F, that is, J is closed with hull $h_F(J)$ consisting exactly of one element f_0, then the quotient algebra F/J is a *primary* or *local algebra*, in the sense that $\mathfrak{M}(F/J)\cong h_F(J)$ consists of just one point (cf. [18, p. 351, Definition 6.3]). In fact, the converse is true, as well (*A. Mallios*); so one has the following characterization

> *a closed ideal I of a topological algebra E is primary iff the quotient algebra E/I is local (: primary).*

Thus, in the setting of Theorems 2.4 and 4.1, we obtain, up to homeomorphisms, the relations

$$\mathfrak{M}(\mathfrak{A}_0)=(\mathfrak{M}(E)\times\mathfrak{M}(F))\bigcup_{\mathfrak{M}(E)\times\{f_0\}}(\mathfrak{M}(A)\times\{f_0\}),$$

$$Ch(\mathfrak{A}_0)=\Big(Ch(E)\times(Ch(F)\backslash\{f_0\})\Big)\vee\Big(Ch(A)\times\{f_0\}\Big),$$

and

$$\partial(\mathfrak{A}_0)=\Big(\partial(E)\times\partial(F)\Big)\vee\Big(\partial(A)\times\{f_0\}\Big).$$

On the other hand, one can compute the Šilov boundary of $\mathfrak{A}_0$, independently of its Choquet boundary, based on a relevant result of *A. Mallios* [18, p. 195, Theorem 2.2], establishing a connection between the Šilov boundaries of two appropriate topological algebras, related by a continuous algebra morphism. Indeed, we use here a more general form of this result, given in [9, p. 38, Lemma 4.2]. That is, we have.

Lemma 4.1.. *Let $E\,,F$ be unital spectral algebras* (cf. (1.1)) *with $\mathfrak{M}(E)$ hereditarily Weierstrass, and $\phi : E \to F$ a continuous algebra morphism, preserving the spectral radii; that is,*

$$r_E(x) = r_F(\phi(x)), \quad x \in E.$$

Then, one has

$$\partial(E) \subseteq \overline{{}^t\phi(\partial(F))}, \tag{4.8}$$

with ${}^t\phi : \mathfrak{M}(F) \to \mathfrak{M}(E)$ the transpose map of ϕ. In particular, if E has continuous Gel'fand map, then $\partial(E)$ is characterized by the property of being the largest closed subset of $\mathfrak{M}(E)$ satisfying (4.8)*, for any given triad $(E\,,\phi\,,F)$.* ■

On the basis of the preceding discussion, we obtain now the next generalization of a result of *Sidney* for Banach function algebras (see [21, p. 136, Proposition 3.5]).

Theorem 4.2.. *Let $E\,,F$ be unital topological algebras, A a separating subalgebra of E containing the constants and J a 2-sided ideal of F. Moreover, let $E \underset{\tau}{\otimes} F$ be the corresponding unital topological tensor product algebra and $\mathfrak{A}_0$ the subalgebra of it defined by* (2.9)*, with continuous Gel'fand map $\mathcal{G}_{\mathfrak{A}_0}$, such that*

$$\mathfrak{M}(\mathfrak{A}_0 / E \underset{\tau}{\otimes} F) = \mathfrak{M}(\mathfrak{A}_0\hat{}\,|_{\mathfrak{M}(E)\times h_F(J)}),$$

and

$$\sup_{f\in\mathfrak{M}(E)} |f(x)| = \sup_{f\in\mathfrak{M}(A)} |f(x)|, \quad x \in A.$$

Finally, let $A \underset{\tau}{\otimes} F$, $E \underset{\tau}{\otimes} J$, $\mathfrak{A}_0$, $E \underset{\tau}{\otimes} F$ be spectral algebras, with spectra hereditarily Weierstrass spaces but $\mathfrak{M}(E \underset{\tau}{\otimes} F)$, such that

$$\begin{aligned} r_{A \underset{\tau}{\otimes} F}(\omega) &= r_{\mathfrak{A}_0}(\omega), \quad \omega \in A \underset{\tau}{\otimes} F \subseteq \mathfrak{A}_0, \\ r_{\mathfrak{A}_0}(z) &= r_{E \underset{\tau}{\otimes} F}(z), \quad z \in \mathfrak{A}_0 \subseteq E \underset{\tau}{\otimes} F, \\ r_{E \underset{\tau}{\otimes} J}(s) &= r_{\mathfrak{A}_0}(s), \quad s \in E \underset{\tau}{\otimes} J \subseteq \mathfrak{A}_0. \end{aligned} \tag{4.9}$$

Then, one has

$$\Big(\partial(E) \times \partial(J)\Big) \cup \Big(\partial(A) \times \partial(F)\Big) \subseteq \partial(\mathfrak{A}_0) \subseteq \partial(E) \times \partial(F). \tag{4.10}$$

Proof. The hypothesis for A implies the injectivity of the continuous maps

$$\mathfrak{M}(E \underset{\tau}{\otimes} F) \underset{\to}{\subseteq}_{\eta} \mathfrak{M}(\mathfrak{A}_0) \underset{\to}{\subseteq}_{\kappa} \mathfrak{M}(A \underset{\tau}{\otimes} F), \tag{4.11}$$

as well as, of the following one

$$\mathfrak{M}(\mathfrak{A}_0) \underset{\to}{\subseteq}_{\lambda} \mathfrak{M}(E \underset{\tau}{\otimes} J),$$

since

$$(i \times j) \circ \lambda = \kappa,$$

with $i \times j : \mathfrak{M}(E) \times \mathfrak{M}(J) \underset{\to}{\subseteq} \mathfrak{M}(A) \times \mathfrak{M}(F)$. Thus, by Lemma 4.1 and (1.10), we obtain the relations

$$\begin{aligned} \partial(A) \times \partial(F) &\underset{\text{homeo}}{\cong} \partial(A \underset{\tau}{\otimes} F) \subseteq \overline{\kappa(\partial(\mathfrak{A}_0))} \underset{\text{homeo}}{\cong} \partial(\mathfrak{A}_0), \\ \partial(E) \times \partial(J) &\underset{\text{homeo}}{\cong} \partial(E \underset{\tau}{\otimes} J) \subseteq \overline{\lambda(\partial(\mathfrak{A}_0))} \underset{\text{homeo}}{\cong} \partial(\mathfrak{A}_0), \\ \partial(\mathfrak{A}_0) &\subseteq \overline{\eta(\partial(E \underset{\tau}{\otimes} F))} \underset{\text{homeo}}{\cong} \partial(E \underset{\tau}{\otimes} F) \underset{\text{homeo}}{\cong} \partial(E) \times \partial(F), \end{aligned}$$

providing the desired relation (4.10). ■

Now, given two subsets $A\,,B$ of a topological space X, one obtains that

$$A \subseteq \overline{A\backslash B} \Leftrightarrow \mathrm{int}_A(B) = \emptyset,$$

having equality when A is closed. Thus, we get the next.

Lemma 4.2.. *Let E be a topological algebra and I an ideal of E. Then, the two following assertions are equivalent:*
1) $\partial(E) = \overline{\partial(E)\backslash h_E(I)} = \partial(E)\backslash h_E(I)$.
2) $int_{\partial(E)}(h_E(I)) = \emptyset$.
In particular, if I is 2-sided, then 1) is equivalent to
3) $\partial(E) \underset{homeo}{\cong} \partial(I)$. ■

A combination of Theorem 4.2 with Lemma 4.2 provides now the next extension of [21, p. 136, Corollary 3.6].

Theorem 4.3.. *Considering the context of Theorem 4.2, assume that*

$$int_{\partial(F)}(h_F(J)) = \emptyset.$$

Then, one has

$$\partial(\mathfrak{A}_0) = \partial(E) \times \partial(F) = \partial(E) \times \partial(J),$$

up to homeomorphisms.

Proof. By virtue of (4.11) and

$$r_{E \underset{\tau}{\otimes} F}(\omega) = r_{A \underset{\tau}{\otimes} F}(\omega), \quad \omega \in A \underset{\tau}{\otimes} F,$$

resulting from the first two equalities in (4.9), we get (Lemma 4.1)

$$\partial(A) \times \partial(F) \underset{\text{homeo}}{\cong} \partial(A \underset{\tau}{\otimes} F) \subseteq \partial(E \underset{\tau}{\otimes} F) \underset{\text{homeo}}{\cong} \partial(E) \times \partial(F),$$

so that, in view also of Lemma 4.2, (4.10) implies already the assertion. ■

Acknowledgements. I wish to express my sincere thanks to the State Scholarships Foundation of Greece (IKY) for partial financial support of the present research work. My thanks are also due to Prof. *A. Mallios* for his continuing interest in my work, and for several helpful and penetrating discussions on the subject matter of this paper. He also proposed the self-explanatory and still suggestive term, "*spectral geometry*", pertaining thus to "*geometric*" (: structural) properties of the *spectrum* of a given topological algebra.

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