Positive solutions of three-point boundary value problems for n-th order differential equations

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Abstract

In this paper, we establish the existence and non-existence results of positive solutions for the (n-1,1) three-point boundary value problems consisting of the equation

$$u^{(n)} + \lambda a(t) f(u(t)) = 0, \quad t \in (0,1)$$

and one of the following boundary value conditions

$$u(1) = \beta u(\eta), \ \ u^{(i)}(0) = 0 \ \text{ for } \ i = 1, 2, \cdots, n-1$$

and

$$u^{(n-1)}(1) = \beta u^{(n-1)}(\eta), \ u^{(i)}(0) = 0 \ \text{ for } \ i = 0, 1, \cdot, n-2,$$

where $\eta \in [0,1)$, $\beta \in [0,1)$ and $a: \ (0,1) \to R$ may change sign. $f(0) > 0$, $\lambda > 0$ is a parameter. Our approach is based on the Leray-Schauder fixed point Theorem. This paper is motivated by Eloe and Henderson [6].

1 Introduction

Three-point boundary value problems for the differential equations were presented by Il'in and Moiseev [10,11]. Motivated by the study of Il'in and Moiseev, in recent

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years, Gupta in [1,2] and Ma in [3,4,5] studied certain three-point boundary value problems for nonlinear second order ordinary differential equations. On the other hand, the solvability of boundary value problems for higher order ordinary differential equations has been discussed extensively in the literature in the past ten years, we refer to the monograph [8] and the recent paper [6]. To the best of our knowledge, existence and nonexistence theorems of positive solutions for three-point boundary value problem of higher order ordinary differential equations, however, have not been found in the known literature especially when the coefficient changes sign.

In this paper, we study the existence of positive solutions of the following (n-1,1) three-point boundary value problem consisting of the differential equation

$$u^{(n)} + \lambda a(t) f(u(t)) = 0, \quad t \in (0,1) \tag{1}$$

and one of the following boundary value conditions

$$u(1) = \beta u(\eta), \;\; u^{(i)}(0) = 0 \;\text{ for }\; i = 1, 2, \cdots, n-1, \tag{2}$$

and

$$u^{(n-1)}(1) = \beta u^{(n-1)}(\eta), \; u^{(i)}(0) = 0 \;\text{ for }\; i = 0, 1, \cdot, n-2, \tag{3}$$

where $\eta \in (0,1)$, $\beta \in [0,1)$ and $a: \; (0,1) \to R$. $f(0) > 0$, $\lambda > 0$ is a parameter. For the case where $\beta = 0$, (1)-(2) becomes

$$\begin{cases} u^{(n)} + \lambda a(t) f(u) = 0, & 0 < t < 1, \\ u^{(i)}(0) = u(1) = 0, & i = 1, 2, \cdots, n-1. \end{cases} \tag{4}$$

BVP(4) was studied by Eloe nd Henderson [6]. In [6], it was proved that BVP(4) has positive solutions under the following assumptions (A) and (B) or (A) and (C).

(A): $a: \; [0,1] \to [0, +\infty)$, $f: \; [0, +\infty) \to [0, +\infty)$ are continuous.

(B): $\lim_{x \to 0} \frac{f(x)}{x} = 0$ and $\lim_{x \to +\infty} \frac{f(x)}{x} = +\infty$(super-linear).

(C): $\lim_{x \to 0} \frac{f(x)}{x} = +\infty$ and $\lim_{x \to +\infty} \frac{f(x)}{x} = 0$(sub-linear).

BVP(1)-(2) also contains as special case the following BVP

$$\begin{cases} u''(t) + \lambda f(t, u) = 0, & 0 < t < 1, \\ u(0) = u(1) - \beta u(\eta) = 0. \end{cases} \tag{5}$$

In [5], Ma proved that BVP(5) has positive solutions under the conditions $0 < \beta < 1/\eta$, (A) and (B) or (A) and (C). Very recently, the authors in [9] proved that BVP(5) has at least three positive solutions by imposing conditions on f.

In this paper, we make the following assumptions.

(A_1): $M = \frac{1}{(1-\beta)(n-1)!} > 0$.

(A_2): $f: \; [0, +\infty) \to [0, +\infty)$ is continuous and $f(0) > 0$.

(A_3): $a: \; [0,1] \to R$ is continuous and there is $k > 1$ such that

$$\int_0^1 G(t,s) a^+(s) ds \geq k \int_0^1 G(t,s) a^-(s) ds \;\text{ for }\; t \in \; [0,1]$$

where $a^+(t) = \max\{0, a(t)\}$ and $a^-(t) = \max\{0, -a(t)\}$, $G(t,s)$ is defined by

$$G(t,s) = \frac{1}{(1-\beta)(n-1)!}\begin{cases} -(1-\beta)(t-s)^{n-1} + (1-s)^{n-1} & \\ -(1-\beta)\beta(\eta-s)^{n-1}, & 0 \le s \le t \le \eta < 1 \\ & \textit{or } 0 \le s \le \eta < t \le 1, \\ (1-s)^{n-1} - \beta(1-\beta)(\eta-s)^{n-1}, & 0 \le t \le s \le \eta < 1, \\ (1-s)^{n-1}, & 0 \le t \le \eta \le s \le 1 \\ & \textit{or } 0 \le \eta \le t \le s \le 1, \\ -(1-\beta)(t-s)^{n-1} + (1-s)^{n-1}, & 0 \le \eta \le s \le t \le 1 \end{cases}$$

for BVP(1) and (2) and

$$G(t,s) = \frac{1}{(1-\beta)(n-1)!}\begin{cases} (\beta-1)(t-s)^{n-1} + t^{n-1}(1-s) & \\ -\beta t^{n-1}(\eta-s), & 0 \le s \le \eta \le t \le< 1 \\ & \textit{or } 0 \le s \le t \le \eta < 1, \\ (\beta-1)(t-s)^{n-1} + t^{n-1}(1-s), & 0 \le \eta \le\le s \le t \le 1, \\ t^{n-1}(1-s), & 0 \le t \le \eta \le s \le 1 \\ & \textit{or } 0 \le \eta \le t \le s \le 1, \\ t^{n-1}(1-s) - \beta t^{n-1}(\eta-s), & 0 \le t \le s \le \eta < 1 \end{cases}$$

for BVP(1) and (3), respectively.

2 Main Results

In this section, we present the main results of this paper. The proofs of Theorems will be given in Section 3.

Theorem 1. *Let $(A_1)-(A_3)$ hold. Then there is a positive number λ^* such that BVP(1) and (2) has at least one positive solution for $\lambda \in (0, \lambda^*)$.*

Theorem 2. *Let $(A_1), (A_2)$ and (A_3) hold. Then there is a positive number λ^* such that BVP(1) and (3) has at least one positive solution for $\lambda \in (0, \lambda^*)$.*

Theorem 3. *Suppose the following conditions are satisfied:*
(i) $\min_{t\in[0,1]} \int_0^1 G(t,s)a^-(s)ds > 0$;
(ii) *There are constants $1 \le \theta < k$ and $\mu > 0$ such that, for any $a > 0$,*

$$\mu a \le f(x) \le \theta\mu a \quad \textit{for} \quad 0 \le x \le a;$$

(iii) $\lambda > \left((k-\theta)\mu \min_{t\in[0,1]} \int_0^1 G(t,s)a^-(s)ds\right)^{-1}$.
Then BVP(1) and (2) has no positive solution.

Theorem 4. *Suppose the following conditions are satisfied:*
(i) $\min_{t\in[0,1]} \int_0^1 G(t,s)a^-(s)ds > 0$;
(ii) *There are constants* $1 \leq \theta < k$ *and* $\mu > 0$ *such that, for any* $a > 0$,

$$\mu a \leq f(x) \leq \theta\mu a \quad for \quad 0 \leq x \leq a;$$

(iii) $\lambda > \left((k-\theta)\mu \min_{t\in[0,1]} \int_0^1 G(t,s)a^-(s)ds\right)^{-1}$.
Then $BVP(1)$ *and* (2) *has no positive solution.*

Our Theorems are new and different from [3-6] and [9] and are easy to check, Particularly, we don't need the assumptions that f is either super-linear or sub-linear, which was supposed in [3-6].

By the way, the proofs of the theorems is based on the Leray-Schauder fixed point Theorem and motivated by [12,13]. In [12], Hai studied the existence of positive solutions for elliptic equation

$$\Delta u + \lambda a(t)g(u) = 0, \quad u|_{\partial\Omega} = 0$$

where a may change sign. We note that the techniques in our paper are well know for elliptic problems and Sturm-Liuville problem, see [7,8] and the references cited therein.

3 Proofs of Theorems

In order prove Theorem 1, we need the following Lemmas.
Lemma 1. *Suppose that* $M \neq 0$. *Then for* $y \in C[0,1]$, *the problem*

$$\begin{cases} u^{(n)} + y(t) = 0, & t \in (0,1), \\ u(1) = \beta u(\eta), \ u^{(i)}(0) = 0 \ \ for \ \ i = 1,2,\cdots,n-1 \end{cases} \tag{6}$$

has unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds$$

where $G(t,s)$ *is defined in section 1.*
Proof . To the purpose, we let

$$u(t) = -\int_0^t \frac{t-s)^{n-1}}{(n-1)!}y(s)ds + B + \sum_{i=1}^{n-1} A_i t^i. \tag{7}$$

Since $u^{(i)}(0) = 0$ for $i = 1,2,\cdots,n-1$, one gets $A_i = 0$ for $i = 1,2,\cdots,n-1$. Now, we solve B. By $u(1) = \beta u(\eta)$, it follows that

$$-\beta\int_0^\eta \frac{(\eta-s)^{n-1}}{(n-1)!}y(s)ds + \beta B = -\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y(s)ds + B.$$

Solving the above equations, we get

$$B = \frac{1}{(1-\beta)(n-1)!}\left[\int_0^1 (1-s)^{n-1}y(s)ds - \beta\int_0^\eta (\eta-s)^{n-1}y(s)ds\right].$$

Substituting A_i and B into (7), one has

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y(s)ds \\ &\quad +\frac{1}{(1-\beta)(n-1)!}[\int_0^1 (1-s)^{n-1}y(s)ds - \beta\int_0^\eta (\eta-s)^{n-1}y(s)ds] \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

Lemma 2. *Let $M > 0$. If $y \in C[0,1]$ and $y(t) \geq 0$, then the unique solution of* (6) *satisfies $u(t) \geq 0$ for all $t \in [0,1]$.*
Proof . It suffices to prove that

$$G(t,s) \geq 0, \quad \text{for} \quad (t,s) \in [0,1]\times[0,1]. \tag{8}$$

This is simple and then omitted.
Lemma 3. *Suppose that $(A_1) - (A_3)$ hold. Then for every $0 < \delta < 1$, there exists a positive number $\overline{\lambda}$ such that, for $\lambda \in (0,\overline{\lambda})$, the equation*

$$\begin{cases} u^{(n)} + \lambda a^+(t)f(u(t)) = 0, & t \in (0,1), \\ u(1) = \beta u(\eta),\ u^{(i)}(0) = 0 \ \ for \ \ i = 1,2,\cdots,n-1 \end{cases} \tag{9}$$

has a positive solution $\overline{u}_\lambda$ with $||\overline{u}_\lambda|| \to 0$ as $\lambda \to 0$ and

$$\overline{u}_\lambda \geq \lambda\delta f(0)||p(t)||, \tag{10}$$

where

$$p(t) = \int_0^1 G(t,s)a^+(s)ds.$$

Proof . We know that $p(t) \geq 0$ for $t \in [0,1]$ and (9) is equivalent to the integral equation

$$u(t) = \lambda\int_0^1 G(t,s)a^+(s)f(u(s))ds := Tu(t) \tag{11}$$

where $u \in X := C[0,1]$. It is easy to prove that T is completely continuous, $TX \subset X$ and the fixed points of T are solutions of (9). We shall apply the Leray-Schauder fixed point Theorem to prove T has at least one fixed point for small λ.
Let $\epsilon > 0$ be such that

$$f(t) \geq \delta f(0) \quad \text{for} \quad 0 \leq t \leq \epsilon. \tag{12}$$

Suppose that

$$0 < \lambda < \frac{\epsilon}{2||p||\overline{f}(\epsilon)} := \overline{\lambda},$$

where $\overline{f}(t) = \max_{0\leq s\leq t} f(s)$. Since

$$\lim_{t\to 0^+}\frac{\overline{f}(t)}{t} = +\infty,$$

together with $\overline{f}(\epsilon)/\epsilon < 1/(2||p||\lambda)$, then there is $r_\lambda \in (0,\epsilon)$ such that

$$\frac{\overline{f}(r_\lambda)}{r_\lambda} = \frac{1}{2\lambda||p||}.$$

We note that this implies $r_\lambda \to 0$ as $\lambda \to 0$.

Now, consider the homotopy equation

$$u = \theta T u, \quad \theta \in (0,1).$$

Let $u \in X$ and $\theta \in (0,1)$ be such that $u = \theta T u$. We claim that $||u|| \neq r_\lambda$. In fact,

$$u(t) = \theta\lambda \int_0^1 G(t,s)a^+(s)f(u(s))ds.$$

Set

$$w(t) = \theta\lambda \int_0^1 G(t,s)a^+(s)\overline{f}(||u||)ds \leq \theta\lambda\overline{f}(||u||)p(t).$$

Then by $f(u) \leq \overline{f}(||u||)$, we know that $u(t) \leq w(t)$ for all $t \in R$. Moreover, we have

$$||u|| \leq \lambda ||p|| \overline{f}(||u||),$$

i.e.,

$$\frac{\overline{f}(||u||)}{||u||} \geq \frac{1}{\lambda ||p||},$$

which implies that $||u|| \neq r_\lambda$. Thus by Leray-Schauder fixed point Theorem, T has a fixed point $\overline{u}_\lambda$ with

$$||\overline{u}_\lambda|| \leq r_\lambda < \epsilon.$$

Moreover, combining (11) and (12), we get

$$\overline{u}_\lambda \geq \lambda\delta f(0)p(t), \quad t \in R. \tag{13}$$

This completes the proof.

Proof of Theorem 1. Let

$$q(t) = \int_0^1 G(t,s)a^-(s)ds. \tag{14}$$

Then $q(t) \geq 0$. Since $p(t)/q(t) \geq k > 1$. Choosing $d \in (0,1)$ such that $kd > 1$, there is $c > 0$ such that $|f(y)| \leq kdf(0)$ for $y \in [0,c]$, then

$$q(t)|f(y)| \leq dp(t)f(0) \quad t \in R \ \ y \in [0,c].$$

Fix $\delta \in (d,1)$ and let $\lambda^* > 0$ be such that

$$||\overline{u_\lambda}|| + \lambda\delta f(0)||p|| \leq c, \quad \lambda \in (0,\lambda^*), \tag{15}$$

where $\overline{u}_\lambda$ is given by Lemma 3 and

$$|f(x) - f(y)| \leq f(0)\frac{\delta - d}{2} \tag{16}$$

for $x, y \in [-c,c]$ with $|x - y| \leq \lambda^*\delta f(0)||p||$.

Let $\lambda \in (0,\lambda^*)$, we consider, for $y \in C[0,1]$, the following equation

$$\begin{cases} w^{(n)} + \lambda a^+(t)[f(\overline{u}_\lambda + y) - f(\overline{u}_\lambda)] - \lambda a^-(t)f(\overline{u}_\lambda + y) = 0, 0 < t < 1, \\ w(1) = \beta w(\eta), \ w^{(i)}(0) = 0 \ \text{ for } \ i = 1,2,\cdots,n-1. \end{cases} \tag{17}$$

For each $y \in C[0,1]$, let $w = Ty$ be the solution of (17). We look for a solution y_λ of the form $\overline{u}_\lambda + y_\lambda$ such that y_λ solves the following equation

$$\begin{cases} y^{(n)} + \lambda a^+(t)[f(\overline{u}_\lambda + y) - f(\overline{u}_\lambda)] - \lambda a^-(t) f(\overline{u}_\lambda + y) = 0, 0 < t < 1, \\ y(1) = \beta y(\eta),\ y^{(i)}(0) = 0 \ \text{ for } \ i = 1, 2, \cdots, n-1. \end{cases}$$

It is easy to check that T is completely continuous. Let $y \in X$ and $\theta \in (0,1)$ be such that $y = \theta Ty$, then we have

$$y^{(n)} + \lambda\theta a^+(t)[f(\overline{u}_\lambda + y) - f(\overline{u}_\lambda)] - \lambda\theta a^-(t) f(\overline{u}_\lambda + y) = 0, 0 < t < 1.$$

We claim that $||y|| \neq \lambda\delta f(0)||p||$. Suppose to the contrary that $||y|| = \lambda\delta f(0)||p||$. Then by (15) and (16), we get

$$||\overline{u}_\lambda + y|| \leq ||\overline{u}_\lambda|| + ||y|| \leq c \tag{18}$$

and

$$|f(\overline{u}_\lambda + y) - f(\overline{u}_\lambda) \leq f(0)\frac{\delta - d}{2}. \tag{19}$$

Using (12) and $q(t)|f(y)| \leq dp(t)f(0)$, we get

$$\begin{aligned} |y(t)| &= \lambda \bigg| \int_0^1 G(t,s)a^+(s)[f(\overline{u}_\lambda(s) + y(s)) - f(\overline{u}_\lambda(s))]ds \\ &\quad + \lambda \int_0^1 G(t,s)a^-(s) f(\overline{u}_\lambda(s) + y(s))ds \bigg| \\ &\leq \lambda \int_0^1 G(t,s)a^+(s) f(0)\frac{\delta - d}{2} ds + \lambda \int_0^1 G(t,s)a^-(s)\frac{p(t)}{q(t)} df(0) ds \\ &\leq \lambda \frac{\delta - d}{2} p(t) f(0) + \lambda d f(0) p(t) \\ &= \lambda \frac{\delta + d}{2} f(0) p(t). \end{aligned}$$

In particular,

$$||y|| \leq \lambda \frac{\delta + d}{2} f(0)||p|| < \lambda\delta f(0)||p|| \tag{20}$$

a contradiction and the claim is proved. Thus by Leray-Schauder fixed point Theorem, T has a fixed point y_λ with

$$||y_\lambda|| \leq \lambda\delta f(0)||p||.$$

Using Lemma 3 and (20), we obtain

$$\begin{aligned} u_\lambda(t) &\geq \overline{u}_\lambda - ||y_\lambda|| \\ &\geq \lambda\delta f(0)p(t) - \lambda\frac{\delta + d}{2} f(0)p(t) \\ &= \lambda\frac{\delta - d}{2} f(0)p(t) > 0, \end{aligned}$$

i.e. u_λ is a positive T-periodic solution. The proof of Theorem 1 is complete.

Proof of Theorem 2. Similarly, let $y \in C[0,1]$. then the unique solution of the equation

$$\begin{cases} u^{(n)} + y(t) = 0, \quad t \in (0,1), \\ u^{(n-1)}(1) = \beta u^{(n-1)}(\eta), \ u^{(i)}(0) = 0 \ \text{ for } \ i = 0,1,2,\cdots,n-2 \end{cases} \tag{21}$$

has unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds$$

where $G(t,s)$ is defined as follows.

$$G(t,s) = \frac{1}{(1-\beta)(n-1)!}\begin{cases} (\beta-1)(t-s)^{n-1} + t^{n-1}(1-s) & \\ -\beta t^{n-1}(\eta - s), & 0 \le s \le \eta \le t \le< 1 \\ & \textit{or } 0 \le s \le t \le \eta < 1, \\ (\beta-1)(t-s)^{n-1} + t^{n-1}(1-s), & 0 \le \eta \le\le s \le t \le 1, \\ t^{n-1}(1-s), & 0 \le t \le \eta \le s \le 1 \\ & \textit{or } 0 \le \eta \le t \le s \le 1, \\ t^{n-1}(1-s) - \beta t^{n-1}(\eta - s), & 0 \le t \le s \le \eta < 1. \end{cases}$$

It is easy to see that if $y(t) \ge 0$, then $u(t) \ge 0$ for all $t \in [0,1]$. The remainder of the proof is similar to those in Theorem 1 and then omitted.

Proof of Theorem 3. To the contrary, $u(t)$ is a positive solution of BVP(1) and (2), suppose $||u|| = a > 0$. Then $0 \le u(t) \le a$ for $t \in [0,1]$. Hence we have

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s)a(s)f(u(s))ds \\ &= \lambda \left(\int_0^1 G(t,s)a^+(s)f(u(s))ds - \int_0^1 G(t,s)a^-(s)f(u(s))ds \right) \\ &\ge \lambda \left(\mu a \int_0^1 G(t,s)a^+(s)ds - \theta \mu a \int_0^1 G(t,s)a^-(s)ds \right) \\ &\ge \lambda \left(k\mu a \int_0^1 G(t,s)a^-(s)ds - \theta \mu a \int_0^1 G(t,s)a^-(s)ds \right) \\ &= \lambda(k-\theta)\mu a \int_0^1 G(t,s)a^-(s)ds \\ &> a = ||u||, \end{aligned}$$

which is a contradiction. Hence BVP(1) and (2) has no positive solution.

Proof of Theorem 4. It is similar to that of Theorem 3 and is omitted.

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